PUZZLER

Soft contact lenses are comfortable to wear because they attract the proteins in the wearer's tears, incorporating the complex molecules right into the lenses. They become, in a sense, part of the wearer. Some types of makeup exploit this same attractive force to adhere to the skin. What is the nature of this force? *(Charles D. Winters)*

chapter

23 Electric Fields

Chapter Outline

The electromagnetic force between charged particles is one of the fundamental forces of nature. We begin this chapter by describing some of the basic properties of electric forces. We then discuss Coulomb's law, which is the fundamental law governing the force between any two charged particles. Next, we introduce the concept of an electric field associated with a charge distribution and describe its effect on other charged particles. We then show how to use Coulomb's law to calculate the electric field for a given charge distribution. We conclude the chapter with a discussion of the motion of a charged particle in a uniform electric field.

23.1 PROPERTIES OF ELECTRIC CHARGES

11.2 A number of simple experiments demonstrate the existence of electric forces and charges. For example, after running a comb through your hair on a dry day, you will find that the comb attracts bits of paper. The attractive force is often strong enough to suspend the paper. The same effect occurs when materials such as glass or rubber are rubbed with silk or fur.

Another simple experiment is to rub an inflated balloon with wool. The balloon then adheres to a wall, often for hours. When materials behave in this way, they are said to be *electrified,* or to have become **electrically charged.** You can easily electrify your body by vigorously rubbing your shoes on a wool rug. The electric charge on your body can be felt and removed by lightly touching (and startling) a friend. Under the right conditions, you will see a spark when you touch, and both of you will feel a slight tingle. (Experiments such as these work best on a dry day because an excessive amount of moisture in the air can cause any charge you build up to "leak" from your body to the Earth.)

In a series of simple experiments, it is found that there are two kinds of electric charges, which were given the names **positive** and **negative** by Benjamin Franklin (1706–1790). To verify that this is true, consider a hard rubber rod that has been rubbed with fur and then suspended by a nonmetallic thread, as shown in Figure 23.1. When a glass rod that has been rubbed with silk is brought near the rubber rod, the two attract each other (Fig. 23.1a). On the other hand, if two charged rubber rods (or two charged glass rods) are brought near each other, as shown in Figure 23.1b, the two repel each other. This observation shows that the rubber and glass are in two different states of electrification. On the basis of these observations, we conclude that **like charges repel one another and unlike charges attract one another.**

QuickLab

Rub an inflated balloon against your hair and then hold the balloon near a thin stream of water running from a faucet. What happens? (A rubbed plastic pen or comb will also work.)

Using the convention suggested by Franklin, the electric charge on the glass rod is called positive and that on the rubber rod is called negative. Therefore, any charged object attracted to a charged rubber rod (or repelled by a charged glass rod) must have a positive charge, and any charged object repelled by a charged rubber rod (or attracted to a charged glass rod) must have a negative charge.

Attractive electric forces are responsible for the behavior of a wide variety of commercial products. For example, the plastic in many contact lenses, *etafilcon,* is made up of molecules that electrically attract the protein molecules in human tears. These protein molecules are absorbed and held by the plastic so that the lens ends up being primarily composed of the wearer's tears. Because of this, the wearer's eye does not treat the lens as a foreign object, and it can be worn comfortably. Many cosmetics also take advantage of electric forces by incorporating materials that are electrically attracted to skin or hair, causing the pigments or other chemicals to stay put once they are applied.

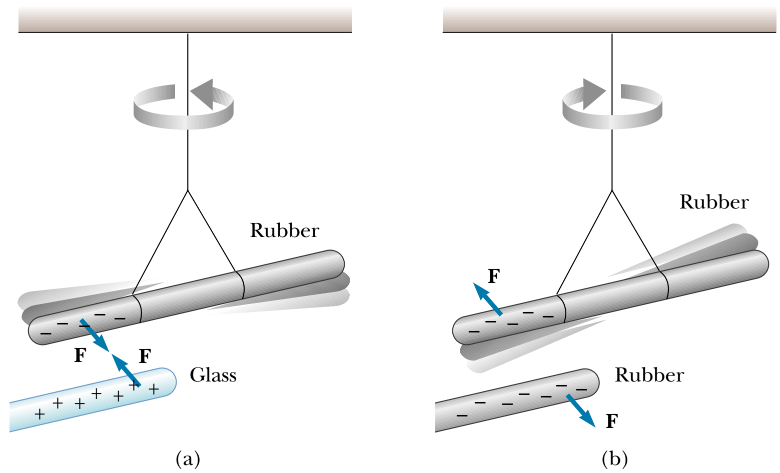


Figure 23.1 (a) A negatively charged rubber rod suspended by a thread is attracted to a positively charged glass rod. (b) A negatively charged rubber rod is repelled by another negatively charged rubber rod.

Charge is conserved

Another important aspect of Franklin's model of electricity is the implication that **electric charge is always conserved.** That is, when one object is rubbed against another, charge is not created in the process. The electrified state is due to a *transfer* of charge from one object to the other. One object gains some amount of negative charge while the other gains an equal amount of positive charge. For example, when a glass rod is rubbed with silk, the silk obtains a negative charge that is equal in magnitude to the positive charge on the glass rod. We now know from our understanding of atomic structure that negatively charged electrons are transferred from the glass to the silk in the rubbing process. Similarly, when rubber is rubbed with fur, electrons are transferred from the fur to the rubber, giving the rubber a net negative charge and the fur a net positive charge. This process is consistent with the fact that neutral, uncharged matter contains as many positive charges (protons within atomic nuclei) as negative charges (electrons).

Figure 23.2 Rubbing a balloon against your hair on a dry day causes the balloon and your hair to become charged.

Quick Quiz 23.1

If you rub an inflated balloon against your hair, the two materials attract each other, as shown in Figure 23.2. Is the amount of charge present in the balloon and your hair after rubbing (a) less than, (b) the same as, or (c) more than the amount of charge present before rubbing?

Charge is quantized

In 1909, Robert Millikan (1868–1953) discovered that electric charge always occurs as some integral multiple of a fundamental amount of charge e. In modern terms, the electric charge q is said to be **quantized,** where q is the standard symbol used for charge. That is, electric charge exists as discrete "packets," and we can write $q = Ne$, where N is some integer. Other experiments in the same period showed that the electron has a charge $-e$ and the proton has a charge of equal magnitude but opposite sign $+e$. Some particles, such as the neutron, have no charge. A neutral atom must contain as many protons as electrons.

Because charge is a conserved quantity, the net charge in a closed region remains the same. If charged particles are created in some process, they are always created in pairs whose members have equal-magnitude charges of opposite sign.

From our discussion thus far, we conclude that electric charge has the following important properties:

Properties of electric charge

- Two kinds of charges occur in nature, with the property that unlike charges attract one another and like charges repel one another.
- Charge is conserved.
- Charge is quantized.

23.2 INSULATORS AND CONDUCTORS

11.3 It is convenient to classify substances in terms of their ability to conduct electric charge:

Electrical **conductors** are materials in which electric charges move freely, whereas electrical **insulators** are materials in which electric charges cannot move freely.

Materials such as glass, rubber, and wood fall into the category of electrical insulators. When such materials are charged by rubbing, only the area rubbed becomes charged, and the charge is unable to move to other regions of the material.

Metals are good conductors

In contrast, materials such as copper, aluminum, and silver are good electrical conductors. When such materials are charged in some small region, the charge readily distributes itself over the entire surface of the material. If you hold a copper rod in your hand and rub it with wool or fur, it will not attract a small piece of paper. This might suggest that a metal cannot be charged. However, if you attach a wooden handle to the rod and then hold it by that handle as you rub the rod, the rod will remain charged and attract the piece of paper. The explanation for this is as follows: Without the insulating wood, the electric charges produced by rubbing readily move from the copper through your body and into the Earth. The insulating wooden handle prevents the flow of charge into your hand.

Semiconductors are a third class of materials, and their electrical properties are somewhere between those of insulators and those of conductors. Silicon and germanium are well-known examples of semiconductors commonly used in the fabrication of a variety of electronic devices, such as transistors and light-emitting diodes. The electrical properties of semiconductors can be changed over many orders of magnitude by the addition of controlled amounts of certain atoms to the materials.

When a conductor is connected to the Earth by means of a conducting wire or pipe, it is said to be **grounded.** The Earth can then be considered an infinite "sink" to which electric charges can easily migrate. With this in mind, we can understand how to charge a conductor by a process known as **induction.**

Charging by induction

To understand induction, consider a neutral (uncharged) conducting sphere insulated from ground, as shown in Figure 23.3a. When a negatively charged rubber rod is brought near the sphere, the region of the sphere nearest the rod obtains an excess of positive charge while the region farthest from the rod obtains an equal excess of negative charge, as shown in Figure 23.3b. (That is, electrons in the region nearest the rod migrate to the opposite side of the sphere. This occurs even if the rod never actually touches the sphere.) If the same experiment is performed with a conducting wire connected from the sphere to ground (Fig. 23.3c), some of the electrons in the conductor are so strongly repelled by the presence of

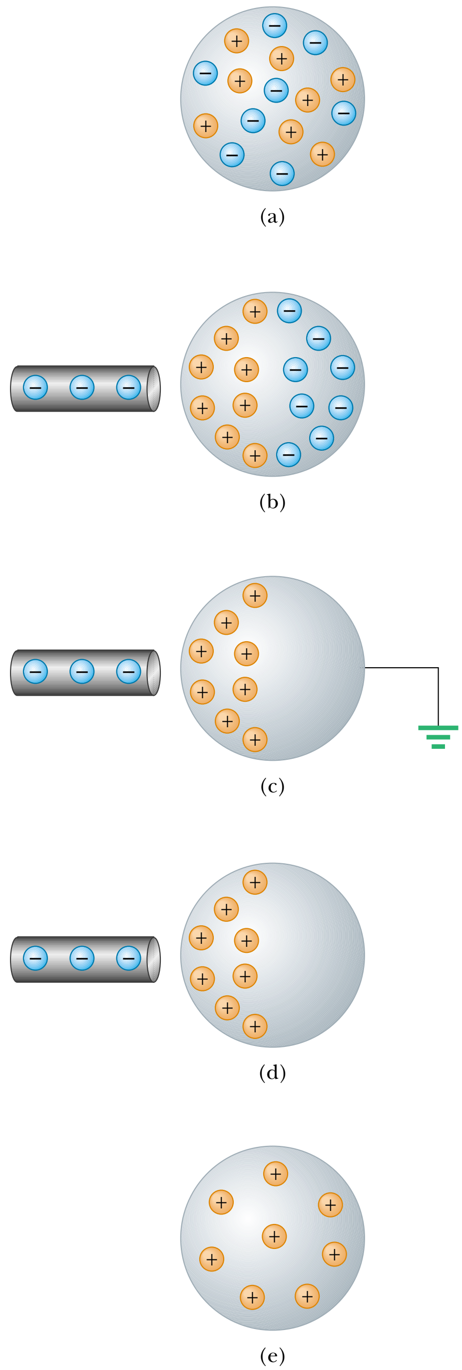
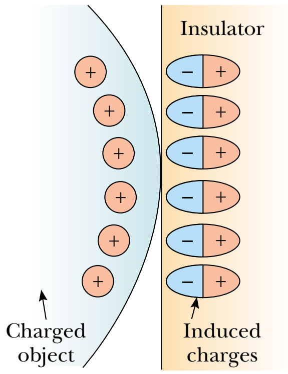


Figure 23.3 Charging a metallic object by *induction* (that is, the two objects never touch each other). (a) A neutral metallic sphere, with equal numbers of positive and negative charges. (b) The charge on the neutral sphere is redistributed when a charged rubber rod is placed near the sphere. (c) When the sphere is grounded, some of its electrons leave through the ground wire. (d) When the ground connection is removed, the sphere has excess positive charge that is nonuniformly distributed. (e) When the rod is removed, the excess positive charge becomes uniformly distributed over the surface of the sphere.



(a)

(b)

Figure 23.4 (a) The charged object on the left induces charges on the surface of an insulator. (b) A charged comb attracts bits of paper because charges are displaced in the paper.

QuickLab

Tear some paper into very small pieces. Comb your hair and then bring the comb close to the paper pieces. Notice that they are accelerated toward the comb. How does the magnitude of the electric force compare with the magnitude of the gravitational force exerted on the paper? Keep watching and you might see a few pieces jump away from the comb. They don't just fall away; they are repelled. What causes this?

the negative charge in the rod that they move out of the sphere through the ground wire and into the Earth. If the wire to ground is then removed (Fig. 23.3d), the conducting sphere contains an excess of *induced* positive charge. When the rubber rod is removed from the vicinity of the sphere (Fig. 23.3e), this induced positive charge remains on the ungrounded sphere. Note that the charge remaining on the sphere is uniformly distributed over its surface because of the repulsive forces among the like charges. Also note that the rubber rod loses none of its negative charge during this process.

Charging an object by induction requires no contact with the body inducing the charge. This is in contrast to charging an object by rubbing (that is, by *conduction*), which does require contact between the two objects.

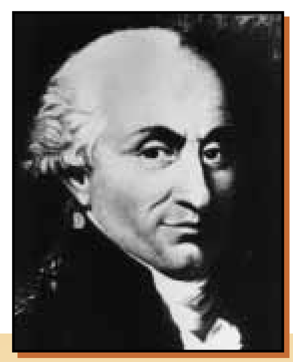
A process similar to induction in conductors takes place in insulators. In most neutral molecules, the center of positive charge coincides with the center of negative charge. However, in the presence of a charged object, these centers inside each molecule in an insulator may shift slightly, resulting in more positive charge on one side of the molecule than on the other. This realignment of charge within individual molecules produces an induced charge on the surface of the insulator, as shown in Figure 23.4. Knowing about induction in insulators, you should be able to explain why a comb that has been rubbed through hair attracts bits of electrically neutral paper and why a balloon that has been rubbed against your clothing is able to stick to an electrically neutral wall.

Quick Quiz 23.2

Object A is attracted to object B. If object B is known to be positively charged, what can we say about object A? (a) It is positively charged. (b) It is negatively charged. (c) It is electrically neutral. (d) Not enough information to answer.

23.3 COULOMB'S LAW

11.4 Charles Coulomb (1736–1806) measured the magnitudes of the electric forces between charged objects using the torsion balance, which he invented (Fig. 23.5).



Charles Coulomb (1736–1806)
Coulomb's major contribution to science was in the field of electrostatics and magnetism. During his lifetime, he also investigated the strengths of materials and determined the forces that affect objects on beams, thereby contributing to the field of structural mechanics. In the field of ergonomics, his research provided a fundamental understanding of the ways in which people and animals can best do work.
(Photo courtesy of AIP Niels Bohr Library/E. Scott Barr Collection)

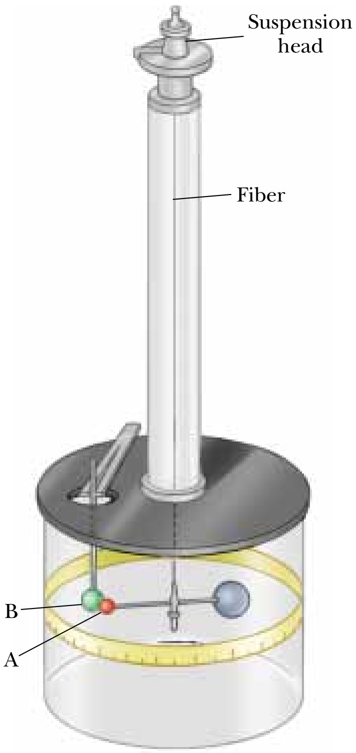


Figure 23.5 Coulomb's torsion balance, used to establish the inverse-square law for the electric force between two charges.

Coulomb confirmed that the electric force between two small charged spheres is proportional to the inverse square of their separation distance r—that is, $F_e \propto 1/r^2$. The operating principle of the torsion balance is the same as that of the apparatus used by Cavendish to measure the gravitational constant (see Section 14.2), with the electrically neutral spheres replaced by charged ones. The electric force between charged spheres A and B in Figure 23.5 causes the spheres to either attract or repel each other, and the resulting motion causes the suspended fiber to twist. Because the restoring torque of the twisted fiber is proportional to the angle through which the fiber rotates, a measurement of this angle provides a quantitative measure of the electric force of attraction or repulsion. Once the spheres are charged by rubbing, the electric force between them is very large compared with the gravitational attraction, and so the gravitational force can be neglected.

Coulomb's experiments showed that the **electric force** between two stationary charged particles

- is inversely proportional to the square of the separation r between the particles and directed along the line joining them;
- is proportional to the product of the charges q_1 and q_2 on the two particles;
- is attractive if the charges are of opposite sign and repulsive if the charges have the same sign.

From these observations, we can express **Coulomb's law** as an equation giving the magnitude of the electric force (sometimes called the *Coulomb force*) between two point charges:

$$F_e = k_e \frac{|q_1||q_2|}{r^2} \tag{23.1}$$

where k_e is a constant called the **Coulomb constant.** In his experiments, Coulomb was able to show that the value of the exponent of r was 2 to within an uncertainty of a few percent. Modern experiments have shown that the exponent is 2 to within an uncertainty of a few parts in 10^{16}.

The value of the Coulomb constant depends on the choice of units. The SI unit of charge is the **coulomb** (C). The Coulomb constant k_e in SI units has the value

Coulomb constant

$$k_e = 8.987\,5 \times 10^9 \text{ N}\cdot\text{m}^2/\text{C}^2$$

This constant is also written in the form

$$k_e = \frac{1}{4\pi\epsilon_0}$$

where the constant ϵ_0 (lowercase Greek epsilon) is known as the *permittivity of free space* and has the value $8.854\,2 \times 10^{-12}\text{ C}^2/\text{N}\cdot\text{m}^2$.

The smallest unit of charge known in nature is the charge on an electron or proton,[1] which has an absolute value of

Charge on an electron or proton

$$|e| = 1.602\,19 \times 10^{-19}\text{ C}$$

Therefore, 1 C of charge is approximately equal to the charge of 6.24×10^{18} electrons or protons. This number is very small when compared with the number of

[1] No unit of charge smaller than e has been detected as a free charge; however, recent theories propose the existence of particles called *quarks* having charges $e/3$ and $2e/3$. Although there is considerable experimental evidence for such particles inside nuclear matter, *free* quarks have never been detected. We discuss other properties of quarks in Chapter 46 of the extended version of this text.

TABLE 23.1 Charge and Mass of the Electron, Proton, and Neutron

Particle	Charge (C)	Mass (kg)
Electron (e)	$-1.602\,191\,7 \times 10^{-19}$	$9.109\,5 \times 10^{-31}$
Proton (p)	$+1.602\,191\,7 \times 10^{-19}$	$1.672\,61 \times 10^{-27}$
Neutron (n)	0	$1.674\,92 \times 10^{-27}$

free electrons[2] in 1 cm^3 of copper, which is of the order of 10^{23}. Still, 1 C is a substantial amount of charge. In typical experiments in which a rubber or glass rod is charged by friction, a net charge of the order of 10^{-6} C is obtained. In other words, only a very small fraction of the total available charge is transferred between the rod and the rubbing material.

The charges and masses of the electron, proton, and neutron are given in Table 23.1.

EXAMPLE 23.1 The Hydrogen Atom

The electron and proton of a hydrogen atom are separated (on the average) by a distance of approximately 5.3×10^{-11} m. Find the magnitudes of the electric force and the gravitational force between the two particles.

Solution From Coulomb's law, we find that the attractive electric force has the magnitude

$$F_e = k_e \frac{|e|^2}{r^2} = \left(8.99 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right) \frac{(1.60 \times 10^{-19}\ \text{C})^2}{(5.3 \times 10^{-11}\ \text{m})^2}$$

$$= \boxed{8.2 \times 10^{-8}\ \text{N}}$$

Using Newton's law of gravitation and Table 23.1 for the particle masses, we find that the gravitational force has the magnitude

$$F_g = G\frac{m_e m_p}{r^2}$$

$$= \left(6.7 \times 10^{-11} \frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)$$

$$= \times \frac{(9.11 \times 10^{-31}\ \text{kg})(1.67 \times 10^{-27}\ \text{kg})}{(5.3 \times 10^{-11}\ \text{m})^2}$$

$$= \boxed{3.6 \times 10^{-47}\ \text{N}}$$

The ratio $F_e/F_g \approx 2 \times 10^{39}$. Thus, the gravitational force between charged atomic particles is negligible when compared with the electric force. Note the similarity of form of Newton's law of gravitation and Coulomb's law of electric forces. Other than magnitude, what is a fundamental difference between the two forces?

When dealing with Coulomb's law, you must remember that force is a vector quantity and must be treated accordingly. Thus, the law expressed in vector form for the electric force exerted by a charge q_1 on a second charge q_2, written $\mathbf{F}_{12}$, is

$$\mathbf{F}_{12} = k_e \frac{q_1 q_2}{r^2}\,\hat{\mathbf{r}} \qquad \textbf{(23.2)}$$

where $\hat{\mathbf{r}}$ is a unit vector directed from q_1 to q_2, as shown in Figure 23.6a. Because the electric force obeys Newton's third law, the electric force exerted by q_2 on q_1 is

[2] A metal atom, such as copper, contains one or more outer electrons, which are weakly bound to the nucleus. When many atoms combine to form a metal, the so-called *free electrons* are these outer electrons, which are not bound to any one atom. These electrons move about the metal in a manner similar to that of gas molecules moving in a container.

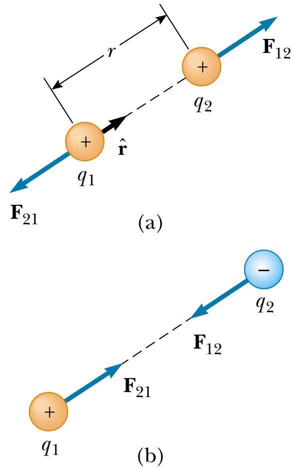


Figure 23.6 Two point charges separated by a distance r exert a force on each other that is given by Coulomb's law. The force $\mathbf{F}_{21}$ exerted by q_2 on q_1 is equal in magnitude and opposite in direction to the force $\mathbf{F}_{12}$ exerted by q_1 on q_2. (a) When the charges are of the same sign, the force is repulsive. (b) When the charges are of opposite signs, the force is attractive.

equal in magnitude to the force exerted by q_1 on q_2 and in the opposite direction; that is, $\mathbf{F}_{21} = -\mathbf{F}_{12}$. Finally, from Equation 23.2, we see that if q_1 and q_2 have the same sign, as in Figure 23.6a, the product q_1q_2 is positive and the force is repulsive. If q_1 and q_2 are of opposite sign, as shown in Figure 23.6b, the product q_1q_2 is negative and the force is attractive. Noting the sign of the product q_1q_2 is an easy way of determining the direction of forces acting on the charges.

Quick Quiz 23.3

Object A has a charge of $+2\ \mu\text{C}$, and object B has a charge of $+6\ \mu\text{C}$. Which statement is true?

(a) $\mathbf{F}_{AB} = -3\mathbf{F}_{BA}$. (b) $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$. (c) $3\mathbf{F}_{AB} = -\mathbf{F}_{BA}$.

When more than two charges are present, the force between any pair of them is given by Equation 23.2. Therefore, the resultant force on any one of them equals the vector sum of the forces exerted by the various individual charges. For example, if four charges are present, then the resultant force exerted by particles 2, 3, and 4 on particle 1 is

$$\mathbf{F}_1 = \mathbf{F}_{21} + \mathbf{F}_{31} + \mathbf{F}_{41}$$

EXAMPLE 23.2 Find the Resultant Force

Consider three point charges located at the corners of a right triangle as shown in Figure 23.7, where $q_1 = q_3 = 5.0\ \mu\text{C}$, $q_2 = -2.0\ \mu\text{C}$, and $a = 0.10$ m. Find the resultant force exerted on q_3.

Solution First, note the direction of the individual forces exerted by q_1 and q_2 on q_3. The force F_{23} exerted by q_2 on q_3 is attractive because q_2 and q_3 have opposite signs. The force F_{13} exerted by q_1 on q_3 is repulsive because both charges are positive.

The magnitude of $\mathbf{F}_{23}$ is

$$F_{23} = k_e \frac{|q_2||q_3|}{a^2}$$

$$= \left(8.99 \times 10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right) \frac{(2.0 \times 10^{-6}\ \text{C})(5.0 \times 10^{-6}\ \text{C})}{(0.10\ \text{m})^2}$$

$$= 9.0\ \text{N}$$

Note that because q_3 and q_2 have opposite signs, $\mathbf{F}_{23}$ is to the left, as shown in Figure 23.7.

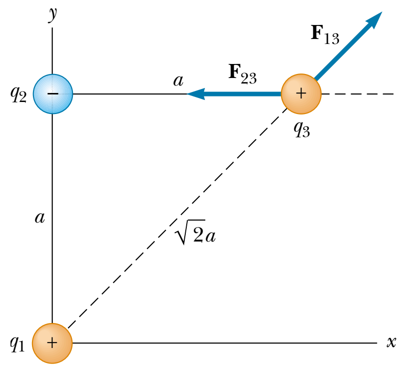


Figure 23.7 The force exerted by q_1 on q_3 is $\mathbf{F}_{13}$. The force exerted by q_2 on q_3 is $\mathbf{F}_{23}$. The resultant force $\mathbf{F}_3$ exerted on q_3 is the vector sum $\mathbf{F}_{13} + \mathbf{F}_{23}$.

The magnitude of the force exerted by q_1 on q_3 is

$$F_{13} = k_e \frac{|q_1||q_3|}{(\sqrt{2}a)^2}$$

$$= \left(8.99 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right) \frac{(5.0 \times 10^{-6}\ \text{C})(5.0 \times 10^{-6}\ \text{C})}{2(0.10\ \text{m})^2}$$

$$= 11\ \text{N}$$

The force $\mathbf{F}_{13}$ is repulsive and makes an angle of 45° with the x axis. Therefore, the x and y components of $\mathbf{F}_{13}$ are equal, with magnitude given by $F_{13}\cos 45° = 7.9$ N.

The force $\mathbf{F}_{23}$ is in the negative x direction. Hence, the x and y components of the resultant force acting on q_3 are

$$F_{3x} = F_{13x} + F_{23} = 7.9\ \text{N} - 9.0\ \text{N} = -1.1\ \text{N}$$

$$F_{3y} = F_{13y} = 7.9\ \text{N}$$

We can also express the resultant force acting on q_3 in unit-vector form as

$$\mathbf{F}_3 = \boxed{(-1.1\mathbf{i} + 7.9\mathbf{j})\ \text{N}}$$

Exercise Find the magnitude and direction of the resultant force $\mathbf{F}_3$.

Answer 8.0 N at an angle of 98° with the x axis.

EXAMPLE 23.3 Where Is the Resultant Force Zero?

Three point charges lie along the x axis as shown in Figure 23.8. The positive charge $q_1 = 15.0\ \mu$C is at $x = 2.00$ m, the positive charge $q_2 = 6.00\ \mu$C is at the origin, and the resultant force acting on q_3 is zero. What is the x coordinate of q_3?

Solution Because q_3 is negative and q_1 and q_2 are positive, the forces $\mathbf{F}_{13}$ and $\mathbf{F}_{23}$ are both attractive, as indicated in Figure 23.8. From Coulomb's law, $\mathbf{F}_{13}$ and $\mathbf{F}_{23}$ have magnitudes

$$F_{13} = k_e \frac{|q_1||q_3|}{(2.00 - x)^2} \qquad F_{23} = k_e \frac{|q_2||q_3|}{x^2}$$

For the resultant force on q_3 to be zero, $\mathbf{F}_{23}$ must be equal in magnitude and opposite in direction to $\mathbf{F}_{13}$, or

$$k_e \frac{|q_2||q_3|}{x^2} = k_e \frac{|q_1||q_3|}{(2.00 - x)^2}$$

Noting that k_e and q_3 are common to both sides and so can be dropped, we solve for x and find that

$$(2.00 - x)^2 |q_2| = x^2 |q_1|$$

$$(4.00 - 4.00x + x^2)(6.00 \times 10^{-6}\ \text{C}) = x^2(15.0 \times 10^{-6}\ \text{C})$$

Solving this quadratic equation for x, we find that $\boxed{x = 0.775\ \text{m.}}$ Why is the negative root not acceptable?

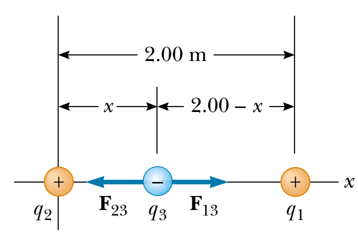


Figure 23.8 Three point charges are placed along the x axis. If the net force acting on q_3 is zero, then the force $\mathbf{F}_{13}$ exerted by q_1 on q_3 must be equal in magnitude and opposite in direction to the force $\mathbf{F}_{23}$ exerted by q_2 on q_3.

EXAMPLE 23.4 Find the Charge on the Spheres

Two identical small charged spheres, each having a mass of 3.0×10^{-2} kg, hang in equilibrium as shown in Figure 23.9a. The length of each string is 0.15 m, and the angle θ is 5.0°. Find the magnitude of the charge on each sphere.

Solution From the right triangle shown in Figure 23.9a, we see that $\sin\theta = a/L$. Therefore,

$$a = L\sin\theta = (0.15\ \text{m})\sin 5.0° = 0.013\ \text{m}$$

The separation of the spheres is $2a = 0.026$ m.

The forces acting on the left sphere are shown in Figure 23.9b. Because the sphere is in equilibrium, the forces in the

horizontal and vertical directions must separately add up to zero:

$$(1) \qquad \sum F_x = T \sin\theta - F_e = 0$$

$$(2) \qquad \sum F_y = T \cos\theta - mg = 0$$

From Equation (2), we see that $T = mg/\cos\theta$; thus, T can be

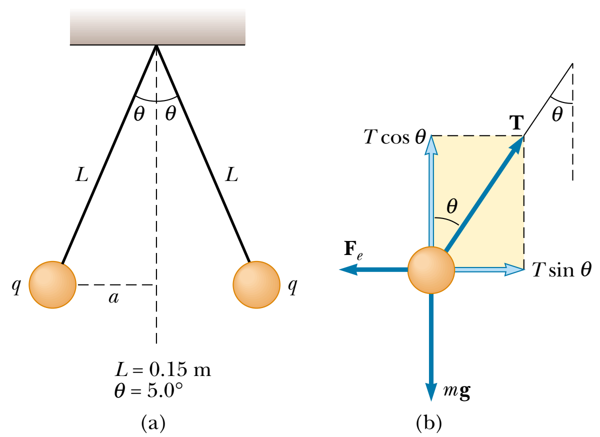


Figure 23.9 (a) Two identical spheres, each carrying the same charge q, suspended in equilibrium. (b) The free-body diagram for the sphere on the left.

eliminated from Equation (1) if we make this substitution. This gives a value for the magnitude of the electric force F_e:

$$\begin{aligned}(3) \qquad F_e &= mg \tan\theta \\ &= (3.0 \times 10^{-2}\ \text{kg})(9.80\ \text{m/s}^2)\tan 5.0° \\ &= 2.6 \times 10^{-2}\ \text{N}\end{aligned}$$

From Coulomb's law (Eq. 23.1), the magnitude of the electric force is

$$F_e = k_e \frac{|q|^2}{r^2}$$

where $r = 2a = 0.026$ m and $|q|$ is the magnitude of the charge on each sphere. (Note that the term $|q|^2$ arises here because the charge is the same on both spheres.) This equation can be solved for $|q|^2$ to give

$$|q|^2 = \frac{F_e r^2}{k_e} = \frac{(2.6 \times 10^{-2}\ \text{N})(0.026\ \text{m})^2}{8.99 \times 10^9\ \text{N}\cdot\text{m}^2/\text{C}^2}$$

$$|q| = \boxed{4.4 \times 10^{-8}\ \text{C}}$$

Exercise If the charge on the spheres were negative, how many electrons would have to be added to them to yield a net charge of -4.4×10^{-8} C?

Answer 2.7×10^{11} electrons.

QuickLab

For this experiment you need two 20-cm strips of transparent tape (mass of each ≈ 65 mg). Fold about 1 cm of tape over at one end of each strip to create a handle. Press both pieces of tape side by side onto a table top, rubbing your finger back and forth across the strips. Quickly pull the strips off the surface so that they become charged. Hold the tape handles together and the strips will repel each other, forming an inverted "V" shape. Measure the angle between the pieces, and estimate the excess charge on each strip. Assume that the charges act as if they were located at the center of mass of each strip.

23.4 THE ELECTRIC FIELD

11.5 Two field forces have been introduced into our discussions so far—the gravitational force and the electric force. As pointed out earlier, field forces can act through space, producing an effect even when no physical contact between the objects occurs. The gravitational field $\mathbf{g}$ at a point in space was defined in Section 14.6 to be equal to the gravitational force $\mathbf{F}_g$ acting on a test particle of mass m divided by that mass: $\mathbf{g} \equiv \mathbf{F}_g/m$. A similar approach to electric forces was developed by Michael Faraday and is of such practical value that we shall devote much attention to it in the next several chapters. In this approach, an **electric field** is said to exist in the region of space around a charged object. When another charged object enters this electric field, an electric force acts on it. As an example, consider Figure 23.10, which shows a small positive test charge q_0 placed near a second object carrying a much greater positive charge Q. We define the strength (in other words, the magnitude) of the electric field at the location of the test charge to be the electric force *per unit charge,* or to be more specific

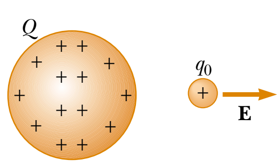


Figure 23.10 A small positive test charge q_0 placed near an object carrying a much larger positive charge Q experiences an electric field $\mathbf{E}$ directed as shown.

the electric field E at a point in space is defined as the electric force $\mathbf{F}_e$ acting on a positive test charge q_0 placed at that point divided by the magnitude of the test charge:

$$\mathbf{E} \equiv \frac{\mathbf{F}_e}{q_0} \tag{23.3}$$

Definition of electric field

Note that **E** is the field produced by some charge *external* to the test charge—it is not the field produced by the test charge itself. Also, note that the existence of an electric field is a property of its source. For example, every electron comes with its own electric field.

The vector **E** has the SI units of newtons per coulomb (N/C), and, as Figure 23.10 shows, its direction is the direction of the force a positive test charge experiences when placed in the field. We say that **an electric field exists at a point if a test charge at rest at that point experiences an electric force.** Once the magnitude and direction of the electric field are known at some point, the electric force exerted on *any* charged particle placed at that point can be calculated from

This dramatic photograph captures a lightning bolt striking a tree near some rural homes.

TABLE 23.2 Typical Electric Field Values

Source	E (N/C)
Fluorescent lighting tube	10
Atmosphere (fair weather)	100
Balloon rubbed on hair	1 000
Atmosphere (under thundercloud)	10 000
Photocopier	100 000
Spark in air	$>$3 000 000
Near electron in hydrogen atom	5×10^{11}

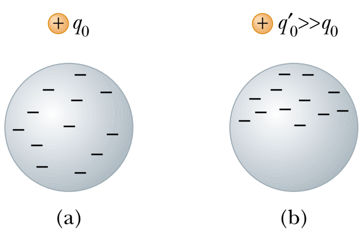


Figure 23.11 (a) For a small enough test charge q_0, the charge distribution on the sphere is undisturbed. (b) When the test charge q_0' is greater, the charge distribution on the sphere is disturbed as the result of the proximity of q_0'.

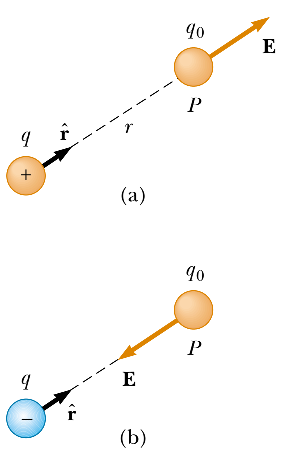


Figure 23.12 A test charge q_0 at point P is a distance r from a point charge q. (a) If q is positive, then the electric field at P points radially outward from q. (b) If q is negative, then the electric field at P points radially inward toward q.

Equation 23.3. Furthermore, the electric field is said to exist at some point (even empty space) **regardless of whether a test charge is located at that point.** (This is analogous to the gravitational field set up by any object, which is said to exist at a given point regardless of whether some other object is present at that point to "feel" the field.) The electric field magnitudes for various field sources are given in Table 23.2.

When using Equation 23.3, we must assume that the test charge q_0 is small enough that it does not disturb the charge distribution responsible for the electric field. If a vanishingly small test charge q_0 is placed near a uniformly charged metallic sphere, as shown in Figure 23.11a, the charge on the metallic sphere, which produces the electric field, remains uniformly distributed. If the test charge is great enough ($q_0' \gg q_0$), as shown in Figure 23.11b, the charge on the metallic sphere is redistributed and the ratio of the force to the test charge is different: ($F_e'/q_0' \neq F_e/q_0$). That is, because of this redistribution of charge on the metallic sphere, the electric field it sets up is different from the field it sets up in the presence of the much smaller q_0.

To determine the direction of an electric field, consider a point charge q located a distance r from a test charge q_0 located at a point P, as shown in Figure 23.12. According to Coulomb's law, the force exerted by q on the test charge is

$$\mathbf{F}_e = k_e \frac{q q_0}{r^2} \hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is a unit vector directed from q toward q_0. Because the electric field at P, the position of the test charge, is defined by $\mathbf{E} = \mathbf{F}_e/q_0$, we find that at P, the electric field created by q is

$$\mathbf{E} = k_e \frac{q}{r^2} \hat{\mathbf{r}} \tag{23.4}$$

If q is positive, as it is in Figure 23.12a, the electric field is directed radially outward from it. If q is negative, as it is in Figure 23.12b, the field is directed toward it.

To calculate the electric field at a point P due to a group of point charges, we first calculate the electric field vectors at P individually using Equation 23.4 and then add them vectorially. In other words,

> at any point P, the total electric field due to a group of charges equals the vector sum of the electric fields of the individual charges.

This superposition principle applied to fields follows directly from the superposition property of electric forces. Thus, the electric field of a group of charges can

This metallic sphere is charged by a generator so that it carries a net electric charge. The high concentration of charge on the sphere creates a strong electric field around the sphere. The charges then leak through the gas surrounding the sphere, producing a pink glow.

be expressed as

$$\mathbf{E} = k_e \sum_i \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i \tag{23.5}$$

where r_i is the distance from the ith charge q_i to the point P (the location of the test charge) and $\hat{\mathbf{r}}_i$ is a unit vector directed from q_i toward P.

Quick Quiz 23.4

A charge of $+3\ \mu$C is at a point P where the electric field is directed to the right and has a magnitude of 4×10^6 N/C. If the charge is replaced with a -3-μC charge, what happens to the electric field at P?

Example 23.5 Electric Field Due to Two Charges

A charge $q_1 = 7.0\ \mu$C is located at the origin, and a second charge $q_2 = -5.0\ \mu$C is located on the x axis, 0.30 m from the origin (Fig. 23.13). Find the electric field at the point P, which has coordinates (0, 0.40) m.

Solution First, let us find the magnitude of the electric field at P due to each charge. The fields $\mathbf{E}_1$ due to the 7.0-μC charge and $\mathbf{E}_2$ due to the -5.0-μC charge are shown in Figure 23.13. Their magnitudes are

$$E_1 = k_e \frac{|q_1|}{{r_1}^2} = \left(8.99 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right) \frac{(7.0 \times 10^{-6}\ \text{C})}{(0.40\ \text{m})^2}$$
$$= 3.9 \times 10^5\ \text{N/C}$$

$$E_2 = k_e \frac{|q_2|}{{r_2}^2} = \left(8.99 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right) \frac{(5.0 \times 10^{-6}\ \text{C})}{(0.50\ \text{m})^2}$$
$$= 1.8 \times 10^5\ \text{N/C}$$

The vector $\mathbf{E}_1$ has only a y component. The vector $\mathbf{E}_2$ has an x component given by $E_2 \cos\theta = \frac{3}{5}E_2$ and a negative y component given by $-E_2 \sin\theta = -\frac{4}{5}E_2$. Hence, we can express the vectors as

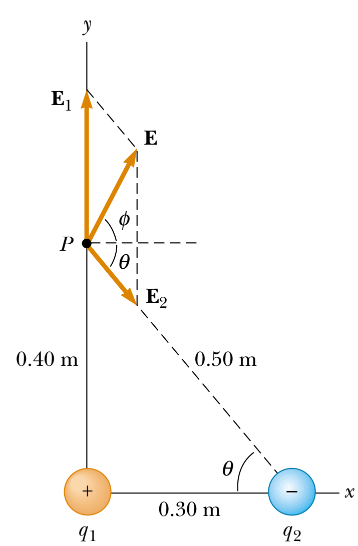


Figure 23.13 The total electric field $\mathbf{E}$ at P equals the vector sum $\mathbf{E}_1 + \mathbf{E}_2$, where $\mathbf{E}_1$ is the field due to the positive charge q_1 and $\mathbf{E}_2$ is the field due to the negative charge q_2.

$$\mathbf{E}_1 = 3.9 \times 10^5 \mathbf{j} \text{ N/C}$$

$$\mathbf{E}_2 = (1.1 \times 10^5 \mathbf{i} - 1.4 \times 10^5 \mathbf{j}) \text{ N/C}$$

The resultant field $\mathbf{E}$ at P is the superposition of $\mathbf{E}_1$ and $\mathbf{E}_2$:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \boxed{(1.1 \times 10^5 \mathbf{i} + 2.5 \times 10^5 \mathbf{j}) \text{ N/C}}$$

From this result, we find that $\mathbf{E}$ has a magnitude of 2.7×10^5 N/C and makes an angle ϕ of 66° with the positive x axis.

Exercise Find the electric force exerted on a charge of 2.0×10^{-8} C located at P.

Answer 5.4×10^{-3} N in the same direction as $\mathbf{E}$.

EXAMPLE 23.6 Electric Field of a Dipole

An **electric dipole** is defined as a positive charge q and a negative charge $-q$ separated by some distance. For the dipole shown in Figure 23.14, find the electric field $\mathbf{E}$ at P due to the charges, where P is a distance $y \gg a$ from the origin.

Solution At P, the fields $\mathbf{E}_1$ and $\mathbf{E}_2$ due to the two charges are equal in magnitude because P is equidistant from the charges. The total field is $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$, where

$$E_1 = E_2 = k_e \frac{q}{r^2} = k_e \frac{q}{y^2 + a^2}$$

The y components of $\mathbf{E}_1$ and $\mathbf{E}_2$ cancel each other, and the x components add because they are both in the positive x direction. Therefore, $\mathbf{E}$ is parallel to the x axis and has a magnitude equal to $2E_1 \cos\theta$. From Figure 23.14 we see that $\cos\theta = a/r = a/(y^2 + a^2)^{1/2}$. Therefore,

$$E = 2E_1 \cos\theta = 2k_e \frac{q}{(y^2 + a^2)} \frac{a}{(y^2 + a^2)^{1/2}}$$

$$= k_e \frac{2qa}{(y^2 + a^2)^{3/2}}$$

Because $y \gg a$, we can neglect a^2 and write

$$\boxed{E \approx k_e \frac{2qa}{y^3}}$$

Thus, we see that, at distances far from a dipole but along the perpendicular bisector of the line joining the two charges, the magnitude of the electric field created by the dipole varies as $1/r^3$, whereas the more slowly varying field of a point charge varies as $1/r^2$ (see Eq. 23.4). This is because at distant points, the fields of the two charges of equal magnitude and opposite sign almost cancel each other. The $1/r^3$ variation in E for the dipole also is obtained for a distant point along the x axis (see Problem 21) and for any general distant point.

The electric dipole is a good model of many molecules, such as hydrochloric acid (HCl). As we shall see in later chapters, neutral atoms and molecules behave as dipoles when placed in an external electric field. Furthermore, many molecules, such as HCl, are permanent dipoles. The effect of such dipoles on the behavior of materials subjected to electric fields is discussed in Chapter 26.

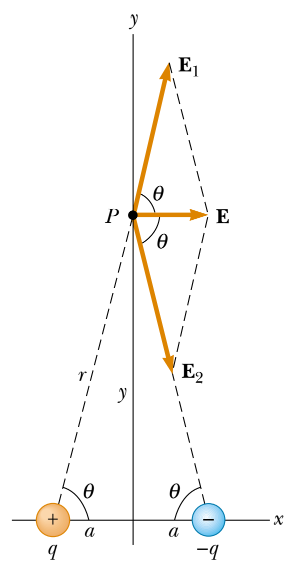


Figure 23.14 The total electric field $\mathbf{E}$ at P due to two charges of equal magnitude and opposite sign (an electric dipole) equals the vector sum $\mathbf{E}_1 + \mathbf{E}_2$. The field $\mathbf{E}_1$ is due to the positive charge q, and $\mathbf{E}_2$ is the field due to the negative charge $-q$.

23.5 ELECTRIC FIELD OF A CONTINUOUS CHARGE DISTRIBUTION

Very often the distances between charges in a group of charges are much smaller than the distance from the group to some point of interest (for example, a point where the electric field is to be calculated). In such situations, the system of

charges is smeared out, or *continuous*. That is, the system of closely spaced charges is equivalent to a total charge that is continuously distributed along some line, over some surface, or throughout some volume.

A continuous charge distribution

To evaluate the electric field created by a continuous charge distribution, we use the following procedure: First, we divide the charge distribution into small elements, each of which contains a small charge Δq, as shown in Figure 23.15. Next, we use Equation 23.4 to calculate the electric field due to one of these elements at a point P. Finally, we evaluate the total field at P due to the charge distribution by summing the contributions of all the charge elements (that is, by applying the superposition principle).

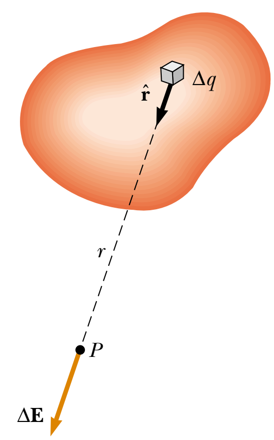


Figure 23.15 The electric field at P due to a continuous charge distribution is the vector sum of the fields $\Delta\mathbf{E}$ due to all the elements Δq of the charge distribution.

The electric field at P due to one element carrying charge Δq is

$$\Delta\mathbf{E} = k_e \frac{\Delta q}{r^2}\,\hat{\mathbf{r}}$$

where r is the distance from the element to point P and $\hat{\mathbf{r}}$ is a unit vector directed from the charge element toward P. The total electric field at P due to all elements in the charge distribution is approximately

$$\mathbf{E} \approx k_e \sum_i \frac{\Delta q_i}{r_i^{\,2}}\,\hat{\mathbf{r}}_i$$

where the index i refers to the ith element in the distribution. Because the charge distribution is approximately continuous, the total field at P in the limit $\Delta q_i \rightarrow 0$ is

$$\mathbf{E} = k_e \lim_{\Delta q_i \to 0} \sum_i \frac{\Delta q_i}{r_i^{\,2}}\,\hat{\mathbf{r}}_i = k_e \int \frac{dq}{r^2}\,\hat{\mathbf{r}} \qquad \textbf{(23.6)}$$

Electric field of a continuous charge distribution

where the integration is over the entire charge distribution. This is a vector operation and must be treated appropriately.

We illustrate this type of calculation with several examples, in which we assume the charge is uniformly distributed on a line, on a surface, or throughout a volume. When performing such calculations, it is convenient to use the concept of a charge density along with the following notations:

- If a charge Q is uniformly distributed throughout a volume V, the **volume charge density** ρ is defined by

  $$\rho \equiv \frac{Q}{V}$$

  Volume charge density

  where ρ has units of coulombs per cubic meter ($\mathrm{C/m^3}$).

- If a charge Q is uniformly distributed on a surface of area A, the **surface charge density** σ (lowercase Greek sigma) is defined by

  $$\sigma \equiv \frac{Q}{A}$$

  Surface charge density

  where σ has units of coulombs per square meter ($\mathrm{C/m^2}$).

- If a charge Q is uniformly distributed along a line of length ℓ, the **linear charge density** λ is defined by

  $$\lambda \equiv \frac{Q}{\ell}$$

  Linear charge density

  where λ has units of coulombs per meter (C/m).

- If the charge is nonuniformly distributed over a volume, surface, or line, we have to express the charge densities as

$$\rho = \frac{dQ}{dV} \qquad \sigma = \frac{dQ}{dA} \qquad \lambda = \frac{dQ}{d\ell}$$

where dQ is the amount of charge in a small volume, surface, or length element.

EXAMPLE 23.7 The Electric Field Due to a Charged Rod

A rod of length ℓ has a uniform positive charge per unit length λ and a total charge Q. Calculate the electric field at a point P that is located along the long axis of the rod and a distance a from one end (Fig. 23.16).

Solution Let us assume that the rod is lying along the x axis, that dx is the length of one small segment, and that dq is the charge on that segment. Because the rod has a charge per unit length λ, the charge dq on the small segment is $dq = \lambda\, dx$.

The field $d\mathbf{E}$ due to this segment at P is in the negative x direction (because the source of the field carries a positive charge Q), and its magnitude is

$$dE = k_e \frac{dq}{x^2} = k_e \lambda \frac{dx}{x^2}$$

Because every other element also produces a field in the negative x direction, the problem of summing their contributions is particularly simple in this case. The total field at P due to all segments of the rod, which are at different distances from P, is given by Equation 23.6, which in this case becomes[3]

$$E = \int_a^{\ell + a} k_e \lambda \frac{dx}{x^2}$$

where the limits on the integral extend from one end of the rod ($x = a$) to the other ($x = \ell + a$). The constants k_e and λ can be removed from the integral to yield

$$E = k_e \lambda \int_a^{\ell + a} \frac{dx}{x^2} = k_e \lambda \left[-\frac{1}{x} \right]_a^{\ell + a}$$

$$= k_e \lambda \left(\frac{1}{a} - \frac{1}{\ell + a} \right) = \boxed{\frac{k_e Q}{a(\ell + a)}}$$

where we have used the fact that the total charge $Q = \lambda \ell$.

If P is far from the rod ($a \gg \ell$), then the ℓ in the denominator can be neglected, and $E \approx k_e Q / a^2$. This is just the form you would expect for a point charge. Therefore, at large values of a/ℓ, the charge distribution appears to be a point charge of magnitude Q. The use of the limiting technique ($a/\ell \to \infty$) often is a good method for checking a theoretical formula.

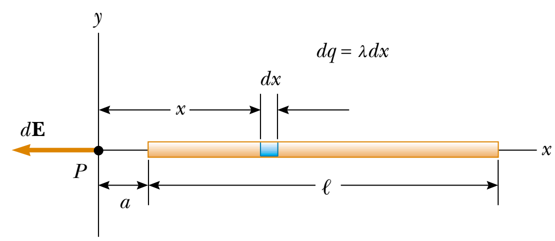


Figure 23.16 The electric field at P due to a uniformly charged rod lying along the x axis. The magnitude of the field at P due to the segment of charge dq is $k_e dq/x^2$. The total field at P is the vector sum over all segments of the rod.

EXAMPLE 23.8 The Electric Field of a Uniform Ring of Charge

A ring of radius a carries a uniformly distributed positive total charge Q. Calculate the electric field due to the ring at a point P lying a distance x from its center along the central axis perpendicular to the plane of the ring (Fig. 23.17a).

Solution The magnitude of the electric field at P due to the segment of charge dq is

$$dE = k_e \frac{dq}{r^2}$$

This field has an x component $dE_x = dE \cos\theta$ along the axis and a component $dE_\perp$ perpendicular to the axis. As we see in Figure 23.17b, however, the resultant field at P must lie along the x axis because the perpendicular components of all the

[3] It is important that you understand how to carry out integrations such as this. First, express the charge element dq in terms of the other variables in the integral (in this example, there is one variable, x, and so we made the change $dq = \lambda\, dx$). The integral must be over scalar quantities; therefore, you must express the electric field in terms of components, if necessary. (In this example the field has only an x component, so we do not bother with this detail.) Then, reduce your expression to an integral over a single variable (or to multiple integrals, each over a single variable). In examples that have spherical or cylindrical symmetry, the single variable will be a radial coordinate.

various charge segments sum to zero. That is, the perpendicular component of the field created by any charge element is canceled by the perpendicular component created by an element on the opposite side of the ring. Because $r = (x^2 + a^2)^{1/2}$ and $\cos\theta = x/r$, we find that

$$dE_x = dE\cos\theta = \left(k_e \frac{dq}{r^2}\right)\frac{x}{r} = \frac{k_e x}{(x^2 + a^2)^{3/2}}\, dq$$

All segments of the ring make the same contribution to the field at P because they are all equidistant from this point. Thus, we can integrate to obtain the total field at P:

$$E_x = \int \frac{k_e x}{(x^2 + a^2)^{3/2}}\, dq = \frac{k_e x}{(x^2 + a^2)^{3/2}} \int dq$$

$$= \boxed{\frac{k_e x}{(x^2 + a^2)^{3/2}}\, Q}$$

This result shows that the field is zero at $x = 0$. Does this finding surprise you?

Exercise Show that at great distances from the ring ($x \gg a$) the electric field along the axis shown in Figure 23.17 approaches that of a point charge of magnitude Q.

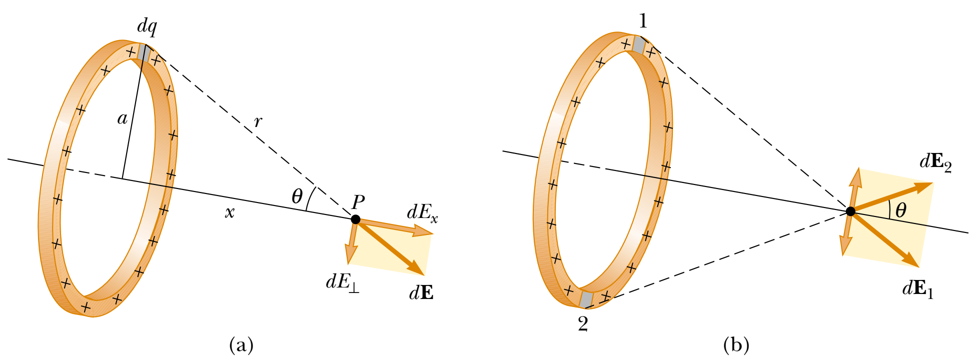


Figure 23.17 A uniformly charged ring of radius a. (a) The field at P on the x axis due to an element of charge dq. (b) The total electric field at P is along the x axis. The perpendicular component of the field at P due to segment 1 is canceled by the perpendicular component due to segment 2.

EXAMPLE 23.9 The Electric Field of a Uniformly Charged Disk

A disk of radius R has a uniform surface charge density σ. Calculate the electric field at a point P that lies along the central perpendicular axis of the disk and a distance x from the center of the disk (Fig. 23.18).

Solution If we consider the disk as a set of concentric rings, we can use our result from Example 23.8—which gives the field created by a ring of radius a—and sum the contributions of all rings making up the disk. By symmetry, the field at an axial point must be along the central axis.

The ring of radius r and width dr shown in Figure 23.18 has a surface area equal to $2\pi r\, dr$. The charge dq on this ring is equal to the area of the ring multiplied by the surface charge density: $dq = 2\pi\sigma r\, dr$. Using this result in the equation given for E_x in Example 23.8 (with a replaced by r), we have for the field due to the ring

$$dE = \frac{k_e x}{(x^2 + r^2)^{3/2}}\,(2\pi\sigma r\, dr)$$

To obtain the total field at P, we integrate this expression over the limits $r = 0$ to $r = R$, noting that x is a constant. This gives

$$E = k_e x\pi\sigma \int_0^R \frac{2r\, dr}{(x^2 + r^2)^{3/2}}$$

$$= k_e x\pi\sigma \int_0^R (x^2 + r^2)^{-3/2}\, d(r^2)$$

$$= k_e x\pi\sigma \left[\frac{(x^2 + r^2)^{-1/2}}{-1/2}\right]_0^R$$

$$= 2\pi k_e\sigma\left(\frac{x}{|x|} - \frac{x}{(x^2 + R^2)^{1/2}}\right)$$

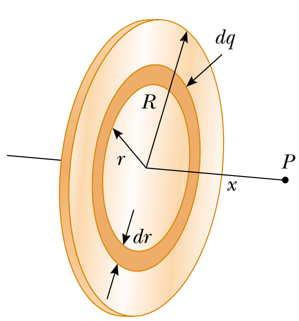


Figure 23.18 A uniformly charged disk of radius R. The electric field at an axial point P is directed along the central axis, perpendicular to the plane of the disk.

This result is valid for all values of x. We can calculate the field close to the disk along the axis by assuming that $R \gg x$; thus, the expression in parentheses reduces to unity:

$$E \approx 2\pi k_e \sigma = \boxed{\frac{\sigma}{2\epsilon_0}}$$

where $\epsilon_0 = 1/(4\pi k_e)$ is the permittivity of free space. As we shall find in the next chapter, we obtain the same result for the field created by a uniformly charged infinite sheet.

23.6 ELECTRIC FIELD LINES

11.5

A convenient way of visualizing electric field patterns is to draw lines that follow the same direction as the electric field vector at any point. These lines, called **electric field lines,** are related to the electric field in any region of space in the following manner:

- The electric field vector **E** is tangent to the electric field line at each point.
- The number of lines per unit area through a surface perpendicular to the lines is proportional to the magnitude of the electric field in that region. Thus, E is great when the field lines are close together and small when they are far apart.

These properties are illustrated in Figure 23.19. The density of lines through surface A is greater than the density of lines through surface B. Therefore, the electric field is more intense on surface A than on surface B. Furthermore, the fact that the lines at different locations point in different directions indicates that the field is nonuniform.

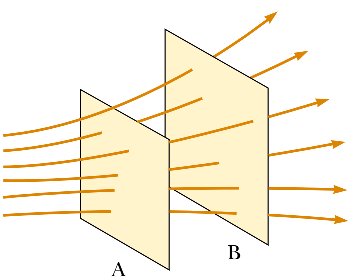


Figure 23.19 Electric field lines penetrating two surfaces. The magnitude of the field is greater on surface A than on surface B.

Representative electric field lines for the field due to a single positive point charge are shown in Figure 23.20a. Note that in this two-dimensional drawing we show only the field lines that lie in the plane containing the point charge. The lines are actually directed radially outward from the charge in all directions; thus, instead of the flat "wheel" of lines shown, you should picture an entire sphere of lines. Because a positive test charge placed in this field would be repelled by the positive point charge, the lines are directed radially away from the positive point

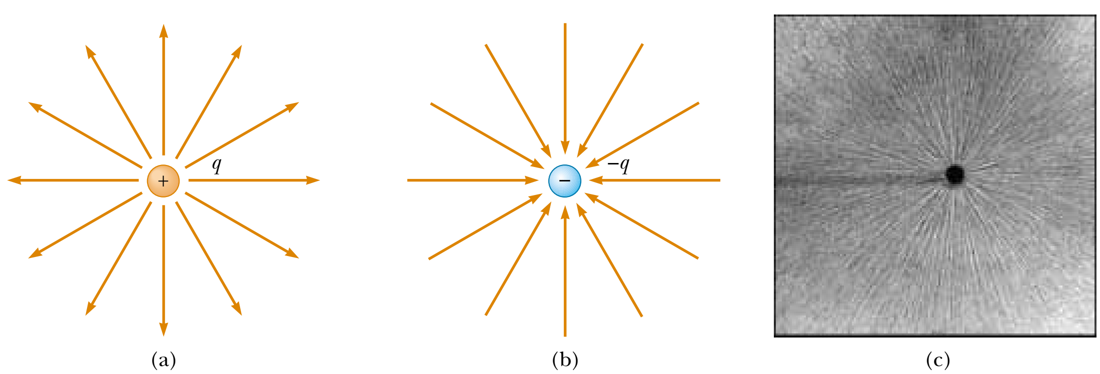


Figure 23.20 The electric field lines for a point charge. (a) For a positive point charge, the lines are directed radially outward. (b) For a negative point charge, the lines are directed radially inward. Note that the figures show only those field lines that lie in the plane containing the charge. (c) The dark areas are small pieces of thread suspended in oil, which align with the electric field produced by a small charged conductor at the center.

charge. The electric field lines representing the field due to a single negative point charge are directed toward the charge (Fig. 23.20b). In either case, the lines are along the radial direction and extend all the way to infinity. Note that the lines become closer together as they approach the charge; this indicates that the strength of the field increases as we move toward the source charge.

The rules for drawing electric field lines are as follows:

- The lines must begin on a positive charge and terminate on a negative charge.
- The number of lines drawn leaving a positive charge or approaching a negative charge is proportional to the magnitude of the charge.
- No two field lines can cross.

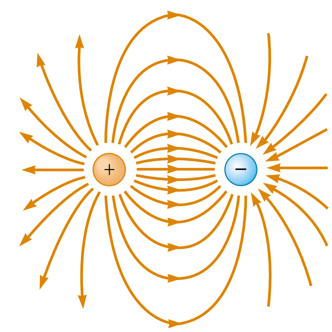
Rules for drawing electric field lines

Is this visualization of the electric field in terms of field lines consistent with Equation 23.4, the expression we obtained for E using Coulomb's law? To answer this question, consider an imaginary spherical surface of radius r concentric with a point charge. From symmetry, we see that the magnitude of the electric field is the same everywhere on the surface of the sphere. The number of lines N that emerge from the charge is equal to the number that penetrate the spherical surface. Hence, the number of lines per unit area on the sphere is $N/4\pi r^2$ (where the surface area of the sphere is $4\pi r^2$). Because E is proportional to the number of lines per unit area, we see that E varies as $1/r^2$; this finding is consistent with Equation 23.4.

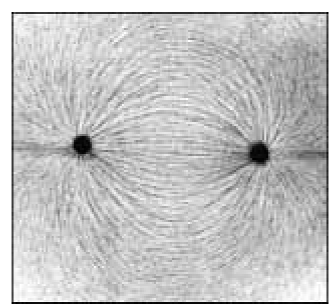
As we have seen, we use electric field lines to qualitatively describe the electric field. One problem with this model is that we always draw a finite number of lines from (or to) each charge. Thus, it appears as if the field acts only in certain directions; this is not true. Instead, the field is *continuous*—that is, it exists at every point. Another problem associated with this model is the danger of gaining the wrong impression from a two-dimensional drawing of field lines being used to describe a three-dimensional situation. Be aware of these shortcomings every time you either draw or look at a diagram showing electric field lines.

We choose the number of field lines starting from any positively charged object to be $C'q$ and the number of lines ending on any negatively charged object to be $C'|q|$, where C' is an arbitrary proportionality constant. Once C' is chosen, the number of lines is fixed. For example, if object 1 has charge Q_1 and object 2 has charge Q_2, then the ratio of number of lines is $N_2/N_1 = Q_2/Q_1$.

The electric field lines for two point charges of equal magnitude but opposite signs (an electric dipole) are shown in Figure 23.21. Because the charges are of equal magnitude, the number of lines that begin at the positive charge must equal the number that terminate at the negative charge. At points very near the charges, the lines are nearly radial. The high density of lines between the charges indicates a region of strong electric field.



(a)



(b)

Figure 23.21 (a) The electric field lines for two point charges of equal magnitude and opposite sign (an electric dipole). The number of lines leaving the positive charge equals the number terminating at the negative charge. (b) The dark lines are small pieces of thread suspended in oil, which align with the electric field of a dipole.

Figure 23.22 shows the electric field lines in the vicinity of two equal positive point charges. Again, the lines are nearly radial at points close to either charge, and the same number of lines emerge from each charge because the charges are equal in magnitude. At great distances from the charges, the field is approximately equal to that of a single point charge of magnitude $2q$.

Finally, in Figure 23.23 we sketch the electric field lines associated with a positive charge $+2q$ and a negative charge $-q$. In this case, the number of lines leaving $+2q$ is twice the number terminating at $-q$. Hence, only half of the lines that leave the positive charge reach the negative charge. The remaining half terminate

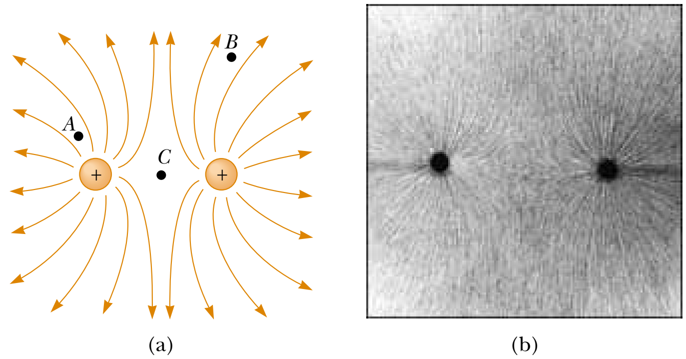


Figure 23.22 (a) The electric field lines for two positive point charges. (The locations *A*, *B*, and *C* are discussed in Quick Quiz 23.5.) (b) Pieces of thread suspended in oil, which align with the electric field created by two equal-magnitude positive charges.

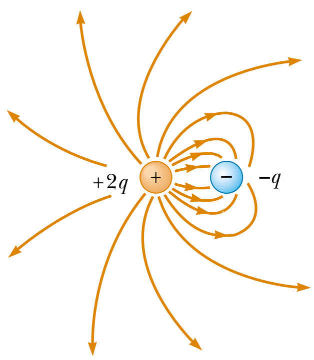


Figure 23.23 The electric field lines for a point charge $+2q$ and a second point charge $-q$. Note that two lines leave $+2q$ for every one that terminates on $-q$.

on a negative charge we assume to be at infinity. At distances that are much greater than the charge separation, the electric field lines are equivalent to those of a single charge $+q$.

Quick Quiz 23.5

Rank the magnitude of the electric field at points *A*, *B*, and *C* shown in Figure 23.22a (greatest magnitude first).

23.7 MOTION OF CHARGED PARTICLES IN A UNIFORM ELECTRIC FIELD

When a particle of charge q and mass m is placed in an electric field $\mathbf{E}$, the electric force exerted on the charge is $q\mathbf{E}$. If this is the only force exerted on the particle, it must be the net force and so must cause the particle to accelerate. In this case, Newton's second law applied to the particle gives

$$\mathbf{F}_e = q\mathbf{E} = m\mathbf{a}$$

The acceleration of the particle is therefore

$$\mathbf{a} = \frac{q\mathbf{E}}{m} \tag{23.7}$$

If $\mathbf{E}$ is uniform (that is, constant in magnitude and direction), then the acceleration is constant. If the particle has a positive charge, then its acceleration is in the direction of the electric field. If the particle has a negative charge, then its acceleration is in the direction opposite the electric field.

EXAMPLE 23.10 An Accelerating Positive Charge

A positive point charge q of mass m is released from rest in a uniform electric field $\mathbf{E}$ directed along the x axis, as shown in Figure 23.24. Describe its motion.

Solution The acceleration is constant and is given by $q\mathbf{E}/m$. The motion is simple linear motion along the x axis. Therefore, we can apply the equations of kinematics in one

dimension (see Chapter 2):

$$x_f = x_i + v_{xi}t + \tfrac{1}{2}a_x t^2$$

$$v_{xf} = v_{xi} + a_x t$$

$$v_{xf}^{\,2} = v_{xi}^{\,2} + 2a_x(x_f - x_i)$$

Taking $x_i = 0$ and $v_{xi} = 0$, we have

$$x_f = \tfrac{1}{2}a_x t^2 = \frac{qE}{2m}\,t^2$$

$$v_{xf} = a_x t = \frac{qE}{m}\,t$$

$$v_{xf}^{\,2} = 2a_x x_f = \left(\frac{2qE}{m}\right)x_f$$

The kinetic energy of the charge after it has moved a distance $x = x_f - x_i$ is

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}m\left(\frac{2qE}{m}\right)x = qEx$$

We can also obtain this result from the work–kinetic energy theorem because the work done by the electric force is $F_e x = qEx$ and $W = \Delta K$.

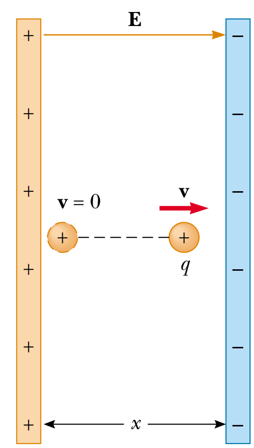


Figure 23.24 A positive point charge q in a uniform electric field **E** undergoes constant acceleration in the direction of the field.

The electric field in the region between two oppositely charged flat metallic plates is approximately uniform (Fig. 23.25). Suppose an electron of charge $-e$ is projected horizontally into this field with an initial velocity $v_i\mathbf{i}$. Because the electric field **E** in Figure 23.25 is in the positive y direction, the acceleration of the electron is in the negative y direction. That is,

$$\mathbf{a} = -\frac{eE}{m}\,\mathbf{j} \qquad \textbf{(23.8)}$$

Because the acceleration is constant, we can apply the equations of kinematics in two dimensions (see Chapter 4) with $v_{xi} = v_i$ and $v_{yi} = 0$. After the electron has been in the electric field for a time t, the components of its velocity are

$$v_x = v_i = \text{constant} \qquad \textbf{(23.9)}$$

$$v_y = a_y t = -\frac{eE}{m}\,t \qquad \textbf{(23.10)}$$

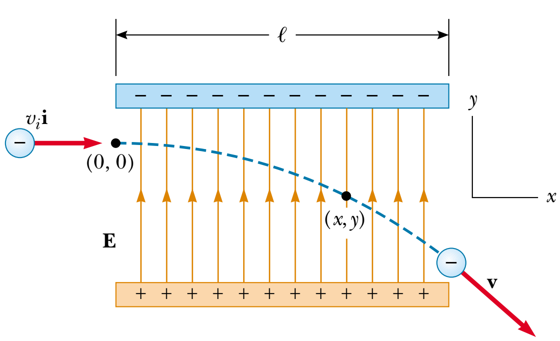


Figure 23.25 An electron is projected horizontally into a uniform electric field produced by two charged plates. The electron undergoes a downward acceleration (opposite **E**), and its motion is parabolic while it is between the plates.

Its coordinates after a time t in the field are

$$x = v_i t \tag{23.11}$$

$$y = \tfrac{1}{2}a_y t^2 = -\tfrac{1}{2}\frac{eE}{m}t^2 \tag{23.12}$$

Substituting the value $t = x/v_i$ from Equation 23.11 into Equation 23.12, we see that y is proportional to x^2. Hence, the trajectory is a parabola. After the electron leaves the field, it continues to move in a straight line in the direction of $\mathbf{v}$ in Figure 23.25, obeying Newton's first law, with a speed $v > v_i$.

Note that we have neglected the gravitational force acting on the electron. This is a good approximation when we are dealing with atomic particles. For an electric field of 10^4 N/C, the ratio of the magnitude of the electric force eE to the magnitude of the gravitational force mg is of the order of 10^{14} for an electron and of the order of 10^{11} for a proton.

EXAMPLE 23.11 An Accelerated Electron

An electron enters the region of a uniform electric field as shown in Figure 23.25, with $v_i = 3.00 \times 10^6$ m/s and $E = 200$ N/C. The horizontal length of the plates is $\ell = 0.100$ m. (a) Find the acceleration of the electron while it is in the electric field.

Solution The charge on the electron has an absolute value of 1.60×10^{-19} C, and $m = 9.11 \times 10^{-31}$ kg. Therefore, Equation 23.8 gives

$$\mathbf{a} = -\frac{eE}{m}\mathbf{j} = -\frac{(1.60 \times 10^{-19}\text{ C})(200\text{ N/C})}{9.11 \times 10^{-31}\text{ kg}}\mathbf{j}$$

$$= \boxed{-3.51 \times 10^{13}\,\mathbf{j}\text{ m/s}^2}$$

(b) Find the time it takes the electron to travel through the field.

Solution The horizontal distance across the field is $\ell = 0.100$ m. Using Equation 23.11 with $x = \ell$, we find that the time spent in the electric field is

$$t = \frac{\ell}{v_i} = \frac{0.100\text{ m}}{3.00 \times 10^6\text{ m/s}} = \boxed{3.33 \times 10^{-8}\text{ s}}$$

(c) What is the vertical displacement y of the electron while it is in the field?

Solution Using Equation 23.12 and the results from parts (a) and (b), we find that

$$y = \tfrac{1}{2}a_y t^2 = \tfrac{1}{2}(-3.51 \times 10^{13}\text{ m/s}^2)(3.33 \times 10^{-8}\text{ s})^2$$

$$= -0.019\,5\text{ m} = \boxed{-1.95\text{ cm}}$$

If the separation between the plates is less than this, the electron will strike the positive plate.

Exercise Find the speed of the electron as it emerges from the field.

Answer 3.22×10^6 m/s.

The Cathode Ray Tube

The example we just worked describes a portion of a cathode ray tube (CRT). This tube, illustrated in Figure 23.26, is commonly used to obtain a visual display of electronic information in oscilloscopes, radar systems, television receivers, and computer monitors. The CRT is a vacuum tube in which a beam of electrons is accelerated and deflected under the influence of electric or magnetic fields. The electron beam is produced by an assembly called an *electron gun* located in the neck of the tube. These electrons, if left undisturbed, travel in a straight-line path until they strike the front of the CRT, the "screen," which is coated with a material that emits visible light when bombarded with electrons.

In an oscilloscope, the electrons are deflected in various directions by two sets of plates placed at right angles to each other in the neck of the tube. (A television

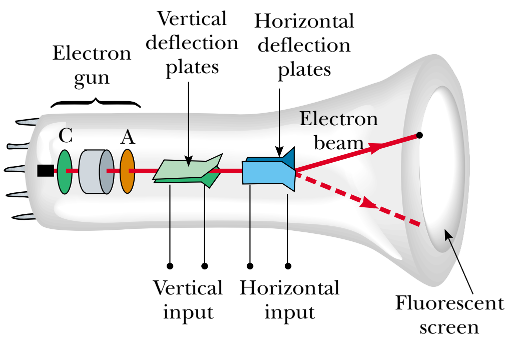


Figure 23.26 Schematic diagram of a cathode ray tube. Electrons leaving the hot cathode C are accelerated to the anode A. In addition to accelerating electrons, the electron gun is also used to focus the beam of electrons, and the plates deflect the beam.

CRT steers the beam with a magnetic field, as discussed in Chapter 29.) An external electric circuit is used to control the amount of charge present on the plates. The placing of positive charge on one horizontal plate and negative charge on the other creates an electric field between the plates and allows the beam to be steered from side to side. The vertical deflection plates act in the same way, except that changing the charge on them deflects the beam vertically.

SUMMARY

Electric charges have the following important properties:

- Unlike charges attract one another, and like charges repel one another.
- Charge is conserved.
- Charge is quantized—that is, it exists in discrete packets that are some integral multiple of the electronic charge.

Conductors are materials in which charges move freely. **Insulators** are materials in which charges do not move freely.

Coulomb's law states that the electric force exerted by a charge q_1 on a second charge q_2 is

$$\mathbf{F}_{12} = k_e \frac{q_1 q_2}{r^2}\,\hat{\mathbf{r}} \tag{23.2}$$

where r is the distance between the two charges and $\hat{\mathbf{r}}$ is a unit vector directed from q_1 to q_2. The constant k_e, called the Coulomb constant, has the value $k_e = 8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}$.

The smallest unit of charge known to exist in nature is the charge on an electron or proton, $|e| = 1.602\,19 \times 10^{-19}$ C.

The electric field $\mathbf{E}$ at some point in space is defined as the electric force $\mathbf{F}_e$ that acts on a small positive test charge placed at that point divided by the magnitude of the test charge q_0:

$$\mathbf{E} \equiv \frac{\mathbf{F}_e}{q_0} \tag{23.3}$$

At a distance r from a point charge q, the electric field due to the charge is given by

$$\mathbf{E} = k_e \frac{q}{r^2}\,\hat{\mathbf{r}} \tag{23.4}$$

where $\hat{\mathbf{r}}$ is a unit vector directed from the charge to the point in question. The

electric field is directed radially outward from a positive charge and radially inward toward a negative charge.

The electric field due to a group of point charges can be obtained by using the superposition principle. That is, the total electric field at some point equals the vector sum of the electric fields of all the charges:

$$\mathbf{E} = k_e \sum_i \frac{q_i}{r_i^2} \hat{\mathbf{r}}_i \tag{23.5}$$

The electric field at some point of a continuous charge distribution is

$$\mathbf{E} = k_e \int \frac{dq}{r^2} \hat{\mathbf{r}} \tag{23.6}$$

where dq is the charge on one element of the charge distribution and r is the distance from the element to the point in question.

Electric field lines describe an electric field in any region of space. The number of lines per unit area through a surface perpendicular to the lines is proportional to the magnitude of $\mathbf{E}$ in that region.

A charged particle of mass m and charge q moving in an electric field $\mathbf{E}$ has an acceleration

$$\mathbf{a} = \frac{q\mathbf{E}}{m} \tag{23.7}$$

Problem-Solving Hints

Finding the Electric Field

- **Units:** In calculations using the Coulomb constant k_e $(= 1/4\pi\epsilon_0)$, charges must be expressed in coulombs and distances in meters.
- **Calculating the electric field of point charges:** To find the total electric field at a given point, first calculate the electric field at the point due to each individual charge. The resultant field at the point is the vector sum of the fields due to the individual charges.
- **Continuous charge distributions:** When you are confronted with problems that involve a continuous distribution of charge, the vector sums for evaluating the total electric field at some point must be replaced by vector integrals. Divide the charge distribution into infinitesimal pieces, and calculate the vector sum by integrating over the entire charge distribution. You should review Examples 23.7 through 23.9.
- **Symmetry:** With both distributions of point charges and continuous charge distributions, take advantage of any symmetry in the system to simplify your calculations.

QUESTIONS

1. Sparks are often observed (or heard) on a dry day when clothes are removed in the dark. Explain.
2. Explain from an atomic viewpoint why charge is usually transferred by electrons.
3. A balloon is negatively charged by rubbing and then clings to a wall. Does this mean that the wall is positively charged? Why does the balloon eventually fall?
4. A light, uncharged metallic sphere suspended from a thread is attracted to a charged rubber rod. After touching the rod, the sphere is repelled by the rod. Explain.

5. Explain what is meant by the term "a neutral atom."
6. Why do some clothes cling together and to your body after they are removed from a dryer?
7. A large metallic sphere insulated from ground is charged with an electrostatic generator while a person standing on an insulating stool holds the sphere. Why is it safe to do this? Why wouldn't it be safe for another person to touch the sphere after it has been charged?
8. What are the similarities and differences between Newton's law of gravitation, $F_g = Gm_1m_2/r^2$, and Coulomb's law, $F_e = k_eq_1q_2/r^2$?
9. Assume that someone proposes a theory that states that people are bound to the Earth by electric forces rather than by gravity. How could you prove this theory wrong?
10. How would you experimentally distinguish an electric field from a gravitational field?
11. Would life be different if the electron were positively charged and the proton were negatively charged? Does the choice of signs have any bearing on physical and chemical interactions? Explain.
12. When defining the electric field, why is it necessary to specify that the magnitude of the test charge be very small (that is, why is it necessary to take the limit of $\mathbf{F}_e/q$ as $q \rightarrow 0$)?
13. Two charged conducting spheres, each of radius a, are separated by a distance $r > 2a$. Is the force on either sphere given by Coulomb's law? Explain. (*Hint:* Refer to Chapter 14 on gravitation.)
14. When is it valid to approximate a charge distribution by a point charge?
15. Is it possible for an electric field to exist in empty space? Explain.
16. Explain why electric field lines never cross. (*Hint:* $\mathbf{E}$ must have a unique direction at all points.)
17. A free electron and free proton are placed in an identical electric field. Compare the electric forces on each particle. Compare their accelerations.
18. Explain what happens to the magnitude of the electric field of a point charge as r approaches zero.
19. A negative charge is placed in a region of space where the electric field is directed vertically upward. What is the direction of the electric force experienced by this charge?
20. A charge $4q$ is a distance r from a charge $-q$. Compare the number of electric field lines leaving the charge $4q$ with the number entering the charge $-q$.
21. In Figure 23.23, where do the extra lines leaving the charge $+2q$ end?
22. Consider two equal point charges separated by some distance d. At what point (other than ∞) would a third test charge experience no net force?
23. A negative point charge $-q$ is placed at the point P near the positively charged ring shown in Figure 23.17. If $x \ll a$, describe the motion of the point charge if it is released from rest.
24. Explain the differences between linear, surface, and volume charge densities, and give examples of when each would be used.
25. If the electron in Figure 23.25 is projected into the electric field with an arbitrary velocity $\mathbf{v}_i$ (at an angle to $\mathbf{E}$), will its trajectory still be parabolic? Explain.
26. It has been reported that in some instances people near where a lightning bolt strikes the Earth have had their clothes thrown off. Explain why this might happen.
27. Why should a ground wire be connected to the metallic support rod for a television antenna?
28. A light strip of aluminum foil is draped over a wooden rod. When a rod carrying a positive charge is brought close to the foil, the two parts of the foil stand apart. Why? What kind of charge is on the foil?
29. Why is it more difficult to charge an object by rubbing on a humid day than on a dry day?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging   □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/**   = Computer useful in solving problem   = Interactive Physics
□ = paired numerical/symbolic problems

Section 23.1 Properties of Electric Charges
Section 23.2 Insulators and Conductors
Section 23.3 Coulomb's Law

1. (a) Calculate the number of electrons in a small, electrically neutral silver pin that has a mass of 10.0 g. Silver has 47 electrons per atom, and its molar mass is 107.87 g/mol. (b) Electrons are added to the pin until the net negative charge is 1.00 mC. How many electrons are added for every 10^9 electrons already present?
2. (a) Two protons in a molecule are separated by a distance of 3.80×10^{-10} m. Find the electric force exerted by one proton on the other. (b) How does the magnitude of this force compare with the magnitude of the gravitational force between the two protons? (c) What must be the charge-to-mass ratio of a particle if the magnitude of the gravitational force between two of these particles equals the magnitude of the electric force between them?

WEB 3. Richard Feynman once said that if two persons stood at arm's length from each other and each person had 1% more electrons than protons, the force of repulsion between them would be enough to lift a "weight" equal to that of the entire Earth. Carry out an order-of-magnitude calculation to substantiate this assertion.

4. Two small silver spheres, each with a mass of 10.0 g, are separated by 1.00 m. Calculate the fraction of the elec-

trons in one sphere that must be transferred to the other to produce an attractive force of 1.00×10^4 N (about 1 ton) between the spheres. (The number of electrons per atom of silver is 47, and the number of atoms per gram is Avogadro's number divided by the molar mass of silver, 107.87 g/mol.)

5. Suppose that 1.00 g of hydrogen is separated into electrons and protons. Suppose also that the protons are placed at the Earth's north pole and the electrons are placed at the south pole. What is the resulting compressional force on the Earth?

6. Two identical conducting small spheres are placed with their centers 0.300 m apart. One is given a charge of 12.0 nC, and the other is given a charge of -18.0 nC. (a) Find the electric force exerted on one sphere by the other. (b) The spheres are connected by a conducting wire. Find the electric force between the two after equilibrium has occurred.

$\boxed{7.}$ Three point charges are located at the corners of an equilateral triangle, as shown in Figure P23.7. Calculate the net electric force on the 7.00-μC charge.

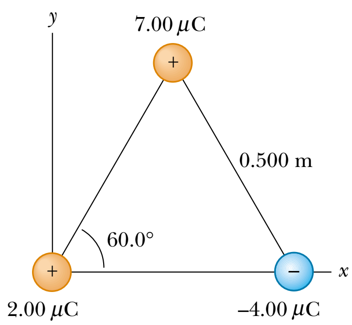


Figure P23.7 Problems 7 and 15.

8. Two small beads having positive charges $3q$ and q are fixed at the opposite ends of a horizontal insulating rod extending from the origin to the point $x = d$. As shown in Figure P23.8, a third small charged bead is free to slide on the rod. At what position is the third bead in equilibrium? Can it be in stable equilibrium?

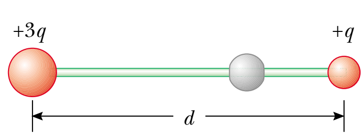


Figure P23.8

9. Review Problem. In the Bohr theory of the hydrogen atom, an electron moves in a circular orbit about a proton, where the radius of the orbit is 0.529×10^{-10} m. (a) Find the electric force between the two. (b) If this force causes the centripetal acceleration of the electron, what is the speed of the electron?

10. Review Problem. Two identical point charges each having charge $+q$ are fixed in space and separated by a distance d. A third point charge $-Q$ of mass m is free to move and lies initially at rest on a perpendicular bisector of the two fixed charges a distance x from the midpoint of the two fixed charges (Fig. P23.10). (a) Show that if x is small compared with d, the motion of $-Q$ is simple harmonic along the perpendicular bisector. Determine the period of that motion. (b) How fast will the charge $-Q$ be moving when it is at the midpoint between the two fixed charges, if initially it is released at a distance $x = a \ll d$ from the midpoint?

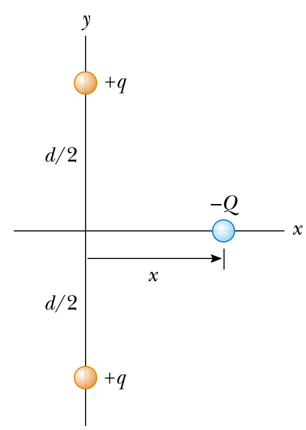


Figure P23.10

Section 23.4 The Electric Field

$\boxed{11.}$ What are the magnitude and direction of the electric field that will balance the weight of (a) an electron and (b) a proton? (Use the data in Table 23.1.)

12. An object having a net charge of 24.0 μC is placed in a uniform electric field of 610 N/C that is directed vertically. What is the mass of this object if it "floats" in the field?

13. In Figure P23.13, determine the point (other than infinity) at which the electric field is zero.

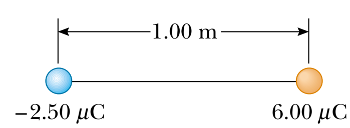


Figure P23.13

14. An airplane is flying through a thundercloud at a height of 2 000 m. (This is a very dangerous thing to do because of updrafts, turbulence, and the possibility of electric discharge.) If there are charge concentrations of $+40.0$ C at a height of 3 000 m within the cloud and of -40.0 C at a height of 1 000 m, what is the electric field E at the aircraft?

15. Three charges are at the corners of an equilateral triangle, as shown in Figure P23.7. (a) Calculate the electric field at the position of the 2.00-μC charge due to the 7.00-μC and -4.00-μC charges. (b) Use your answer to part (a) to determine the force on the 2.00-μC charge.

16. Three point charges are arranged as shown in Figure P23.16. (a) Find the vector electric field that the 6.00-nC and -3.00-nC charges together create at the origin. (b) Find the vector force on the 5.00-nC charge.

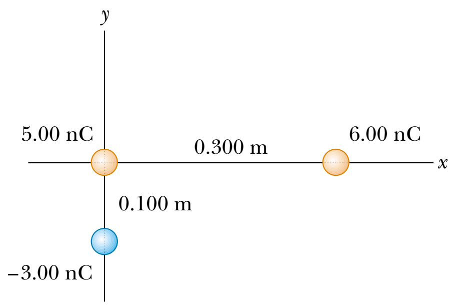


Figure P23.16

WEB **17.** Three equal positive charges q are at the corners of an equilateral triangle of side a, as shown in Figure P23.17. (a) Assume that the three charges together create an electric field. Find the location of a point (other than ∞) where the electric field is zero. (*Hint:* Sketch the field lines in the plane of the charges.) (b) What are the magnitude and direction of the electric field at P due to the two charges at the base?

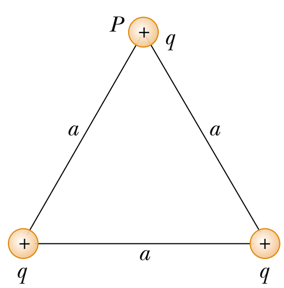


Figure P23.17

18. Two 2.00-μC point charges are located on the x axis. One is at $x = 1.00$ m, and the other is at $x = -1.00$ m. (a) Determine the electric field on the y axis at $y = 0.500$ m. (b) Calculate the electric force on a -3.00-μC charge placed on the y axis at $y = 0.500$ m.

19. Four point charges are at the corners of a square of side a, as shown in Figure P23.19. (a) Determine the magnitude and direction of the electric field at the location of charge q. (b) What is the resultant force on q?

20. A point particle having charge q is located at point (x_0, y_0) in the xy plane. Show that the x and y components of the electric field at point (x, y) due to this charge q are

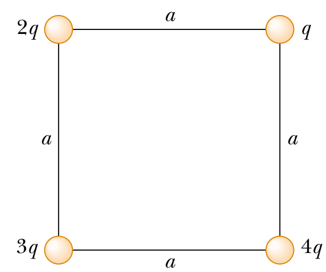


Figure P23.19

$$E_x = \frac{k_e q(x - x_0)}{[(x - x_0)^2 + (y - y_0)^2]^{3/2}}$$

$$E_y = \frac{k_e q(y - y_0)}{[(x - x_0)^2 + (y - y_0)^2]^{3/2}}$$

21. Consider the electric dipole shown in Figure P23.21. Show that the electric field at a *distant* point along the x axis is $E_x \cong 4k_e qa/x^3$.

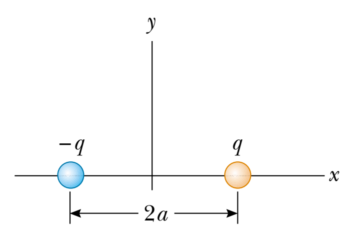


Figure P23.21

22. Consider n equal positive point charges each of magnitude Q/n placed symmetrically around a circle of radius R. (a) Calculate the magnitude of the electric field E at a point a distance x on the line passing through the center of the circle and perpendicular to the plane of the circle. (b) Explain why this result is identical to the one obtained in Example 23.8.

23. Consider an infinite number of identical charges (each of charge q) placed along the x axis at distances a, $2a$, $3a$, $4a$, . . . from the origin. What is the electric field at the origin due to this distribution? *Hint:* Use the fact that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

Section 23.5 Electric Field of a Continuous Charge Distribution

24. A rod 14.0 cm long is uniformly charged and has a total charge of -22.0 μC. Determine the magnitude and direction of the electric field along the axis of the rod at a point 36.0 cm from its center.

25. A continuous line of charge lies along the x axis, extending from $x = +x_0$ to positive infinity. The line carries a uniform linear charge density λ_0. What are the magnitude and direction of the electric field at the origin?

26. A line of charge starts at $x = +x_0$ and extends to positive infinity. If the linear charge density is $\lambda = \lambda_0 x_0/x$, determine the electric field at the origin.

27. A uniformly charged ring of radius 10.0 cm has a total charge of 75.0 μC. Find the electric field on the axis of the ring at (a) 1.00 cm, (b) 5.00 cm, (c) 30.0 cm, and (d) 100 cm from the center of the ring.

28. Show that the maximum field strength E_{max} along the axis of a uniformly charged ring occurs at $x = a/\sqrt{2}$ (see Fig. 23.17) and has the value $Q/(6\sqrt{3}\pi\epsilon_0 a^2)$.

29. A uniformly charged disk of radius 35.0 cm carries a charge density of 7.90×10^{-3} C/m^2. Calculate the electric field on the axis of the disk at (a) 5.00 cm, (b) 10.0 cm, (c) 50.0 cm, and (d) 200 cm from the center of the disk.

30. Example 23.9 derives the exact expression for the electric field at a point on the axis of a uniformly charged disk. Consider a disk of radius $R = 3.00$ cm having a uniformly distributed charge of $+5.20$ μC. (a) Using the result of Example 23.9, compute the electric field at a point on the axis and 3.00 mm from the center. Compare this answer with the field computed from the near-field approximation $E = \sigma/2\epsilon_0$. (b) Using the result of Example 23.9, compute the electric field at a point on the axis and 30.0 cm from the center of the disk. Compare this result with the electric field obtained by treating the disk as a $+5.20$-μC point charge at a distance of 30.0 cm.

31. The electric field along the axis of a uniformly charged disk of radius R and total charge Q was calculated in Example 23.9. Show that the electric field at distances x that are great compared with R approaches that of a point charge $Q = \sigma\pi R^2$. (*Hint:* First show that $x/(x^2 + R^2)^{1/2} = (1 + R^2/x^2)^{-1/2}$, and use the binomial expansion $(1 + \delta)^n \approx 1 + n\delta$ when $\delta \ll 1$.)

32. A piece of Styrofoam having a mass m carries a net charge of $-q$ and floats above the center of a very large horizontal sheet of plastic that has a uniform charge density on its surface. What is the charge per unit area on the plastic sheet?

WEB **33.** A uniformly charged insulating rod of length 14.0 cm is bent into the shape of a semicircle, as shown in Figure P23.33. The rod has a total charge of -7.50 μC. Find the magnitude and direction of the electric field at O, the center of the semicircle.

34. (a) Consider a uniformly charged right circular cylindrical shell having total charge Q, radius R, and height h. Determine the electric field at a point a distance d from the right side of the cylinder, as shown in Figure P23.34. (*Hint:* Use the result of Example 23.8 and treat the cylinder as a collection of ring charges.) (b) Consider now a solid cylinder with the same dimensions and carrying the same charge, which is uniformly distributed through its volume. Use the result of Example 23.9 to find the field it creates at the same point.

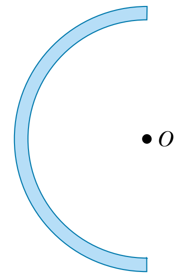


Figure P23.33

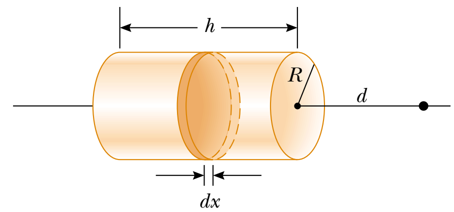


Figure P23.34

35. A thin rod of length ℓ and uniform charge per unit length λ lies along the x axis, as shown in Figure P23.35. (a) Show that the electric field at P, a distance y from the rod, along the perpendicular bisector has no x component and is given by $E = 2k_e\lambda \sin\theta_0/y$. (b) Using your result to part (a), show that the field of a rod of infinite length is $E = 2k_e\lambda/y$. (*Hint:* First calculate the field at P due to an element of length dx, which has a charge $\lambda\, dx$. Then change variables from x to θ, using the facts that $x = y\tan\theta$ and $dx = y\sec^2\theta\, d\theta$, and integrate over θ.)

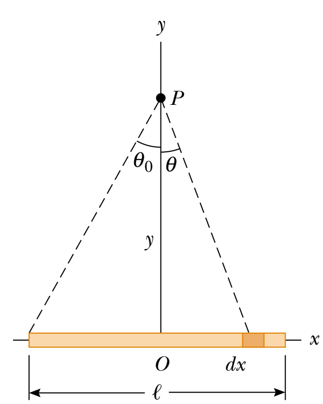
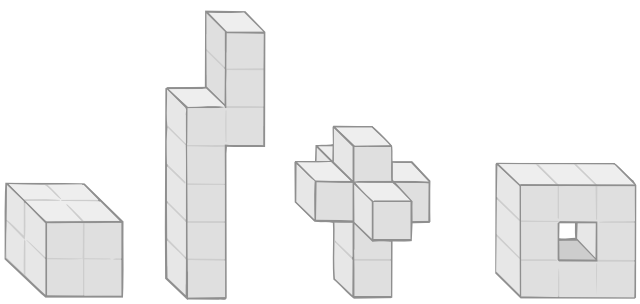


Figure P23.35

36. Three solid plastic cylinders all have a radius of 2.50 cm and a length of 6.00 cm. One (a) carries charge with

uniform density 15.0 nC/m^2 everywhere on its surface. Another (b) carries charge with the same uniform density on its curved lateral surface only. The third (c) carries charge with uniform density 500 nC/m^3 throughout the plastic. Find the charge of each cylinder.

37. Eight solid plastic cubes, each 3.00 cm on each edge, are glued together to form each one of the objects (i, ii, iii, and iv) shown in Figure P23.37. (a) If each object carries charge with a uniform density of 400 nC/m^3 throughout its volume, what is the charge of each object? (b) If each object is given charge with a uniform density of 15.0 nC/m^2 everywhere on its exposed surface, what is the charge on each object? (c) If charge is placed only on the edges where perpendicular surfaces meet, with a uniform density of 80.0 pC/m, what is the charge of each object?



(i) (ii) (iii) (iv)

Figure P23.37

Section 23.6 Electric Field Lines

38. A positively charged disk has a uniform charge per unit area as described in Example 23.9. Sketch the electric field lines in a plane perpendicular to the plane of the disk passing through its center.

39. A negatively charged rod of finite length has a uniform charge per unit length. Sketch the electric field lines in a plane containing the rod.

40. Figure P23.40 shows the electric field lines for two point charges separated by a small distance. (a) Determine the ratio q_1/q_2. (b) What are the signs of q_1 and q_2?

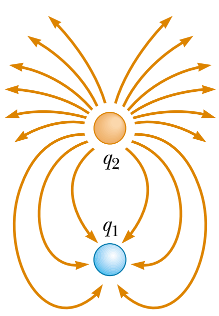


Figure P23.40

Section 23.7 Motion of Charged Particles in a Uniform Electric Field

41. An electron and a proton are each placed at rest in an electric field of 520 N/C. Calculate the speed of each particle 48.0 ns after being released.

42. A proton is projected in the positive x direction into a region of uniform electric field $\mathbf{E} = -6.00 \times 10^5 \mathbf{i}$ N/C. The proton travels 7.00 cm before coming to rest. Determine (a) the acceleration of the proton, (b) its initial speed, and (c) the time it takes the proton to come to rest.

43. A proton accelerates from rest in a uniform electric field of 640 N/C. At some later time, its speed has reached 1.20×10^6 m/s (nonrelativistic, since v is much less than the speed of light). (a) Find the acceleration of the proton. (b) How long does it take the proton to reach this speed? (c) How far has it moved in this time? (d) What is its kinetic energy at this time?

44. The electrons in a particle beam each have a kinetic energy of 1.60×10^{-17} J. What are the magnitude and direction of the electric field that stops these electrons in a distance of 10.0 cm?

WEB **45.** The electrons in a particle beam each have a kinetic energy K. What are the magnitude and direction of the electric field that stops these electrons in a distance d?

46. A positively charged bead having a mass of 1.00 g falls from rest in a vacuum from a height of 5.00 m in a uniform vertical electric field with a magnitude of 1.00×10^4 N/C. The bead hits the ground at a speed of 21.0 m/s. Determine (a) the direction of the electric field (up or down) and (b) the charge on the bead.

47. A proton moves at 4.50×10^5 m/s in the horizontal direction. It enters a uniform vertical electric field with a magnitude of 9.60×10^3 N/C. Ignoring any gravitational effects, find (a) the time it takes the proton to travel 5.00 cm horizontally, (b) its vertical displacement after it has traveled 5.00 cm horizontally, and (c) the horizontal and vertical components of its velocity after it has traveled 5.00 cm horizontally.

48. An electron is projected at an angle of 30.0° above the horizontal at a speed of 8.20×10^5 m/s in a region where the electric field is $\mathbf{E} = 390\mathbf{j}$ N/C. Neglecting the effects of gravity, find (a) the time it takes the electron to return to its initial height, (b) the maximum height it reaches, and (c) its horizontal displacement when it reaches its maximum height.

49. Protons are projected with an initial speed $v_i = 9.55 \times 10^3$ m/s into a region where a uniform electric field $\mathbf{E} = (-720\mathbf{j})$ N/C is present, as shown in Figure P23.49. The protons are to hit a target that lies at a horizontal distance of 1.27 mm from the point where the protons are launched. Find (a) the two projection angles θ that result in a hit and (b) the total time of flight for each trajectory.

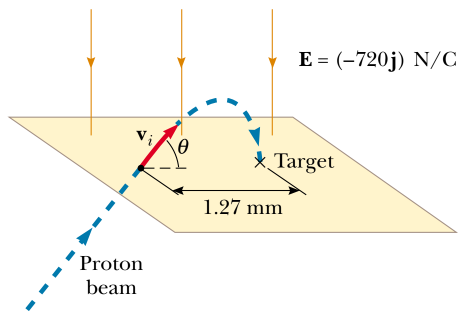


Figure P23.49

ADDITIONAL PROBLEMS

50. Three point charges are aligned along the x axis as shown in Figure P23.50. Find the electric field at (a) the position (2.00, 0) and (b) the position (0, 2.00).

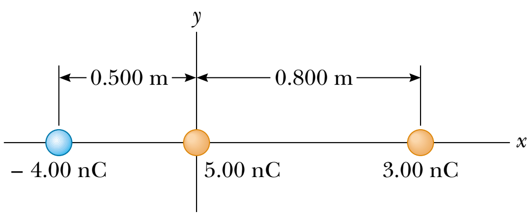


Figure P23.50

51. A uniform electric field of magnitude 640 N/C exists between two parallel plates that are 4.00 cm apart. A proton is released from the positive plate at the same instant that an electron is released from the negative plate. (a) Determine the distance from the positive plate at which the two pass each other. (Ignore the electrical attraction between the proton and electron.) (b) Repeat part (a) for a sodium ion (Na^+) and a chlorine ion (Cl^-).

52. A small, 2.00-g plastic ball is suspended by a 20.0-cm-long string in a uniform electric field, as shown in Figure P23.52. If the ball is in equilibrium when the string makes a 15.0° angle with the vertical, what is the net charge on the ball?

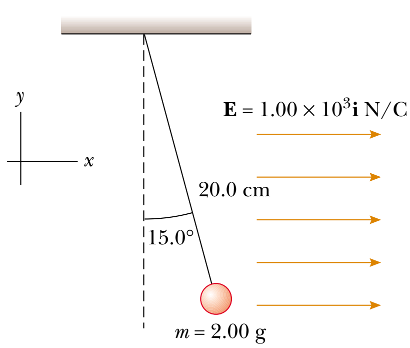


Figure P23.52

WEB 53. A charged cork ball of mass 1.00 g is suspended on a light string in the presence of a uniform electric field, as shown in Figure P23.53. When $\mathbf{E} = (3.00\mathbf{i} + 5.00\mathbf{j}) \times 10^5$ N/C, the ball is in equilibrium at $\theta = 37.0°$. Find (a) the charge on the ball and (b) the tension in the string.

54. A charged cork ball of mass m is suspended on a light string in the presence of a uniform electric field, as shown in Figure P23.53. When $\mathbf{E} = (A\mathbf{i} + B\mathbf{j})$ N/C, where A and B are positive numbers, the ball is in equilibrium at the angle θ. Find (a) the charge on the ball and (b) the tension in the string.

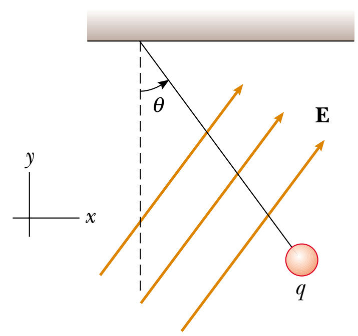


Figure P23.53 Problems 53 and 54.

55. Four identical point charges ($q = +10.0\ \mu$C) are located on the corners of a rectangle, as shown in Figure P23.55. The dimensions of the rectangle are $L = 60.0$ cm and $W = 15.0$ cm. Calculate the magnitude and direction of the net electric force exerted on the charge at the lower left corner by the other three charges.

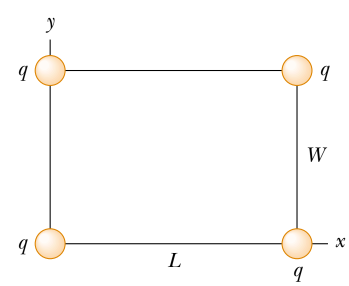


Figure P23.55

56. Three identical small Styrofoam balls ($m = 2.00$ g) are suspended from a fixed point by three nonconducting threads, each with a length of 50.0 cm and with negligi-

ble mass. At equilibrium the three balls form an equilateral triangle with sides of 30.0 cm. What is the common charge q carried by each ball?

57. Two identical metallic blocks resting on a frictionless horizontal surface are connected by a light metallic spring having the spring constant $k = 100$ N/m and an unstretched length of 0.300 m, as shown in Figure P23.57a. A total charge of Q is slowly placed on the system, causing the spring to stretch to an equilibrium length of 0.400 m, as shown in Figure P23.57b. Determine the value of Q, assuming that all the charge resides on the blocks and that the blocks are like point charges.

58. Two identical metallic blocks resting on a frictionless horizontal surface are connected by a light metallic spring having a spring constant k and an unstretched length L_i, as shown in Figure P23.57a. A total charge of Q is slowly placed on the system, causing the spring to stretch to an equilibrium length L, as shown in Figure P23.57b. Determine the value of Q, assuming that all the charge resides on the blocks and that the blocks are like point charges.

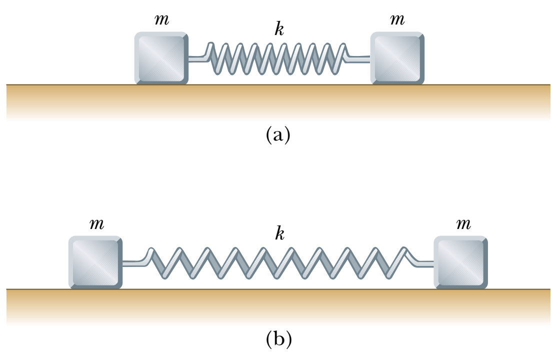


Figure P23.57 Problems 57 and 58.

59. Identical thin rods of length $2a$ carry equal charges, $+Q$, uniformly distributed along their lengths. The rods lie along the x axis with their centers separated by a distance of $b > 2a$ (Fig. P23.59). Show that the magnitude of the force exerted by the left rod on the right one is given by

$$F = \left(\frac{k_e Q^2}{4a^2}\right)\ln\left(\frac{b^2}{b^2 - 4a^2}\right)$$

60. A particle is said to be nonrelativistic as long as its speed is less than one-tenth the speed of light, or less than 3.00×10^7 m/s. (a) How long will an electron remain nonrelativistic if it starts from rest in a region of an electric field of 1.00 N/C? (b) How long will a proton remain nonrelativistic in the same electric field? (c) Electric fields are commonly much larger than

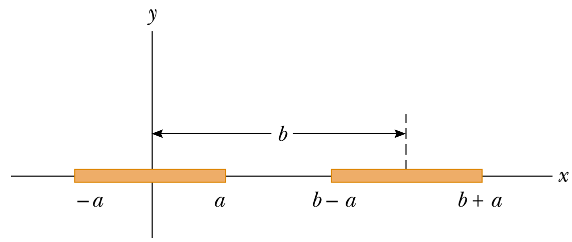


Figure P23.59

1 N/C. Will the charged particle remain nonrelativistic for a shorter or a longer time in a much larger electric field?

61. A line of positive charge is formed into a semicircle of radius $R = 60.0$ cm, as shown in Figure P23.61. The charge per unit length along the semicircle is described by the expression $\lambda = \lambda_0 \cos\theta$. The total charge on the semicircle is 12.0 μC. Calculate the total force on a charge of 3.00 μC placed at the center of curvature.

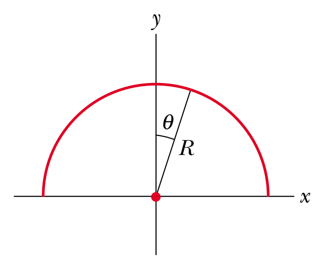


Figure P23.61

62. Two small spheres, each of mass 2.00 g, are suspended by light strings 10.0 cm in length (Fig. P23.62). A uniform electric field is applied in the x direction. The spheres have charges equal to -5.00×10^{-8} C and $+5.00 \times 10^{-8}$ C. Determine the electric field that enables the spheres to be in equilibrium at an angle of $\theta = 10.0°$.

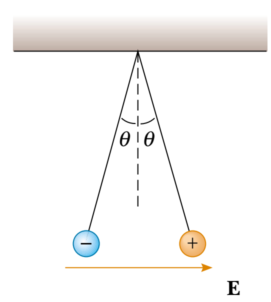


Figure P23.62

63. Two small spheres of mass m are suspended from strings of length ℓ that are connected at a common point. One sphere has charge Q; the other has charge $2Q$. Assume that the angles θ_1 and θ_2 that the strings make with the vertical are small. (a) How are θ_1 and θ_2 related? (b) Show that the distance r between the spheres is

$$r \cong \left(\frac{4k_eQ^2\ell}{mg}\right)^{1/3}$$

64. Three charges of equal magnitude q are fixed in position at the vertices of an equilateral triangle (Fig. P23.64). A fourth charge Q is free to move along the positive x axis under the influence of the forces exerted by the three fixed charges. Find a value for s for which Q is in equilibrium. You will need to solve a transcendental equation.

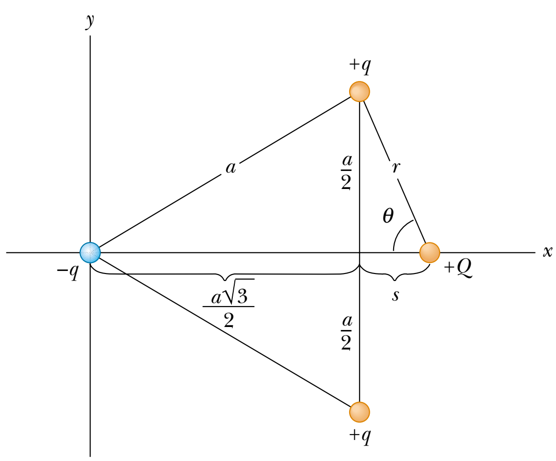


Figure P23.64

65. Review Problem. Four identical point charges, each having charge $+q$, are fixed at the corners of a square of side L. A fifth point charge $-Q$ lies a distance z along the line perpendicular to the plane of the square and passing through the center of the square (Fig. P23.65). (a) Show that the force exerted on $-Q$ by the other four charges is

$$\mathbf{F} = -\frac{4k_eqQz}{\left(z^2 + \frac{L^2}{2}\right)^{3/2}}\,\mathbf{k}$$

Note that this force is directed toward the center of the square whether z is positive ($-Q$ above the square) or negative ($-Q$ below the square). (b) If z is small compared with L, the above expression reduces to $\mathbf{F} \approx -(\text{constant})\,z\mathbf{k}$. Why does this imply that the motion of $-Q$ is simple harmonic, and what would be the period of this motion if the mass of $-Q$ were m?

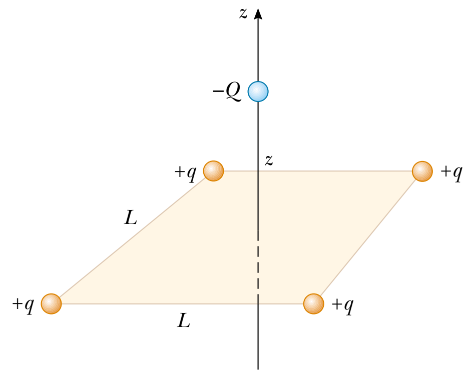


Figure P23.65

66. Review Problem. A 1.00-g cork ball with a charge of 2.00 μC is suspended vertically on a 0.500-m-long light string in the presence of a uniform, downward-directed electric field of magnitude $E = 1.00 \times 10^5$ N/C. If the ball is displaced slightly from the vertical, it oscillates like a simple pendulum. (a) Determine the period of this oscillation. (b) Should gravity be included in the calculation for part (a)? Explain.

67. Three charges of equal magnitude q reside at the corners of an equilateral triangle of side length a (Fig. P23.67). (a) Find the magnitude and direction of the electric field at point P, midway between the negative charges, in terms of k_e, q, and a. (b) Where must a $-4q$ charge be placed so that any charge located at P experiences no net electric force? In part (b), let P be the origin and let the distance between the $+q$ charge and P be 1.00 m.

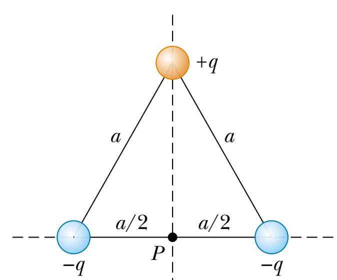


Figure P23.67

68. Two identical beads each have a mass m and charge q. When placed in a hemispherical bowl of radius R with frictionless, nonconducting walls, the beads move, and at equilibrium they are a distance R apart (Fig. P23.68). Determine the charge on each bead.

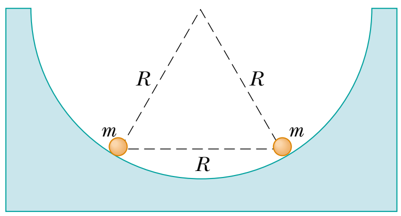


Figure P23.68

69. Eight point charges, each of magnitude q, are located on the corners of a cube of side s, as shown in Figure P23.69. (a) Determine the x, y, and z components of the resultant force exerted on the charge located at point A by the other charges. (b) What are the magnitude and direction of this resultant force?

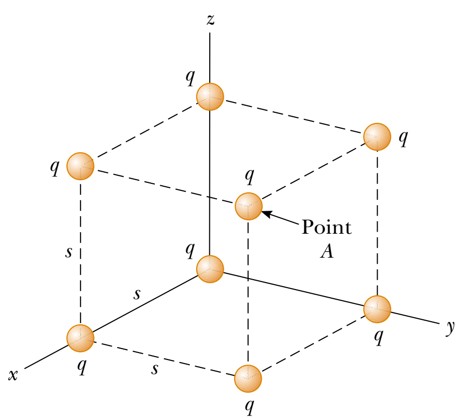


Figure P23.69 Problems 69 and 70.

70. Consider the charge distribution shown in Figure P23.69. (a) Show that the magnitude of the electric field at the center of any face of the cube has a value of $2.18k_e q/s^2$. (b) What is the direction of the electric field at the center of the top face of the cube?

71. A line of charge with a uniform density of 35.0 nC/m lies along the line $y = -15.0$ cm, between the points with coordinates $x = 0$ and $x = 40.0$ cm. Find the electric field it creates at the origin.

72. Three point charges q, $-2q$, and q are located along the x axis, as shown in Figure P23.72. Show that the electric field at P ($y \gg a$) along the y axis is

$$\mathbf{E} = -k_e \frac{3qa^2}{y^4}\,\mathbf{j}$$

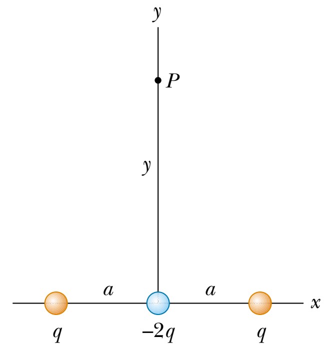


Figure P23.72

This charge distribution, which is essentially that of two electric dipoles, is called an *electric quadrupole.* Note that $\mathbf{E}$ varies as r^{-4} for the quadrupole, compared with variations of r^{-3} for the dipole and r^{-2} for the monopole (a single charge).

$\boxed{73.}$ **Review Problem.** A negatively charged particle $-q$ is placed at the center of a uniformly charged ring, where the ring has a total positive charge Q, as shown in Example 23.8. The particle, confined to move along the x axis, is displaced a *small* distance x along the axis (where $x \ll a$) and released. Show that the particle oscillates with simple harmonic motion with a frequency

$$f = \frac{1}{2\pi}\left(\frac{k_e qQ}{ma^3}\right)^{1/2}$$

74. **Review Problem.** An electric dipole in a uniform electric field is displaced slightly from its equilibrium position, as shown in Figure P23.74, where θ is small and the charges are separated by a distance $2a$. The moment of inertia of the dipole is I. If the dipole is released from this position, show that its angular orientation exhibits simple harmonic motion with a frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{2qaE}{I}}$$

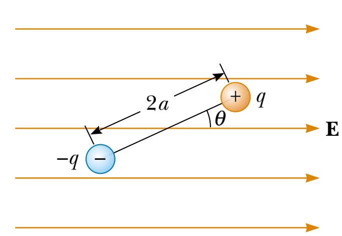
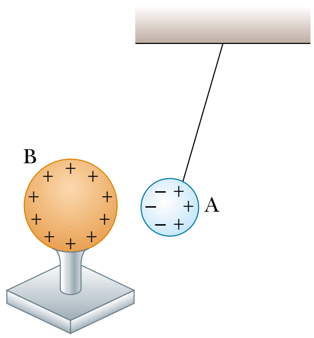


Figure P23.74

Answers to Quick Quizzes

23.1 (b). The amount of charge present after rubbing is the same as that before; it is just distributed differently.

23.2 (d). Object A might be negatively charged, but it also might be electrically neutral with an induced charge separation, as shown in the following figure:



23.3 (b). From Newton's third law, the electric force exerted by object B on object A is equal in magnitude to the force exerted by object A on object B and in the opposite direction—that is, $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$.

23.4 Nothing, if we assume that the source charge producing the field is not disturbed by our actions. Remember that the electric field is created not by the $+3\text{-}\mu C$ charge or by the $-3\text{-}\mu C$ charge but by the source charge (unseen in this case).

23.5 *A*, *B*, and *C*. The field is greatest at point *A* because this is where the field lines are closest together. The absence of lines at point *C* indicates that the electric field there is zero.

PUZZLER

Some railway companies are planning to coat the windows of their commuter trains with a very thin layer of metal. (The coating is so thin you can see through it.) They are doing this in response to rider complaints about other passengers' talking loudly on cellular telephones. How can a metallic coating that is only a few hundred nanometers thick overcome this problem? *(Arthur Tilley/FPG International)*

chapter 24

Gauss's Law

Chapter Outline

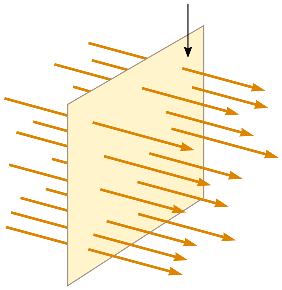


Figure 24.1 Field lines representing a uniform electric field penetrating a plane of area A perpendicular to the field. The electric flux Φ_E through this area is equal to EA.

In the preceding chapter we showed how to use Coulomb's law to calculate the electric field generated by a given charge distribution. In this chapter, we describe *Gauss's law* and an alternative procedure for calculating electric fields. The law is based on the fact that the fundamental electrostatic force between point charges exhibits an inverse-square behavior. Although a consequence of Coulomb's law, Gauss's law is more convenient for calculating the electric fields of highly symmetric charge distributions and makes possible useful qualitative reasoning when we are dealing with complicated problems.

24.1 ELECTRIC FLUX

11.6 The concept of electric field lines is described qualitatively in Chapter 23. We now use the concept of electric flux to treat electric field lines in a more quantitative way.

Consider an electric field that is uniform in both magnitude and direction, as shown in Figure 24.1. The field lines penetrate a rectangular surface of area A, which is perpendicular to the field. Recall from Section 23.6 that the number of lines per unit area (in other words, the *line density*) is proportional to the magnitude of the electric field. Therefore, the total number of lines penetrating the surface is proportional to the product EA. This product of the magnitude of the electric field E and surface area A perpendicular to the field is called the **electric flux** Φ_E (uppercase Greek phi):

$$\Phi_E = EA \tag{24.1}$$

From the SI units of E and A, we see that Φ_E has units of newton–meters squared per coulomb ($\mathrm{N \cdot m^2/C}$). **Electric flux is proportional to the number of electric field lines penetrating some surface.**

EXAMPLE 24.1 Flux Through a Sphere

What is the electric flux through a sphere that has a radius of 1.00 m and carries a charge of $+1.00\ \mu\mathrm{C}$ at its center?

Solution The magnitude of the electric field 1.00 m from this charge is given by Equation 23.4,

$$E = k_e \frac{q}{r^2} = (8.99 \times 10^9\ \mathrm{N \cdot m^2/C^2}) \frac{1.00 \times 10^{-6}\ \mathrm{C}}{(1.00\ \mathrm{m})^2}$$

$$= 8.99 \times 10^3\ \mathrm{N/C}$$

The field points radially outward and is therefore everywhere perpendicular to the surface of the sphere. The flux through the sphere (whose surface area $A = 4\pi r^2 = 12.6\ \mathrm{m^2}$) is thus

$$\Phi_E = EA = (8.99 \times 10^3\ \mathrm{N/C})(12.6\ \mathrm{m^2})$$

$$= \boxed{1.13 \times 10^5\ \mathrm{N \cdot m^2/C}}$$

Exercise What would be the (a) electric field and (b) flux through the sphere if it had a radius of 0.500 m?

Answer (a) $3.60 \times 10^4\ \mathrm{N/C}$; (b) $1.13 \times 10^5\ \mathrm{N \cdot m^2/C}$.

If the surface under consideration is not perpendicular to the field, the flux through it must be less than that given by Equation 24.1. We can understand this by considering Figure 24.2, in which the normal to the surface of area A is at an angle θ to the uniform electric field. Note that the number of lines that cross this area A is equal to the number that cross the area A', which is a projection of area A aligned perpendicular to the field. From Figure 24.2 we see that the two areas are related by $A' = A\cos\theta$. Because the flux through A equals the flux through A', we

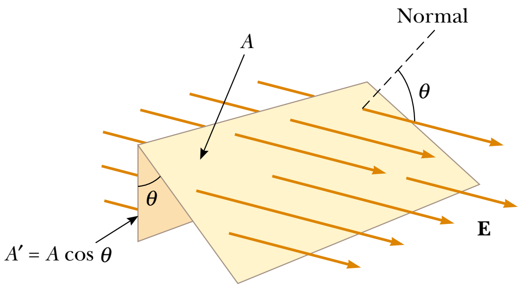


Figure 24.2 Field lines representing a uniform electric field penetrating an area A that is at an angle θ to the field. Because the number of lines that go through the area A' is the same as the number that go through A, the flux through A' is equal to the flux through A and is given by $\Phi_E = EA\cos\theta$.

QuickLab

Shine a desk lamp onto a playing card and notice how the size of the shadow on your desk depends on the orientation of the card with respect to the beam of light. Could a formula like Equation 24.2 be used to describe how much light was being blocked by the card?

conclude that the flux through A is

$$\Phi_E = EA' = EA\cos\theta \qquad \textbf{(24.2)}$$

From this result, we see that the flux through a surface of fixed area A has a maximum value EA when the surface is perpendicular to the field (in other words, when the normal to the surface is parallel to the field, that is, $\theta = 0°$ in Figure 24.2); the flux is zero when the surface is parallel to the field (in other words, when the normal to the surface is perpendicular to the field, that is, $\theta = 90°$).

We assumed a uniform electric field in the preceding discussion. In more general situations, the electric field may vary over a surface. Therefore, our definition of flux given by Equation 24.2 has meaning only over a small element of area. Consider a general surface divided up into a large number of small elements, each of area ΔA. The variation in the electric field over one element can be neglected if the element is sufficiently small. It is convenient to define a vector $\Delta\mathbf{A}_i$ whose magnitude represents the area of the ith element of the surface and whose direction is *defined to be perpendicular* to the surface element, as shown in Figure 24.3. The electric flux $\Delta\Phi_E$ through this element is

$$\Delta\Phi_E = E_i\,\Delta A_i\cos\theta = \mathbf{E}_i\cdot\Delta\mathbf{A}_i$$

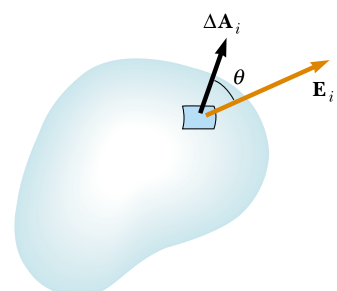


Figure 24.3 A small element of surface area ΔA_i. The electric field makes an angle θ with the vector $\Delta\mathbf{A}_i$, defined as being normal to the surface element, and the flux through the element is equal to $E_i\,\Delta A_i\cos\theta$.

where we have used the definition of the scalar product of two vectors ($\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$). By summing the contributions of all elements, we obtain the total flux through the surface.[1] If we let the area of each element approach zero, then the number of elements approaches infinity and the sum is replaced by an integral. Therefore, the general definition of electric flux is

Definition of electric flux

$$\Phi_E = \lim_{\Delta A_i\to 0}\sum \mathbf{E}_i\cdot\Delta\mathbf{A}_i = \int_{\text{surface}} \mathbf{E}\cdot d\mathbf{A} \qquad \textbf{(24.3)}$$

Equation 24.3 is a *surface integral,* which means it must be evaluated over the surface in question. In general, the value of Φ_E depends both on the field pattern and on the surface.

We are often interested in evaluating the flux through a *closed surface,* which is defined as one that divides space into an inside and an outside region, so that one cannot move from one region to the other without crossing the surface. The surface of a sphere, for example, is a closed surface.

Consider the closed surface in Figure 24.4. The vectors $\Delta\mathbf{A}_i$ point in different directions for the various surface elements, but at each point they are normal to

[1] It is important to note that drawings with field lines have their inaccuracies because a small area element (depending on its location) may happen to have too many or too few field lines penetrating it. We stress that the basic definition of electric flux is $\int \mathbf{E}\cdot d\mathbf{A}$. The use of lines is only an aid for visualizing the concept.

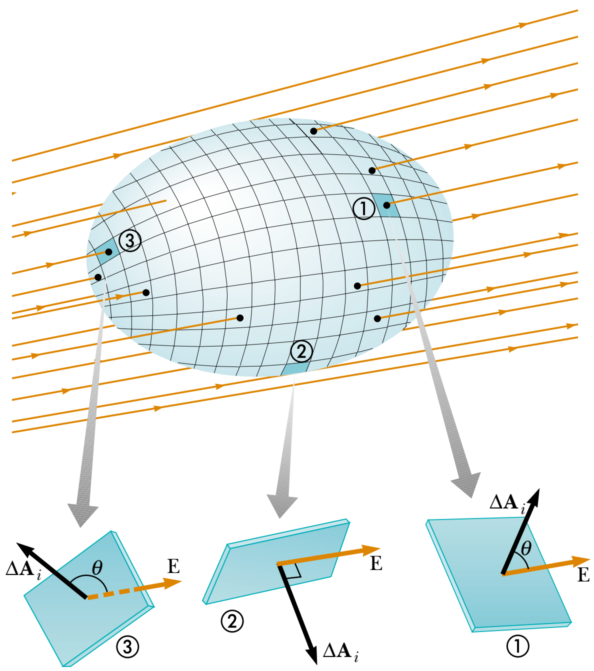
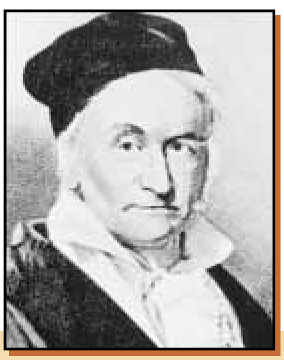


Figure 24.4 A closed surface in an electric field. The area vectors $\Delta\mathbf{A}_i$ are, by convention, normal to the surface and point outward. The flux through an area element can be positive (element ①), zero (element ②), or negative (element ③).



Karl Friedrich Gauss German mathematician and astronomer (1777–1855)

the surface and, by convention, always point outward. At the element labeled ①, the field lines are crossing the surface from the inside to the outside and $\theta < 90°$; hence, the flux $\Delta\Phi_E = \mathbf{E} \cdot \Delta\mathbf{A}_i$ through this element is positive. For element ②, the field lines graze the surface (perpendicular to the vector $\Delta\mathbf{A}_i$); thus, $\theta = 90°$ and the flux is zero. For elements such as ③, where the field lines are crossing the surface from outside to inside, $180° > \theta > 90°$ and the flux is negative because $\cos\theta$ is negative. The *net* flux through the surface is proportional to the net number of lines leaving the surface, where the net number means *the number leaving the surface minus the number entering the surface.* If more lines are leaving than entering, the net flux is positive. If more lines are entering than leaving, the net flux is negative. Using the symbol $\oint$ to represent an integral over a closed surface, we can write the net flux Φ_E through a closed surface as

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E_n \, dA \qquad \textbf{(24.4)}$$

where E_n represents the component of the electric field normal to the surface. Evaluating the net flux through a closed surface can be very cumbersome. However, if the field is normal to the surface at each point and constant in magnitude, the calculation is straightforward, as it was in Example 24.1. The next example also illustrates this point.

EXAMPLE 24.2 Flux Through a Cube

Consider a uniform electric field $\mathbf{E}$ oriented in the x direction. Find the net electric flux through the surface of a cube of edges ℓ, oriented as shown in Figure 24.5.

Solution The net flux is the sum of the fluxes through all faces of the cube. First, note that the flux through four of the faces (③, ④, and the unnumbered ones) is zero because $\mathbf{E}$ is perpendicular to $d\mathbf{A}$ on these faces.

The net flux through faces ① and ② is

$$\Phi_E = \int_1 \mathbf{E} \cdot d\mathbf{A} + \int_2 \mathbf{E} \cdot d\mathbf{A}$$

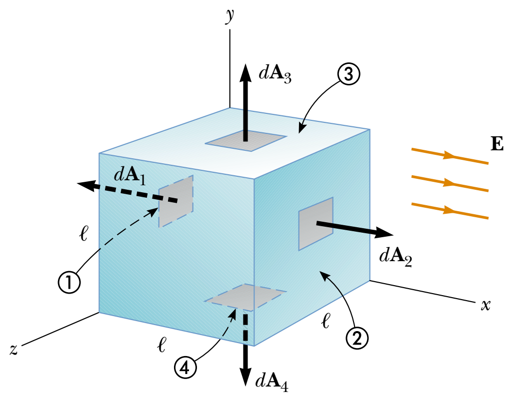


Figure 24.5 A closed surface in the shape of a cube in a uniform electric field oriented parallel to the x axis. The net flux through the closed surface is zero. Side ④ is the bottom of the cube, and side ① is opposite side ②.

For ①, $\mathbf{E}$ is constant and directed inward but $d\mathbf{A}_1$ is directed outward ($\theta = 180°$); thus, the flux through this face is

$$\int_1 \mathbf{E} \cdot d\mathbf{A} = \int_1 E(\cos 180°)\,dA = -E\int_1 dA = -EA = -E\ell^2$$

because the area of each face is $A = \ell^2$.

For ②, $\mathbf{E}$ is constant and outward and in the same direction as $d\mathbf{A}_2 (\theta = 0°)$; hence, the flux through this face is

$$\int_2 \mathbf{E} \cdot d\mathbf{A} = \int_2 E(\cos 0°)\,dA = E\int_2 dA = +EA = E\ell^2$$

Therefore, the net flux over all six faces is

$$\Phi_E = -E\ell^2 + E\ell^2 + 0 + 0 + 0 + 0 = \boxed{0}$$

24.2 GAUSS'S LAW

11.6 In this section we describe a general relationship between the net electric flux through a closed surface (often called a *gaussian surface*) and the charge enclosed by the surface. This relationship, known as *Gauss's law*, is of fundamental importance in the study of electric fields.

Let us again consider a positive point charge q located at the center of a sphere of radius r, as shown in Figure 24.6. From Equation 23.4 we know that the magnitude of the electric field everywhere on the surface of the sphere is $E = k_e q/r^2$. As noted in Example 24.1, the field lines are directed radially outward and hence perpendicular to the surface at every point on the surface. That is, at each surface point, $\mathbf{E}$ is parallel to the vector $\Delta\mathbf{A}_i$ representing a local element of area ΔA_i surrounding the surface point. Therefore,

$$\mathbf{E} \cdot \Delta\mathbf{A}_i = E\,\Delta A_i$$

and from Equation 24.4 we find that the net flux through the gaussian surface is

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \oint E\,dA = E\oint dA$$

where we have moved E outside of the integral because, by symmetry, E is constant over the surface and given by $E = k_e q/r^2$. Furthermore, because the surface is spherical, $\oint dA = A = 4\pi r^2$. Hence, the net flux through the gaussian surface is

$$\Phi_E = \frac{k_e q}{r^2}(4\pi r^2) = 4\pi k_e q$$

Recalling from Section 23.3 that $k_e = 1/(4\pi\epsilon_0)$, we can write this equation in the form

$$\Phi_E = \frac{q}{\epsilon_0} \tag{24.5}$$

We can verify that this expression for the net flux gives the same result as Example 24.1: $\Phi_E = (1.00 \times 10^{-6}\ \text{C})/(8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2) = 1.13 \times 10^5\ \text{N}\cdot\text{m}^2/\text{C}$.

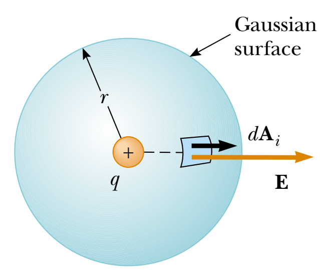


Figure 24.6 A spherical gaussian surface of radius r surrounding a point charge q. When the charge is at the center of the sphere, the electric field is everywhere normal to the surface and constant in magnitude.

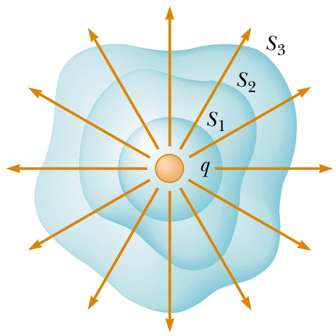


Figure 24.7 Closed surfaces of various shapes surrounding a charge q. The net electric flux is the same through all surfaces.

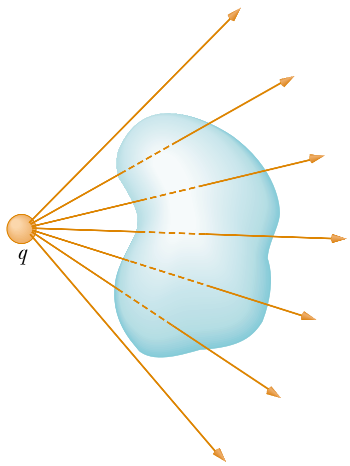


Figure 24.8 A point charge located *outside* a closed surface. The number of lines entering the surface equals the number leaving the surface.

The net electric flux through a closed surface is zero if there is no charge inside

Note from Equation 24.5 that the net flux through the spherical surface is proportional to the charge inside. The flux is independent of the radius r because the area of the spherical surface is proportional to r^2, whereas the electric field is proportional to $1/r^2$. Thus, in the product of area and electric field, the dependence on r cancels.

Now consider several closed surfaces surrounding a charge q, as shown in Figure 24.7. Surface S_1 is spherical, but surfaces S_2 and S_3 are not. From Equation 24.5, the flux that passes through S_1 has the value q/ϵ_0. As we discussed in the previous section, flux is proportional to the number of electric field lines passing through a surface. The construction shown in Figure 24.7 shows that the number of lines through S_1 is equal to the number of lines through the nonspherical surfaces S_2 and S_3. Therefore, we conclude that the net flux through *any* closed surface is independent of the shape of that surface. **The net flux through any closed surface surrounding a point charge q is given by q/ϵ_0.**

Now consider a point charge located *outside* a closed surface of arbitrary shape, as shown in Figure 24.8. As you can see from this construction, any electric field line that enters the surface leaves the surface at another point. The number of electric field lines entering the surface equals the number leaving the surface. Therefore, we conclude that **the net electric flux through a closed surface that surrounds no charge is zero.** If we apply this result to Example 24.2, we can easily see that the net flux through the cube is zero because there is no charge inside the cube.

Quick Quiz 24.1

Suppose that the charge in Example 24.1 is just outside the sphere, 1.01 m from its center. What is the total flux through the sphere?

Let us extend these arguments to two generalized cases: (1) that of many point charges and (2) that of a continuous distribution of charge. We once again use the superposition principle, which states that **the electric field due to many charges is the vector sum of the electric fields produced by the individual charges.** Therefore, we can express the flux through any closed surface as

$$\oint \mathbf{E} \cdot d\mathbf{A} = \oint (\mathbf{E}_1 + \mathbf{E}_2 + \cdots) \cdot d\mathbf{A}$$

where $\mathbf{E}$ is the total electric field at any point on the surface produced by the vector addition of the electric fields at that point due to the individual charges.

Consider the system of charges shown in Figure 24.9. The surface S surrounds only one charge, q_1; hence, the net flux through S is q_1/ϵ_0. The flux through S due to charges q_2 and q_3 outside it is zero because each electric field line that enters S at one point leaves it at another. The surface S' surrounds charges q_2 and q_3; hence, the net flux through it is $(q_2 + q_3)/\epsilon_0$. Finally, the net flux through surface S'' is zero because there is no charge inside this surface. That is, *all* the electric field lines that enter S'' at one point leave at another.

Gauss's law, which is a generalization of what we have just described, states that the net flux through *any* closed surface is

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{in}}}{\epsilon_0} \tag{24.6}$$

Gauss's law

where q_{in} represents the net charge inside the surface and $\mathbf{E}$ represents the electric field at any point on the surface.

A formal proof of Gauss's law is presented in Section 24.6. When using Equation 24.6, you should note that although the charge q_{in} is the net charge inside the gaussian surface, $\mathbf{E}$ represents the *total electric field,* which includes contributions from charges both inside and outside the surface.

Gauss's law is useful for evaluating E when the charge distribution has high symmetry

In principle, Gauss's law can be solved for $\mathbf{E}$ to determine the electric field due to a system of charges or a continuous distribution of charge. In practice, however, this type of solution is applicable only in a limited number of highly symmetric situations. As we shall see in the next section, Gauss's law can be used to evaluate the electric field for charge distributions that have spherical, cylindrical, or planar symmetry. If one chooses the gaussian surface surrounding the charge distribution carefully, the integral in Equation 24.6 can be simplified. You should also note that a gaussian surface is a mathematical construction and need not coincide with any real physical surface.

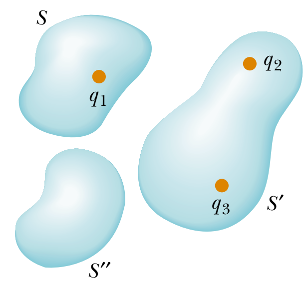


Figure 24.9 The net electric flux through any closed surface depends only on the charge *inside* that surface. The net flux through surface S is q_1/ϵ_0, the net flux through surface S' is $(q_2 + q_3)/\epsilon_0$, and the net flux through surface S'' is zero.

Quick Quiz 24.2

For a gaussian surface through which the net flux is zero, the following four statements *could be true.* Which of the statements *must be true?* (a) There are no charges inside the surface. (b) The net charge inside the surface is zero. (c) The electric field is zero everywhere on the surface. (d) The number of electric field lines entering the surface equals the number leaving the surface.

CONCEPTUAL EXAMPLE 24.3

A spherical gaussian surface surrounds a point charge q. Describe what happens to the total flux through the surface if (a) the charge is tripled, (b) the radius of the sphere is doubled, (c) the surface is changed to a cube, and (d) the charge is moved to another location inside the surface.

Solution (a) The flux through the surface is tripled because flux is proportional to the amount of charge inside the surface.

(b) The flux does not change because all electric field lines from the charge pass through the sphere, regardless of its radius.

(c) The flux does not change when the shape of the gaussian surface changes because all electric field lines from the charge pass through the surface, regardless of its shape.

(d) The flux does not change when the charge is moved to another location inside that surface because Gauss's law refers to the total charge enclosed, regardless of where the charge is located inside the surface.

24.3 APPLICATION OF GAUSS'S LAW TO CHARGED INSULATORS

As mentioned earlier, Gauss's law is useful in determining electric fields when the charge distribution is characterized by a high degree of symmetry. The following examples demonstrate ways of choosing the gaussian surface over which the surface integral given by Equation 24.6 can be simplified and the electric field determined. In choosing the surface, we should always take advantage of the symmetry of the charge distribution so that we can remove E from the integral and solve for it. The goal in this type of calculation is to determine a surface that satisfies one or more of the following conditions:

1. The value of the electric field can be argued by symmetry to be constant over the surface.
2. The dot product in Equation 24.6 can be expressed as a simple algebraic product $E\,dA$ because $\mathbf{E}$ and $d\mathbf{A}$ are parallel.
3. The dot product in Equation 24.6 is zero because $\mathbf{E}$ and $d\mathbf{A}$ are perpendicular.
4. The field can be argued to be zero over the surface.

All four of these conditions are used in examples throughout the remainder of this chapter.

EXAMPLE 24.4 The Electric Field Due to a Point Charge

Starting with Gauss's law, calculate the electric field due to an isolated point charge q.

Solution A single charge represents the simplest possible charge distribution, and we use this familiar case to show how to solve for the electric field with Gauss's law. We choose a spherical gaussian surface of radius r centered on the point charge, as shown in Figure 24.10. The electric field due to a positive point charge is directed radially outward by symmetry and is therefore normal to the surface at every point. Thus, as in condition (2), $\mathbf{E}$ is parallel to $d\mathbf{A}$ at each point. Therefore, $\mathbf{E}\cdot d\mathbf{A} = E\,dA$ and Gauss's law gives

$$\Phi_E = \oint \mathbf{E}\cdot d\mathbf{A} = \oint E\,dA = \frac{q}{\epsilon_0}$$

By symmetry, E is constant everywhere on the surface, which satisfies condition (1), so it can be removed from the integral. Therefore,

$$\oint E\,dA = E\oint dA = E(4\pi r^2) = \frac{q}{\epsilon_0}$$

where we have used the fact that the surface area of a sphere is $4\pi r^2$. Now, we solve for the electric field:

$$E = \frac{q}{4\pi\epsilon_0 r^2} = \boxed{k_e\frac{q}{r^2}}$$

This is the familiar electric field due to a point charge that we developed from Coulomb's law in Chapter 23.

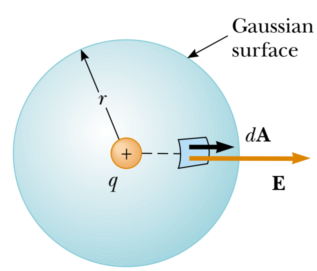


Figure 24.10 The point charge q is at the center of the spherical gaussian surface, and $\mathbf{E}$ is parallel to $d\mathbf{A}$ at every point on the surface.

EXAMPLE 24.5 A Spherically Symmetric Charge Distribution

11.6

An insulating solid sphere of radius a has a uniform volume charge density ρ and carries a total positive charge Q (Fig. 24.11). (a) Calculate the magnitude of the electric field at a point outside the sphere.

Solution Because the charge distribution is spherically symmetric, we again select a spherical gaussian surface of radius r, concentric with the sphere, as shown in Figure 24.11a. For this choice, conditions (1) and (2) are satisfied, as they

were for the point charge in Example 24.4. Following the line of reasoning given in Example 24.4, we find that

$$E = \boxed{k_e \frac{Q}{r^2}} \qquad \text{(for } r > a\text{)}$$

Note that this result is identical to the one we obtained for a point charge. Therefore, we conclude that, for a uniformly charged sphere, the field in the region external to the sphere is *equivalent* to that of a point charge located at the center of the sphere.

(b) Find the magnitude of the electric field at a point inside the sphere.

Solution In this case we select a spherical gaussian surface having radius $r < a$, concentric with the insulated sphere (Fig. 24.11b). Let us denote the volume of this smaller sphere by V'. To apply Gauss's law in this situation, it is important to recognize that the charge q_{in} within the gaussian surface of volume V' is less than Q. To calculate q_{in}, we use the fact that $q_{\text{in}} = \rho V'$:

$$q_{\text{in}} = \rho V' = \rho(\tfrac{4}{3}\pi r^3)$$

By symmetry, the magnitude of the electric field is constant everywhere on the spherical gaussian surface and is normal to the surface at each point—both conditions (1) and (2) are satisfied. Therefore, Gauss's law in the region $r < a$ gives

$$\oint E\,dA = E\oint dA = E(4\pi r^2) = \frac{q_{\text{in}}}{\epsilon_0}$$

Solving for E gives

$$E = \frac{q_{\text{in}}}{4\pi\epsilon_0 r^2} = \frac{\rho\frac{4}{3}\pi r^3}{4\pi\epsilon_0 r^2} = \frac{\rho}{3\epsilon_0} r$$

Because $\rho = Q/\frac{4}{3}\pi a^3$ by definition and since $k_e = 1/(4\pi\epsilon_0)$, this expression for E can be written as

$$E = \frac{Qr}{4\pi\epsilon_0 a^3} = \boxed{\frac{k_e Q}{a^3} r} \qquad \text{(for } r < a\text{)}$$

Note that this result for E differs from the one we obtained in part (a). It shows that $E \to 0$ as $r \to 0$. Therefore, the result eliminates the problem that would exist at $r = 0$ if E varied as $1/r^2$ inside the sphere as it does outside the sphere. That is, if $E \propto 1/r^2$ for $r < a$, the field would be infinite at $r = 0$, which is physically impossible. Note also that the expressions for parts (a) and (b) match when $r = a$.

A plot of E versus r is shown in Figure 24.12.

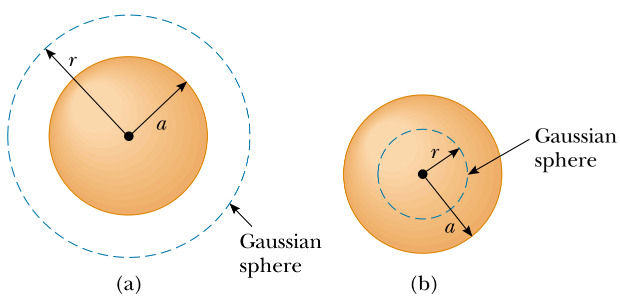


Figure 24.11 A uniformly charged insulating sphere of radius a and total charge Q. (a) The magnitude of the electric field at a point exterior to the sphere is k_eQ/r^2. (b) The magnitude of the electric field inside the insulating sphere is due only to the charge *within* the gaussian sphere defined by the dashed circle and is k_eQr/a^3.

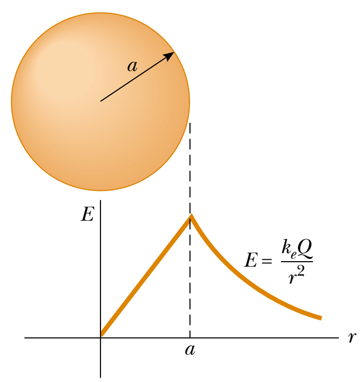


Figure 24.12 A plot of E versus r for a uniformly charged insulating sphere. The electric field inside the sphere ($r < a$) varies linearly with r. The field outside the sphere ($r > a$) is the same as that of a point charge Q located at $r = 0$.

EXAMPLE 24.6 The Electric Field Due to a Thin Spherical Shell

A thin spherical shell of radius a has a total charge Q distributed uniformly over its surface (Fig. 24.13a). Find the electric field at points (a) outside and (b) inside the shell.

Solution (a) The calculation for the field outside the shell is identical to that for the solid sphere shown in Example 24.5a. If we construct a spherical gaussian surface of radius $r > a$ concentric with the shell (Fig. 24.13b), the charge inside this surface is Q. Therefore, the field at a point outside the shell is equivalent to that due to a point charge Q located at the center:

$$E = \boxed{k_e \frac{Q}{r^2}} \qquad \text{(for } r > a\text{)}$$

(b) The electric field inside the spherical shell is zero. This follows from Gauss's law applied to a spherical surface of radius $r < a$ concentric with the shell (Fig. 24.13c). Because

of the spherical symmetry of the charge distribution and because the net charge inside the surface is zero—satisfaction of conditions (1) and (2) again—application of Gauss's law shows that $E = 0$ in the region $r < a$.

We obtain the same results using Equation 23.6 and integrating over the charge distribution. This calculation is rather complicated. Gauss's law allows us to determine these results in a much simpler way.

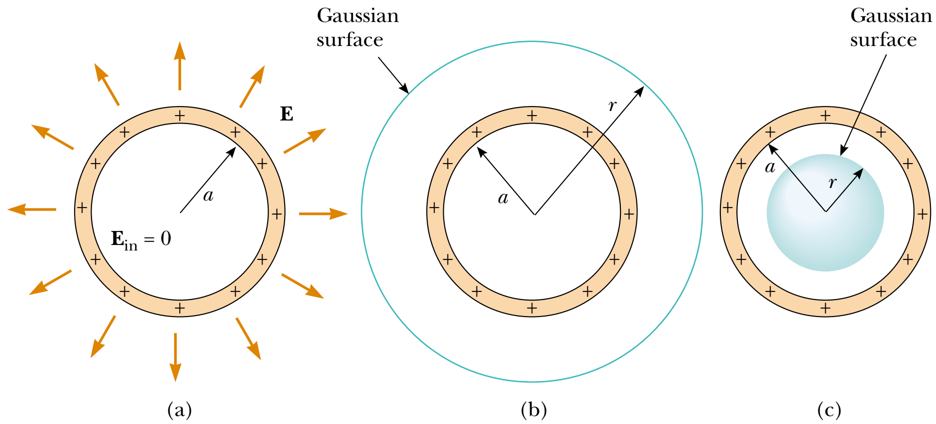


Figure 24.13 (a) The electric field inside a uniformly charged spherical shell is zero. The field outside is the same as that due to a point charge Q located at the center of the shell. (b) Gaussian surface for $r > a$. (c) Gaussian surface for $r < a$.

EXAMPLE 24.7 A Cylindrically Symmetric Charge Distribution

11.7

Find the electric field a distance r from a line of positive charge of infinite length and constant charge per unit length λ (Fig. 24.14a).

Solution The symmetry of the charge distribution requires that $\mathbf{E}$ be perpendicular to the line charge and directed outward, as shown in Figure 24.14a and b. To reflect the symmetry of the charge distribution, we select a cylindrical gaussian surface of radius r and length ℓ that is coaxial with the line charge. For the curved part of this surface, $\mathbf{E}$ is constant in magnitude and perpendicular to the surface at each point—satisfaction of conditions (1) and (2). Furthermore, the flux through the ends of the gaussian cylinder is zero because $\mathbf{E}$ is parallel to these surfaces—the first application we have seen of condition (3).

We take the surface integral in Gauss's law over the entire gaussian surface. Because of the zero value of $\mathbf{E} \cdot d\mathbf{A}$ for the ends of the cylinder, however, we can restrict our attention to only the curved surface of the cylinder.

The total charge inside our gaussian surface is $\lambda\ell$. Applying Gauss's law and conditions (1) and (2), we find that for the curved surface

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = E \oint dA = EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\lambda \ell}{\epsilon_0}$$

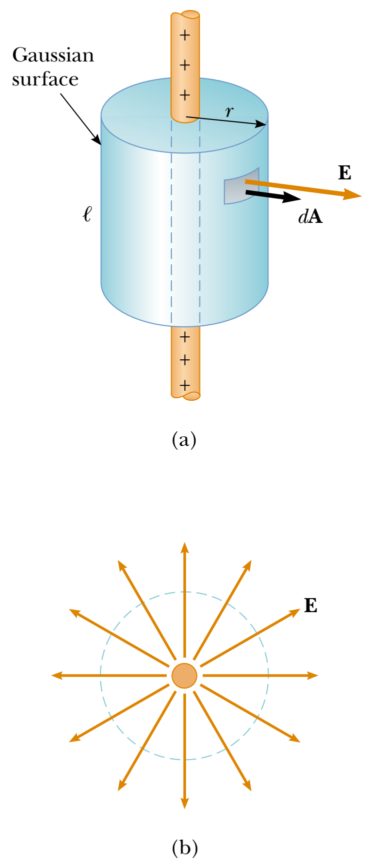


Figure 24.14 (a) An infinite line of charge surrounded by a cylindrical gaussian surface concentric with the line. (b) An end view shows that the electric field at the cylindrical surface is constant in magnitude and perpendicular to the surface.

The area of the curved surface is $A = 2\pi r\ell$; therefore,

$$E(2\pi r\ell) = \frac{\lambda\ell}{\epsilon_0}$$

$$E = \frac{\lambda}{2\pi\epsilon_0 r} = \boxed{2k_e\frac{\lambda}{r}} \tag{24.7}$$

Thus, we see that the electric field due to a cylindrically symmetric charge distribution varies as $1/r$, whereas the field external to a spherically symmetric charge distribution varies as $1/r^2$. Equation 24.7 was also derived in Chapter 23 (see Problem 35[b]), by integration of the field of a point charge.

If the line charge in this example were of finite length, the result for E would not be that given by Equation 24.7. A finite line charge does not possess sufficient symmetry for us to make use of Gauss's law. This is because the magnitude of the electric field is no longer constant over the surface of the gaussian cylinder—the field near the ends of the line would be different from that far from the ends. Thus, condition (1) would not be satisfied in this situation. Furthermore, **E** is not perpendicular to the cylindrical surface at all points—the field vectors near the ends would have a component parallel to the line. Thus, condition (2) would not be satisfied. When there is insufficient symmetry in the charge distribution, as in this situation, it is necessary to use Equation 23.6 to calculate **E**.

For points close to a finite line charge and far from the ends, Equation 24.7 gives a good approximation of the value of the field.

It is left for you to show (see Problem 29) that the electric field inside a uniformly charged rod of finite radius and infinite length is proportional to r.

EXAMPLE 24.8 A Nonconducting Plane of Charge

Find the electric field due to a nonconducting, infinite plane of positive charge with uniform surface charge density σ.

Solution By symmetry, **E** must be perpendicular to the plane and must have the same magnitude at all points equidistant from the plane. The fact that the direction of **E** is away from positive charges indicates that the direction of **E** on one side of the plane must be opposite its direction on the other side, as shown in Figure 24.15. A gaussian surface that reflects the symmetry is a small cylinder whose axis is perpendicular to the plane and whose ends each have an area A and are equidistant from the plane. Because **E** is parallel to the curved surface—and, therefore, perpendicular to $d\mathbf{A}$ everywhere on the surface—condition (3) is satisfied and there is no contribution to the surface integral from this surface. For the flat ends of the cylinder, conditions (1) and (2) are satisfied. The flux through each end of the cylinder is EA; hence, the total flux through the entire gaussian surface is just that through the ends, $\Phi_E = 2EA$.

Noting that the total charge inside the surface is $q_{\text{in}} = \sigma A$, we use Gauss's law and find that

$$\Phi_E = 2EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

$$E = \boxed{\frac{\sigma}{2\epsilon_0}} \tag{24.8}$$

Because the distance from each flat end of the cylinder to the plane does not appear in Equation 24.8, we conclude that $E = \sigma/2\epsilon_0$ at any distance from the plane. That is, the field is uniform everywhere.

An important charge configuration related to this example consists of two parallel planes, one positively charged and the other negatively charged, and each with a surface charge density σ (see Problem 58). In this situation, the electric fields due to the two planes add in the region between the planes, resulting in a field of magnitude σ/ϵ_0, and cancel elsewhere to give a field of zero.

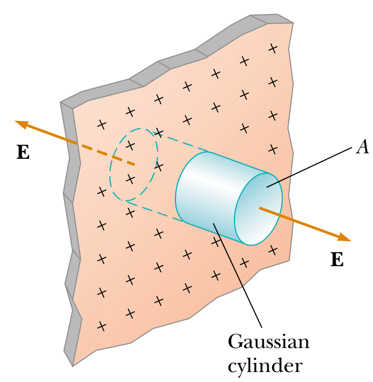


Figure 24.15 A cylindrical gaussian surface penetrating an infinite plane of charge. The flux is EA through each end of the gaussian surface and zero through its curved surface.

CONCEPTUAL EXAMPLE 24.9

Explain why Gauss's law cannot be used to calculate the electric field near an electric dipole, a charged disk, or a triangle with a point charge at each corner.

Solution The charge distributions of all these configurations do not have sufficient symmetry to make the use of Gauss's law practical. We cannot find a closed surface surrounding any of these distributions that satisfies one or more of conditions (1) through (4) listed at the beginning of this section.

24.4 CONDUCTORS IN ELECTROSTATIC EQUILIBRIUM

As we learned in Section 23.2, a good electrical conductor contains charges (electrons) that are not bound to any atom and therefore are free to move about within the material. When there is no net motion of charge within a conductor, the conductor is in **electrostatic equilibrium.** As we shall see, a conductor in electrostatic equilibrium has the following properties:

Properties of a conductor in electrostatic equilibrium

1. The electric field is zero everywhere inside the conductor.
2. If an isolated conductor carries a charge, the charge resides on its surface.
3. The electric field just outside a charged conductor is perpendicular to the surface of the conductor and has a magnitude σ/ϵ_0, where σ is the surface charge density at that point.
4. On an irregularly shaped conductor, the surface charge density is greatest at locations where the radius of curvature of the surface is smallest.

We verify the first three properties in the discussion that follows. The fourth property is presented here without further discussion so that we have a complete list of properties for conductors in electrostatic equilibrium.

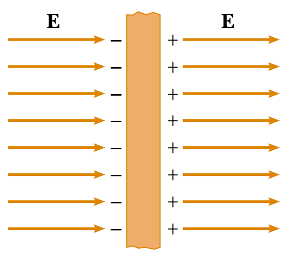


Figure 24.16 A conducting slab in an external electric field **E**. The charges induced on the two surfaces of the slab produce an electric field that opposes the external field, giving a resultant field of zero inside the slab.

We can understand the first property by considering a conducting slab placed in an external field **E** (Fig. 24.16). We can argue that the electric field inside the conductor *must* be zero under the assumption that we have electrostatic equilibrium. If the field were not zero, free charges in the conductor would accelerate under the action of the field. This motion of electrons, however, would mean that the conductor is not in electrostatic equilibrium. Thus, the existence of electrostatic equilibrium is consistent only with a zero field in the conductor.

Let us investigate how this zero field is accomplished. Before the external field is applied, free electrons are uniformly distributed throughout the conductor. When the external field is applied, the free electrons accelerate to the left in Figure 24.16, causing a plane of negative charge to be present on the left surface. The movement of electrons to the left results in a plane of positive charge on the right surface. These planes of charge create an additional electric field inside the conductor that opposes the external field. As the electrons move, the surface charge density increases until the magnitude of the internal field equals that of the external field, and the net result is a net field of zero inside the conductor. The time it takes a good conductor to reach equilibrium is of the order of 10^{-16} s, which for most purposes can be considered instantaneous.

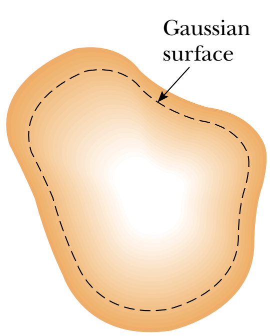
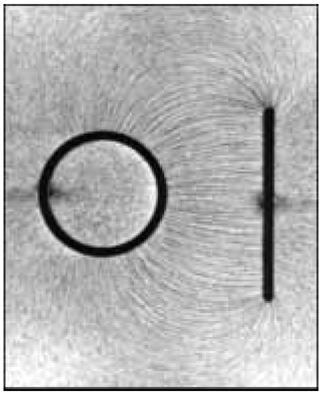


Figure 24.17 A conductor of arbitrary shape. The broken line represents a gaussian surface just inside the conductor.

We can use Gauss's law to verify the second property of a conductor in electrostatic equilibrium. Figure 24.17 shows an arbitrarily shaped conductor. A gaussian surface is drawn inside the conductor and can be as close to the conductor's surface as we wish. As we have just shown, the electric field everywhere inside the conductor is zero when it is in electrostatic equilibrium. Therefore, the electric field must be zero at every point on the gaussian surface, in accordance with condition (4) in Section 24.3. Thus, the net flux through this gaussian surface is zero. From this result and Gauss's law, we conclude that the net charge inside the gaussian sur-



Electric field pattern surrounding a charged conducting plate placed near an oppositely charged conducting cylinder. Small pieces of thread suspended in oil align with the electric field lines. Note that (1) the field lines are perpendicular to both conductors and (2) there are no lines inside the cylinder ($E = 0$).

face is zero. Because there can be no net charge inside the gaussian surface (which is arbitrarily close to the conductor's surface), **any net charge on the conductor must reside on its surface.** Gauss's law does not indicate how this excess charge is distributed on the conductor's surface.

We can also use Gauss's law to verify the third property. We draw a gaussian surface in the shape of a small cylinder whose end faces are parallel to the surface of the conductor (Fig. 24.18). Part of the cylinder is just outside the conductor, and part is inside. The field is normal to the conductor's surface from the condition of electrostatic equilibrium. (If **E** had a component parallel to the conductor's surface, the free charges would move along the surface; in such a case, the conductor would not be in equilibrium.) Thus, we satisfy condition (3) in Section 24.3 for the curved part of the cylindrical gaussian surface—there is no flux through this part of the gaussian surface because **E** is parallel to the surface. There is no flux through the flat face of the cylinder inside the conductor because here $\mathbf{E} = 0$—satisfaction of condition (4). Hence, the net flux through the gaussian surface is that through only the flat face outside the conductor, where the field is perpendicular to the gaussian surface. Using conditions (1) and (2) for this face, the flux is EA, where E is the electric field just outside the conductor and A is the area of the cylinder's face. Applying Gauss's law to this surface, we obtain

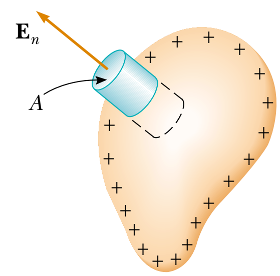


Figure 24.18 A gaussian surface in the shape of a small cylinder is used to calculate the electric field just outside a charged conductor. The flux through the gaussian surface is E_nA. Remember that **E** is zero inside the conductor.

$$\Phi_E = \oint E\,dA = EA = \frac{q_{\text{in}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$$

where we have used the fact that $q_{\text{in}} = \sigma A$. Solving for E gives

$$E = \frac{\sigma}{\epsilon_0} \tag{24.9}$$

Electric field just outside a charged conductor

EXAMPLE 24.10 A Sphere Inside a Spherical Shell

A solid conducting sphere of radius a carries a net positive charge $2Q$. A conducting spherical shell of inner radius b and outer radius c is concentric with the solid sphere and carries a net charge $-Q$. Using Gauss's law, find the electric field in the regions labeled ①, ②, ③, and ④ in Figure 24.19 and the charge distribution on the shell when the entire system is in electrostatic equilibrium.

Solution First note that the charge distributions on both the sphere and the shell are characterized by spherical symmetry around their common center. To determine the electric field at various distances r from this center, we construct a spherical gaussian surface for each of the four regions of interest. Such a surface for region ② is shown in Figure 24.19.

To find E inside the solid sphere (region ①), consider a

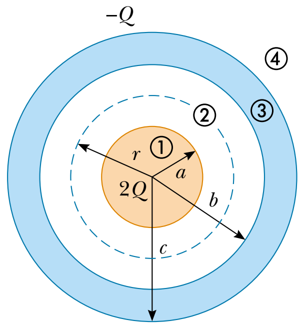


Figure 24.19 A solid conducting sphere of radius a and carrying a charge $2Q$ surrounded by a conducting spherical shell carrying a charge $-Q$.

gaussian surface of radius $r < a$. Because there can be no charge inside a conductor in electrostatic equilibrium, we see that $q_{in} = 0$; thus, on the basis of Gauss's law and symmetry, $E_1 = 0$ for $r < a$.

In region ②—between the surface of the solid sphere and the inner surface of the shell—we construct a spherical gaussian surface of radius r where $a < r < b$ and note that the charge inside this surface is $+2Q$ (the charge on the solid sphere). Because of the spherical symmetry, the electric field lines must be directed radially outward and be constant in magnitude on the gaussian surface. Following Example 24.4 and using Gauss's law, we find that

$$E_2 A = E_2(4\pi r^2) = \frac{q_{in}}{\epsilon_0} = \frac{2Q}{\epsilon_0}$$

$$E_2 = \frac{2Q}{4\pi\epsilon_0 r^2} = \boxed{\frac{2k_e Q}{r^2}} \qquad (\text{for } a < r < b)$$

In region ④, where $r > c$, the spherical gaussian surface we construct surrounds a total charge of $q_{in} = 2Q + (-Q) = Q$. Therefore, application of Gauss's law to this surface gives

$$E_4 = \boxed{\frac{k_e Q}{r^2}} \qquad (\text{for } r > c)$$

In region ③, the electric field must be zero because the spherical shell is also a conductor in equilibrium. If we construct a gaussian surface of radius r where $b < r < c$, we see that q_{in} must be zero because $E_3 = 0$. From this argument, we conclude that the charge on the inner surface of the spherical shell must be $-2Q$ to cancel the charge $+2Q$ on the solid sphere. Because the net charge on the shell is $-Q$, we conclude that its outer surface must carry a charge $+Q$.

Quick Quiz 24.3

How would the electric flux through a gaussian surface surrounding the shell in Example 24.10 change if the solid sphere were off-center but still inside the shell?

Optional Section

24.5 EXPERIMENTAL VERIFICATION OF GAUSS'S LAW AND COULOMB'S LAW

When a net charge is placed on a conductor, the charge distributes itself on the surface in such a way that the electric field inside the conductor is zero. Gauss's law shows that there can be no net charge inside the conductor in this situation. In this section, we investigate an experimental verification of the absence of this charge.

We have seen that Gauss's law is equivalent to Equation 23.6, the expression for the electric field of a distribution of charge. Because this equation arises from Coulomb's law, we can claim theoretically that Gauss's law and Coulomb's law are equivalent. Hence, it is possible to test the validity of both laws by attempting to detect a net charge inside a conductor or, equivalently, a nonzero electric field inside the conductor. If a nonzero field is detected within the conductor, Gauss's law and Coulomb's law are invalid. Many experiments, including

early work by Faraday, Cavendish, and Maxwell, have been performed to detect the field inside a conductor. In all reported cases, no electric field could be detected inside a conductor.

Here is one of the experiments that can be performed.[2] A positively charged metal ball at the end of a silk thread is lowered through a small opening into an uncharged hollow conductor that is insulated from ground (Fig. 24.20a). The positively charged ball induces a negative charge on the inner wall of the hollow conductor, leaving an equal positive charge on the outer wall (Fig. 24.20b). The presence of positive charge on the outer wall is indicated by the deflection of the needle of an electrometer (a device used to measure charge and that measures charge only on the outer surface of the conductor). The ball is then lowered and allowed to touch the inner surface of the hollow conductor (Fig. 24.20c). Charge is transferred between the ball and the inner surface so that neither is charged after contact is made. The needle deflection remains unchanged while this happens, indicating that the charge on the outer surface is unaffected. When the ball is removed, the electrometer reading remains the same (Fig. 24.20d). Furthermore, the ball is found to be uncharged; this verifies that charge was transferred between the ball and the inner surface of the hollow conductor. The overall effect is that the charge that was originally on the ball now appears on the hollow conductor. The fact that the deflection of the needle on the electrometer measuring the charge on the outer surface remained unchanged regardless of what was happening inside the hollow conductor indicates that the net charge on the system always resided on the outer surface of the conductor.

If we now apply another positive charge to the metal ball and place it near the outside of the conductor, it is repelled by the conductor. This demonstrates that $\mathbf{E} \neq 0$ outside the conductor, a finding consistent with the fact that the conductor carries a net charge. If the charged metal ball is now lowered into the interior of the charged hollow conductor, it exhibits no evidence of an electric force. This shows that $\mathbf{E} = 0$ inside the hollow conductor.

This experiment verifies the predictions of Gauss's law and therefore verifies Coulomb's law. The equivalence of Gauss's law and Coulomb's law is due to the inverse-square behavior of the electric force. Thus, we can interpret this experiment as verifying the exponent of 2 in the $1/r^2$ behavior of the electric force. Experiments by Williams, Faller, and Hill in 1971 showed that the exponent of r in Coulomb's law is $(2 + \delta)$, where $\delta = (2.7 \pm 3.1) \times 10^{-16}$!

In the experiment we have described, the charged ball hanging in the hollow conductor would show no deflection even in the case in which an external electric field is applied to the entire system. The field inside the conductor is still zero. This ability of conductors to "block" external electric fields is utilized in many places, from electromagnetic shielding for computer components to thin metal coatings on the glass in airport control towers to keep radar originating outside the tower from disrupting the electronics inside. Cellular telephone users riding trains like the one pictured at the beginning of the chapter have to speak loudly to be heard above the noise of the train. In response to complaints from other passengers, the train companies are considering coating the windows with a thin metallic conductor. This coating, combined with the metal frame of the train car, blocks cellular telephone transmissions into and out of the train.

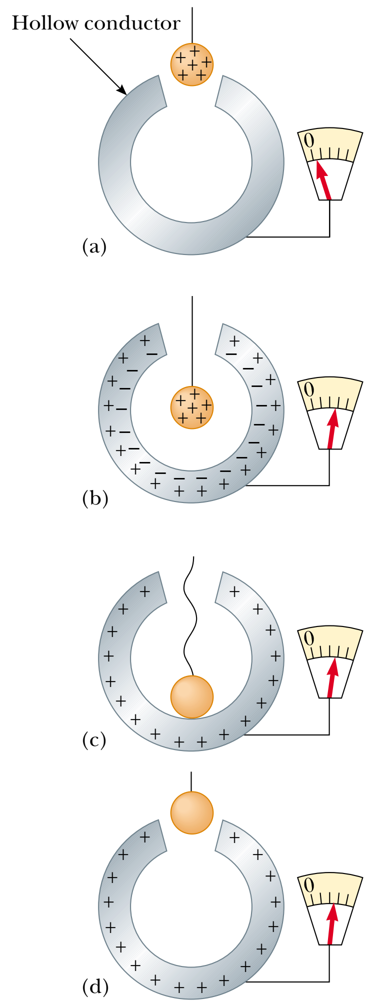


Figure 24.20 An experiment showing that any charge transferred to a conductor resides on its surface in electrostatic equilibrium. The hollow conductor is insulated from ground, and the small metal ball is supported by an insulating thread.

QuickLab

Wrap a radio or cordless telephone in aluminum foil and see if it still works. Does it matter if the foil touches the antenna?

[2] The experiment is often referred to as *Faraday's ice-pail experiment* because Faraday, the first to perform it, used an ice pail for the hollow conductor.

Optional Section

24.6 FORMAL DERIVATION OF GAUSS'S LAW

One way of deriving Gauss's law involves *solid angles.* Consider a spherical surface of radius r containing an area element ΔA. The solid angle $\Delta\Omega$ (uppercase Greek omega) subtended at the center of the sphere by this element is defined to be

$$\Delta\Omega \equiv \frac{\Delta A}{r^2}$$

From this equation, we see that $\Delta\Omega$ has no dimensions because ΔA and r^2 both have dimensions L^2. The dimensionless unit of a solid angle is the **steradian.** (You may want to compare this equation to Equation 10.1b, the definition of the radian.) Because the surface area of a sphere is $4\pi r^2$, the total solid angle subtended by the sphere is

$$\Omega = \frac{4\pi r^2}{r^2} = 4\pi \text{ steradians}$$

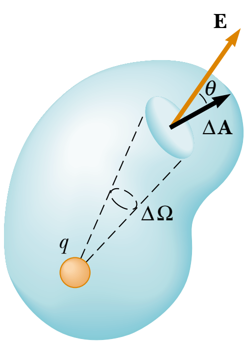


Figure 24.21 A closed surface of arbitrary shape surrounds a point charge q. The net electric flux through the surface is independent of the shape of the surface.

Now consider a point charge q surrounded by a closed surface of arbitrary shape (Fig. 24.21). The total electric flux through this surface can be obtained by evaluating $\mathbf{E} \cdot \Delta\mathbf{A}$ for each small area element ΔA and summing over all elements. The flux through each element is

$$\Delta\Phi_E = \mathbf{E} \cdot \Delta\mathbf{A} = E\,\Delta A \cos\theta = k_e q \frac{\Delta A \cos\theta}{r^2}$$

where r is the distance from the charge to the area element, θ is the angle between the electric field $\mathbf{E}$ and $\Delta\mathbf{A}$ for the element, and $E = k_e q/r^2$ for a point charge. In Figure 24.22, we see that the projection of the area element perpendicular to the radius vector is $\Delta A \cos\theta$. Thus, the quantity $\Delta A \cos\theta/r^2$ is equal to the solid angle $\Delta\Omega$ that the surface element ΔA subtends at the charge q. We also see that $\Delta\Omega$ is equal to the solid angle subtended by the area element of a spherical surface of radius r. Because the total solid angle at a point is 4π steradians, the total flux

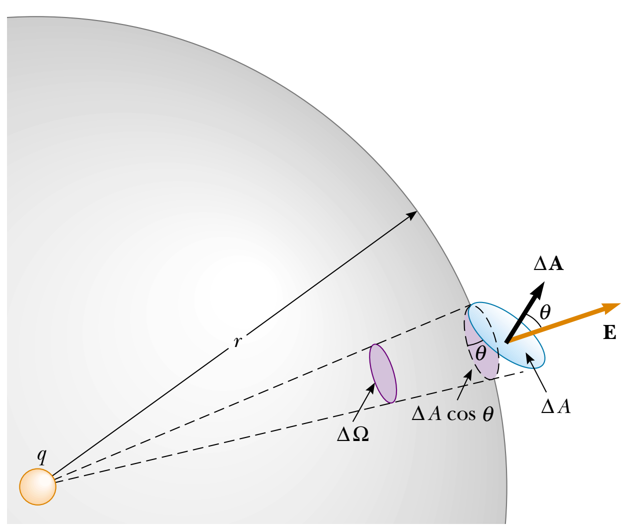


Figure 24.22 The area element ΔA subtends a solid angle $\Delta\Omega = (\Delta A \cos\theta)/r^2$ at the charge q.

through the closed surface is

$$\Phi_E = k_e q \oint \frac{dA \cos\theta}{r^2} = k_e q \oint d\Omega = 4\pi k_e q = \frac{q}{\epsilon_0}$$

Thus we have derived Gauss's law, Equation 24.6. Note that this result is independent of the shape of the closed surface and independent of the position of the charge within the surface.

SUMMARY

Electric flux is proportional to the number of electric field lines that penetrate a surface. If the electric field is uniform and makes an angle θ with the normal to a surface of area A, the electric flux through the surface is

$$\Phi_E = EA\cos\theta \qquad \textbf{(24.2)}$$

In general, the electric flux through a surface is

$$\Phi_E = \int_{\text{surface}} \mathbf{E} \cdot d\mathbf{A} \qquad \textbf{(24.3)}$$

You need to be able to apply Equations 24.2 and 24.3 in a variety of situations, particularly those in which symmetry simplifies the calculation.

Gauss's law says that the net electric flux Φ_E through any closed gaussian surface is equal to the *net* charge inside the surface divided by ϵ_0:

$$\Phi_E = \oint \mathbf{E} \cdot d\mathbf{A} = \frac{q_{\text{in}}}{\epsilon_0} \qquad \textbf{(24.6)}$$

Using Gauss's law, you can calculate the electric field due to various symmetric charge distributions. Table 24.1 lists some typical results.

TABLE 24.1 Typical Electric Field Calculations Using Gauss's Law

Charge Distribution	Electric Field	Location
Insulating sphere of radius R, uniform charge density, and total charge Q	$k_e \dfrac{Q}{r^2}$	$r > R$
	$k_e \dfrac{Q}{R^3} r$	$r < R$
Thin spherical shell of radius R and total charge Q	$k_e \dfrac{Q}{r^2}$	$r > R$
	0	$r < R$
Line charge of infinite length and charge per unit length λ	$2k_e \dfrac{\lambda}{r}$	Outside the line
Nonconducting, infinite charged plane having surface charge density σ	$\dfrac{\sigma}{2\epsilon_0}$	Everywhere outside the plane
Conductor having surface charge density σ	$\dfrac{\sigma}{\epsilon_0}$	Just outside the conductor
	0	Inside the conductor

A conductor in **electrostatic equilibrium** has the following properties:

1. The electric field is zero everywhere inside the conductor.
2. Any net charge on the conductor resides entirely on its surface.
3. The electric field just outside the conductor is perpendicular to its surface and has a magnitude σ/ϵ_0, where σ is the surface charge density at that point.
4. On an irregularly shaped conductor, the surface charge density is greatest where the radius of curvature of the surface is the smallest.

Problem-Solving Hints

Gauss's law, as we have seen, is very powerful in solving problems involving highly symmetric charge distributions. In this chapter, you encountered three kinds of symmetry: planar, cylindrical, and spherical. It is important to review Examples 24.4 through 24.10 and to adhere to the following procedure when using Gauss's law:

- Select a gaussian surface that has a symmetry to match that of the charge distribution and satisfies one or more of the conditions listed in Section 24.3. For point charges or spherically symmetric charge distributions, the gaussian surface should be a sphere centered on the charge as in Examples 24.4, 24.5, 24.6, and 24.10. For uniform line charges or uniformly charged cylinders, your gaussian surface should be a cylindrical surface that is coaxial with the line charge or cylinder as in Example 24.7. For planes of charge, a useful choice is a cylindrical gaussian surface that straddles the plane, as shown in Example 24.8. These choices enable you to simplify the surface integral that appears in Gauss's law and represents the total electric flux through that surface.
- Evaluate the q_{in}/ϵ_0 term in Gauss's law, which amounts to calculating the total electric charge q_{in} inside the gaussian surface. If the charge density is uniform (that is, if λ, σ, or ρ is constant), simply multiply that charge density by the length, area, or volume enclosed by the gaussian surface. If the charge distribution is *nonuniform,* integrate the charge density over the region enclosed by the gaussian surface. For example, if the charge is distributed along a line, integrate the expression $dq = \lambda\, dx$, where dq is the charge on an infinitesimal length element dx. For a plane of charge, integrate $dq = \sigma\, dA$, where dA is an infinitesimal element of area. For a volume of charge, integrate $dq = \rho\, dV$, where dV is an infinitesimal element of volume.
- Once the terms in Gauss's law have been evaluated, solve for the electric field on the gaussian surface if the charge distribution is given in the problem. Conversely, if the electric field is known, calculate the charge distribution that produces the field.

QUESTIONS

1. The Sun is lower in the sky during the winter than it is in the summer. How does this change the flux of sunlight hitting a given area on the surface of the Earth? How does this affect the weather?

2. If the electric field in a region of space is zero, can you conclude no electric charges are in that region? Explain.

3. If more electric field lines are leaving a gaussian surface than entering, what can you conclude about the net charge enclosed by that surface?

4. A uniform electric field exists in a region of space in which there are no charges. What can you conclude about the net electric flux through a gaussian surface placed in this region of space?

5. If the total charge inside a closed surface is known but the distribution of the charge is unspecified, can you use Gauss's law to find the electric field? Explain.
6. Explain why the electric flux through a closed surface with a given enclosed charge is independent of the size or shape of the surface.
7. Consider the electric field due to a nonconducting infinite plane having a uniform charge density. Explain why the electric field does not depend on the distance from the plane in terms of the spacing of the electric field lines.
8. Use Gauss's law to explain why electric field lines must begin or end on electric charges. (*Hint:* Change the size of the gaussian surface.)
9. On the basis of the repulsive nature of the force between like charges and the freedom of motion of charge within the conductor, explain why excess charge on an isolated conductor must reside on its surface.
10. A person is placed in a large, hollow metallic sphere that is insulated from ground. If a large charge is placed on the sphere, will the person be harmed upon touching the inside of the sphere? Explain what will happen if the person also has an initial charge whose sign is opposite that of the charge on the sphere.
11. How would the observations described in Figure 24.20 differ if the hollow conductor were grounded? How would they differ if the small charged ball were an insulator rather than a conductor?
12. What other experiment might be performed on the ball in Figure 24.20 to show that its charge was transferred to the hollow conductor?
13. What would happen to the electrometer reading if the charged ball in Figure 24.20 touched the inner wall of the conductor? the outer wall?
14. You may have heard that one of the safer places to be during a lightning storm is inside a car. Why would this be the case?
15. Two solid spheres, both of radius R, carry identical total charges Q. One sphere is a good conductor, while the other is an insulator. If the charge on the insulating sphere is uniformly distributed throughout its interior volume, how do the electric fields outside these two spheres compare? Are the fields identical inside the two spheres?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 24.1 Electric Flux

1. An electric field with a magnitude of 3.50 kN/C is applied along the x axis. Calculate the electric flux through a rectangular plane 0.350 m wide and 0.700 m long if (a) the plane is parallel to the yz plane; (b) the plane is parallel to the xy plane; and (c) the plane contains the y axis, and its normal makes an angle of 40.0° with the x axis.
2. A vertical electric field of magnitude 2.00×10^4 N/C exists above the Earth's surface on a day when a thunderstorm is brewing. A car with a rectangular size of approximately 6.00 m by 3.00 m is traveling along a roadway sloping downward at 10.0°. Determine the electric flux through the bottom of the car.
3. A 40.0-cm-diameter loop is rotated in a uniform electric field until the position of maximum electric flux is found. The flux in this position is measured to be 5.20×10^5 N·m²/C. What is the magnitude of the electric field?
4. A spherical shell is placed in a uniform electric field. Find the total electric flux through the shell.
5. Consider a closed triangular box resting within a horizontal electric field of magnitude $E = 7.80 \times 10^4$ N/C, as shown in Figure P24.5. Calculate the electric flux through (a) the vertical rectangular surface, (b) the slanted surface, and (c) the entire surface of the box.

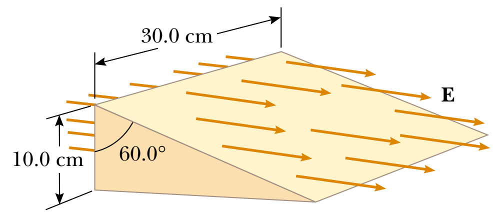


Figure P24.5

6. A uniform electric field $a\mathbf{i} + b\mathbf{j}$ intersects a surface of area A. What is the flux through this area if the surface lies (a) in the yz plane? (b) in the xz plane? (c) in the xy plane?
7. A point charge q is located at the center of a uniform ring having linear charge density λ and radius a, as shown in Figure P24.7. Determine the total electric flux

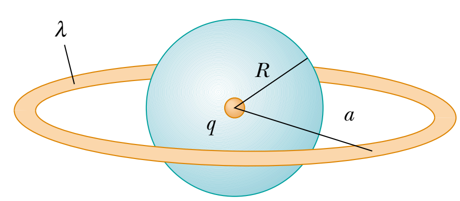


Figure P24.7

through a sphere centered at the point charge and having radius R, where $R < a$.

8. A pyramid with a 6.00-m-square base and height of 4.00 m is placed in a vertical electric field of 52.0 N/C. Calculate the total electric flux through the pyramid's four slanted surfaces.

9. A cone with base radius R and height h is located on a horizontal table. A horizontal uniform field E penetrates the cone, as shown in Figure P24.9. Determine the electric flux that enters the left-hand side of the cone.

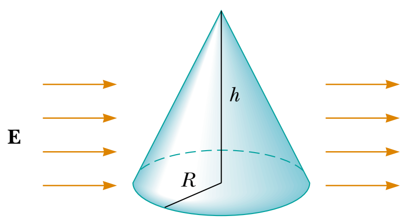


Figure P24.9

Section 24.2 Gauss's Law

10. The electric field everywhere on the surface of a thin spherical shell of radius 0.750 m is measured to be equal to 890 N/C and points radially toward the center of the sphere. (a) What is the net charge within the sphere's surface? (b) What can you conclude about the nature and distribution of the charge inside the spherical shell?

11. The following charges are located inside a submarine: 5.00 μC, -9.00 μC, 27.0 μC, and -84.0 μC. (a) Calculate the net electric flux through the submarine. (b) Is the number of electric field lines leaving the submarine greater than, equal to, or less than the number entering it?

12. Four closed surfaces, S_1 through S_4, together with the charges $-2Q$, Q, and $-Q$ are sketched in Figure P24.12. Find the electric flux through each surface.

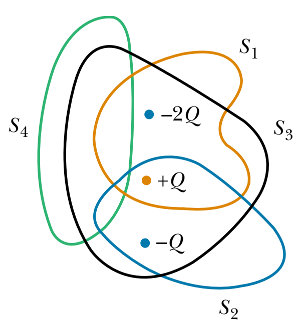


Figure P24.12

13. (a) A point charge q is located a distance d from an infinite plane. Determine the electric flux through the plane due to the point charge. (b) A point charge q is located a *very small* distance from the center of a *very large* square on the line perpendicular to the square and going through its center. Determine the approximate electric flux through the square due to the point charge. (c) Explain why the answers to parts (a) and (b) are identical.

14. Calculate the total electric flux through the paraboloidal surface due to a constant electric field of magnitude E_0 in the direction shown in Figure P24.14.

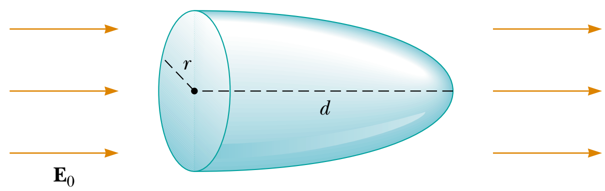


Figure P24.14

WEB **15.** A point charge Q is located just above the center of the flat face of a hemisphere of radius R, as shown in Figure P24.15. What is the electric flux (a) through the curved surface and (b) through the flat face?

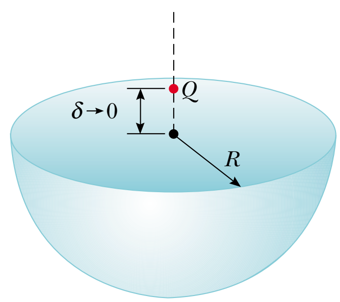


Figure P24.15

16. A point charge of 12.0 μC is placed at the center of a spherical shell of radius 22.0 cm. What is the total electric flux through (a) the surface of the shell and (b) any hemispherical surface of the shell? (c) Do the results depend on the radius? Explain.

17. A point charge of 0.046 2 μC is inside a pyramid. Determine the total electric flux through the surface of the pyramid.

18. An infinitely long line charge having a uniform charge per unit length λ lies a distance d from point O, as shown in Figure P24.18. Determine the total electric flux through the surface of a sphere of radius R centered at O resulting from this line charge. (*Hint:* Consider both cases: when $R < d$, and when $R > d$.)

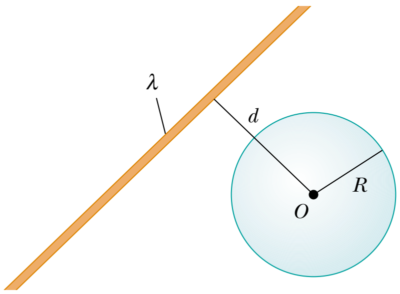


Figure P24.18

19. A point charge $Q = 5.00\ \mu C$ is located at the center of a cube of side $L = 0.100$ m. In addition, six other identical point charges having $q = -1.00\ \mu C$ are positioned symmetrically around Q, as shown in Figure P24.19. Determine the electric flux through one face of the cube.

20. A point charge Q is located at the center of a cube of side L. In addition, six other identical negative point charges are positioned symmetrically around Q, as shown in Figure P24.19. Determine the electric flux through one face of the cube.

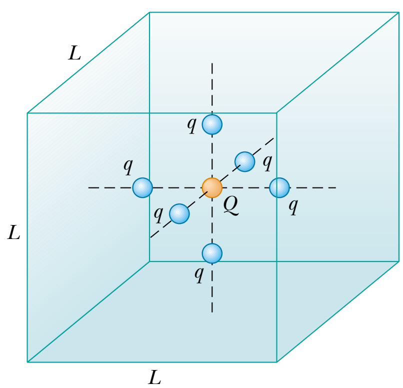


Figure P24.19 Problems 19 and 20.

21. Consider an infinitely long line charge having uniform charge per unit length λ. Determine the total electric flux through a closed right circular cylinder of length L and radius R that is parallel to the line charge, if the distance between the axis of the cylinder and the line charge is d. (*Hint:* Consider both cases: when $R < d$, and when $R > d$.)

22. A 10.0-μC charge located at the origin of a cartesian coordinate system is surrounded by a nonconducting hollow sphere of radius 10.0 cm. A drill with a radius of 1.00 mm is aligned along the z axis, and a hole is drilled in the sphere. Calculate the electric flux through the hole.

23. A charge of 170 μC is at the center of a cube of side 80.0 cm. (a) Find the total flux through each face of the cube. (b) Find the flux through the whole surface of the cube. (c) Would your answers to parts (a) or (b) change if the charge were not at the center? Explain.

24. The total electric flux through a closed surface in the shape of a cylinder is $8.60 \times 10^4\ N \cdot m^2/C$. (a) What is the net charge within the cylinder? (b) From the information given, what can you say about the charge within the cylinder? (c) How would your answers to parts (a) and (b) change if the net flux were $-8.60 \times 10^4\ N \cdot m^2/C$?

25. The line ag is a diagonal of a cube (Fig. P24.25). A point charge q is located on the extension of line ag, very close to vertex a of the cube. Determine the electric flux through each of the sides of the cube that meet at the point a.

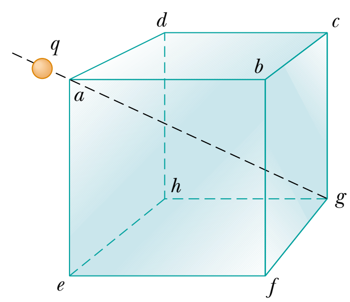


Figure P24.25

Section 24.3 Application of Gauss's Law to Charged Insulators

26. Determine the magnitude of the electric field at the surface of a lead-208 nucleus, which contains 82 protons and 126 neutrons. Assume that the lead nucleus has a volume 208 times that of one proton, and consider a proton to be a sphere of radius 1.20×10^{-15} m.

27. A solid sphere of radius 40.0 cm has a total positive charge of 26.0 μC uniformly distributed throughout its volume. Calculate the magnitude of the electric field (a) 0 cm, (b) 10.0 cm, (c) 40.0 cm, and (d) 60.0 cm from the center of the sphere.

28. A cylindrical shell of radius 7.00 cm and length 240 cm has its charge uniformly distributed on its curved surface. The magnitude of the electric field at a point 19.0 cm radially outward from its axis (measured from the midpoint of the shell) is 36.0 kN/C. Use approximate relationships to find (a) the net charge on the shell and (b) the electric field at a point 4.00 cm from the axis, measured radially outward from the midpoint of the shell.

WEB **29.** Consider a long cylindrical charge distribution of radius R with a uniform charge density ρ. Find the electric field at distance r from the axis where $r < R$.

30. A nonconducting wall carries a uniform charge density of 8.60 μC/cm^2. What is the electric field 7.00 cm in front of the wall? Does your result change as the distance from the wall is varied?

31. Consider a thin spherical shell of radius 14.0 cm with a total charge of 32.0 μC distributed uniformly on its surface. Find the electric field (a) 10.0 cm and (b) 20.0 cm from the center of the charge distribution.

32. In nuclear fission, a nucleus of uranium-238, which contains 92 protons, divides into two smaller spheres, each having 46 protons and a radius of 5.90×10^{-15} m. What is the magnitude of the repulsive electric force pushing the two spheres apart?

33. Fill two rubber balloons with air. Suspend both of them from the same point on strings of equal length. Rub each with wool or your hair, so that they hang apart with a noticeable separation between them. Make order-of-magnitude estimates of (a) the force on each, (b) the charge on each, (c) the field each creates at the center of the other, and (d) the total flux of electric field created by each balloon. In your solution, state the quantities you take as data and the values you measure or estimate for them.

34. An insulating sphere is 8.00 cm in diameter and carries a 5.70-μC charge uniformly distributed throughout its interior volume. Calculate the charge enclosed by a concentric spherical surface with radius (a) $r = 2.00$ cm and (b) $r = 6.00$ cm.

35. A uniformly charged, straight filament 7.00 m in length has a total positive charge of 2.00 μC. An uncharged cardboard cylinder 2.00 cm in length and 10.0 cm in radius surrounds the filament at its center, with the filament as the axis of the cylinder. Using reasonable approximations, find (a) the electric field at the surface of the cylinder and (b) the total electric flux through the cylinder.

36. The charge per unit length on a long, straight filament is -90.0 μC/m. Find the electric field (a) 10.0 cm, (b) 20.0 cm, and (c) 100 cm from the filament, where distances are measured perpendicular to the length of the filament.

37. A large flat sheet of charge has a charge per unit area of 9.00 μC/m^2. Find the electric field just above the surface of the sheet, measured from its midpoint.

Section 24.4 Conductors in Electrostatic Equilibrium

38. On a clear, sunny day, a vertical electrical field of about 130 N/C points down over flat ground. What is the surface charge density on the ground for these conditions?

39. A long, straight metal rod has a radius of 5.00 cm and a charge per unit length of 30.0 nC/m. Find the electric field (a) 3.00 cm, (b) 10.0 cm, and (c) 100 cm from the axis of the rod, where distances are measured perpendicular to the rod.

40. A very large, thin, flat plate of aluminum of area A has a total charge Q uniformly distributed over its surfaces. If the same charge is spread uniformly over the *upper* surface of an otherwise identical glass plate, compare the electric fields just above the center of the upper surface of each plate.

41. A square plate of copper with 50.0-cm sides has no net charge and is placed in a region of uniform electric field of 80.0 kN/C directed perpendicularly to the plate. Find (a) the charge density of each face of the plate and (b) the total charge on each face.

42. A hollow conducting sphere is surrounded by a larger concentric, spherical, conducting shell. The inner sphere has a charge $-Q$, and the outer sphere has a charge $3Q$. The charges are in electrostatic equilibrium. Using Gauss's law, find the charges and the electric fields everywhere.

43. Two identical conducting spheres each having a radius of 0.500 cm are connected by a light 2.00-m-long conducting wire. Determine the tension in the wire if 60.0 μC is placed on one of the conductors. (*Hint:* Assume that the surface distribution of charge on each sphere is uniform.)

44. The electric field on the surface of an irregularly shaped conductor varies from 56.0 kN/C to 28.0 kN/C. Calculate the local surface charge density at the point on the surface where the radius of curvature of the surface is (a) greatest and (b) smallest.

45. A long, straight wire is surrounded by a hollow metal cylinder whose axis coincides with that of the wire. The wire has a charge per unit length of λ, and the cylinder has a net charge per unit length of 2λ. From this information, use Gauss's law to find (a) the charge per unit length on the inner and outer surfaces of the cylinder and (b) the electric field outside the cylinder, a distance r from the axis.

46. A conducting spherical shell of radius 15.0 cm carries a net charge of -6.40 μC uniformly distributed on its surface. Find the electric field at points (a) just outside the shell and (b) inside the shell.

WEB **47.** A thin conducting plate 50.0 cm on a side lies in the xy plane. If a total charge of 4.00×10^{-8} C is placed on the plate, find (a) the charge density on the plate, (b) the electric field just above the plate, and (c) the electric field just below the plate.

48. A conducting spherical shell having an inner radius of a and an outer radius of b carries a net charge Q. If a point charge q is placed at the center of this shell, determine the surface charge density on (a) the inner surface of the shell and (b) the outer surface of the shell.

49. A solid conducting sphere of radius 2.00 cm has a charge 8.00 μC. A conducting spherical shell of inner radius 4.00 cm and outer radius 5.00 cm is concentric with the solid sphere and has a charge -4.00 μC. Find the electric field at (a) $r = 1.00$ cm, (b) $r = 3.00$ cm, (c) $r = 4.50$ cm, and (d) $r = 7.00$ cm from the center of this charge configuration.

50. A positive point charge is at a distance of $R/2$ from the center of an uncharged thin conducting spherical shell of radius R. Sketch the electric field lines set up by this arrangement both inside and outside the shell.

(Optional)
Section 24.5 Experimental Verification of Gauss's Law and Coulomb's Law

Section 24.6 Formal Derivation of Gauss's Law

51. A sphere of radius R surrounds a point charge Q, located at its center. (a) Show that the electric flux through a circular cap of half-angle θ (Fig. P24.51) is

$$\Phi_E = \frac{Q}{2\epsilon_0}(1 - \cos\theta)$$

What is the flux for (b) $\theta = 90°$ and (c) $\theta = 180°$?

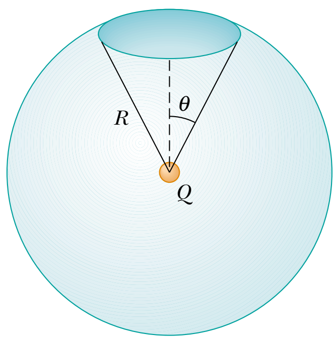


Figure P24.51

ADDITIONAL PROBLEMS

52. A nonuniform electric field is given by the expression $\mathbf{E} = ay\mathbf{i} + bz\mathbf{j} + cx\mathbf{k}$, where a, b, and c are constants. Determine the electric flux through a rectangular surface in the xy plane, extending from $x = 0$ to $x = w$ and from $y = 0$ to $y = h$.

53. A solid insulating sphere of radius a carries a net positive charge $3Q$, uniformly distributed throughout its volume. Concentric with this sphere is a conducting spherical shell with inner radius b and outer radius c, and having a net charge $-Q$, as shown in Figure P24.53. (a) Construct a spherical gaussian surface of radius $r > c$ and find the net charge enclosed by this surface. (b) What is the direction of the electric field at $r > c$? (c) Find the electric field at $r > c$. (d) Find the electric field in the region with radius r where $c > r > b$. (e) Construct a spherical gaussian surface of radius r, where $c > r > b$, and find the net charge enclosed by this surface. (f) Construct a spherical gaussian surface of radius r, where $b > r > a$, and find the net charge enclosed by this surface. (g) Find the electric field in the region $b > r > a$. (h) Construct a spherical gaussian surface of radius $r < a$, and find an expression for the net charge enclosed by this surface, as a function of r. Note that the charge inside this surface is less than $3Q$. (i) Find the electric field in the region $r < a$. (j) Determine the charge on the inner surface of the conducting shell. (k) Determine the charge on the outer surface of the conducting shell. (l) Make a plot of the magnitude of the electric field versus r.

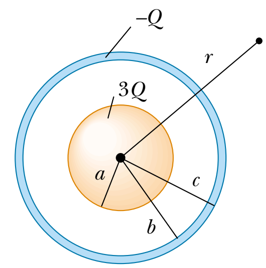


Figure P24.53

54. Consider two identical conducting spheres whose surfaces are separated by a small distance. One sphere is given a large net positive charge, while the other is given a small net positive charge. It is found that the force between them is attractive even though both spheres have net charges of the same sign. Explain how this is possible.

WEB **55.** A solid, insulating sphere of radius a has a uniform charge density ρ and a total charge Q. Concentric with this sphere is an uncharged, conducting hollow sphere whose inner and outer radii are b and c, as shown in Figure P24.55. (a) Find the magnitude of the electric field in the regions $r < a$, $a < r < b$, $b < r < c$, and $r > c$. (b) Determine the induced charge per unit area on the inner and outer surfaces of the hollow sphere.

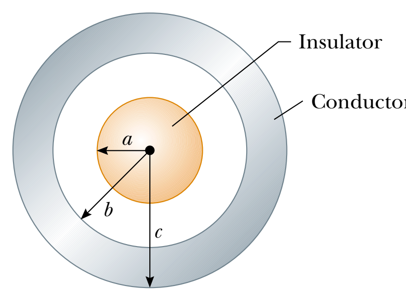


Figure P24.55 Problems 55 and 56.

56. For the configuration shown in Figure P24.55, suppose that $a = 5.00$ cm, $b = 20.0$ cm, and $c = 25.0$ cm. Furthermore, suppose that the electric field at a point 10.0 cm from the center is 3.60×10^3 N/C radially inward, while the electric field at a point 50.0 cm from the center is 2.00×10^2 N/C radially outward. From this information, find (a) the charge on the insulating sphere,

(b) the net charge on the hollow conducting sphere, and (c) the total charge on the inner and outer surfaces of the hollow conducting sphere.

57. An infinitely long cylindrical insulating shell of inner radius a and outer radius b has a uniform volume charge density ρ (C/m^3). A line of charge density λ (C/m) is placed along the axis of the shell. Determine the electric field intensity everywhere.

58. Two infinite, nonconducting sheets of charge are parallel to each other, as shown in Figure P24.58. The sheet on the left has a uniform surface charge density σ, and the one on the right has a uniform charge density $-\sigma$. Calculate the value of the electric field at points (a) to the left of, (b) in between, and (c) to the right of the two sheets. (*Hint:* See Example 24.8.)

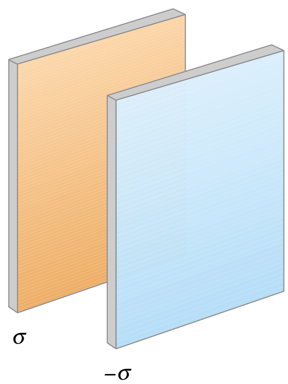


Figure P24.58

WEB **59.** Repeat the calculations for Problem 58 when both sheets have *positive* uniform surface charge densities of value σ.

60. A sphere of radius $2a$ is made of a nonconducting material that has a uniform volume charge density ρ. (Assume that the material does not affect the electric field.) A spherical cavity of radius a is now removed from the sphere, as shown in Figure P24.60. Show that the electric field within the cavity is uniform and is given by $E_x = 0$ and $E_y = \rho a/3\epsilon_0$. (*Hint:* The field within the cavity is the superposition of the field due to the original uncut sphere, plus the field due to a sphere the size of the cavity with a uniform negative charge density $-\rho$.)

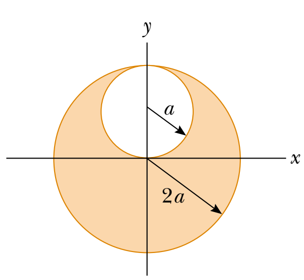


Figure P24.60

61. Review Problem. An early (incorrect) model of the hydrogen atom, suggested by J. J. Thomson, proposed that a positive cloud of charge $+e$ was uniformly distributed throughout the volume of a sphere of radius R, with the electron an equal-magnitude negative point charge $-e$ at the center. (a) Using Gauss's law, show that the electron would be in equilibrium at the center and, if displaced from the center a distance $r < R$, would experience a restoring force of the form $F = -Kr$, where K is a constant. (b) Show that $K = k_e e^2/R^3$. (c) Find an expression for the frequency f of simple harmonic oscillations that an electron of mass m_e would undergo if displaced a short distance ($< R$) from the center and released. (d) Calculate a numerical value for R that would result in a frequency of electron vibration of 2.47×10^{15} Hz, the frequency of the light in the most intense line in the hydrogen spectrum.

62. A closed surface with dimensions $a = b = 0.400$ m and $c = 0.600$ m is located as shown in Figure P24.62. The electric field throughout the region is nonuniform and given by $\mathbf{E} = (3.0 + 2.0x^2)\,\mathbf{i}$ N/C, where x is in meters. Calculate the net electric flux leaving the closed surface. What net charge is enclosed by the surface?

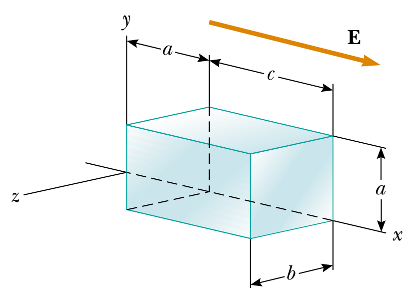


Figure P24.62

63. A solid insulating sphere of radius R has a nonuniform charge density that varies with r according to the expression $\rho = Ar^2$, where A is a constant and $r < R$ is measured from the center of the sphere. (a) Show that the electric field outside ($r > R$) the sphere is $E = AR^5/5\epsilon_0 r^2$. (b) Show that the electric field inside ($r < R$) the sphere is $E = Ar^3/5\epsilon_0$. (*Hint:* Note that the total charge Q on the sphere is equal to the integral of $\rho\, dV$, where r extends from 0 to R; also note that the charge q within a radius $r < R$ is less than Q. To evaluate the integrals, note that the volume element dV for a spherical shell of radius r and thickness dr is equal to $4\pi r^2\, dr$.)

64. A point charge Q is located on the axis of a disk of radius R at a distance b from the plane of the disk (Fig. P24.64). Show that if one fourth of the electric flux from the charge passes through the disk, then $R = \sqrt{3}b$.

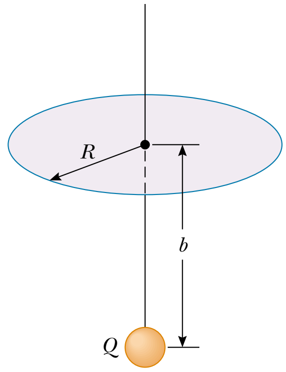


Figure P24.64

65. A spherically symmetric charge distribution has a charge density given by $\rho = a/r$, where a is constant. Find the electric field as a function of r. (*Hint:* Note that the charge within a sphere of radius R is equal to the integral of $\rho\, dV$, where r extends from 0 to R. To evaluate the integral, note that the volume element dV for a spherical shell of radius r and thickness dr is equal to $4\pi r^2\, dr$.)

66. An infinitely long insulating cylinder of radius R has a volume charge density that varies with the radius as

$$\rho = \rho_0\left(a - \frac{r}{b}\right)$$

where ρ_0, a, and b are positive constants and r is the distance from the axis of the cylinder. Use Gauss's law to determine the magnitude of the electric field at radial distances (a) $r < R$ and (b) $r > R$.

67. **Review Problem.** A slab of insulating material (infinite in two of its three dimensions) has a uniform positive charge density ρ. An edge view of the slab is shown in Figure P24.67. (a) Show that the magnitude of the electric field a distance x from its center and inside the slab is $E = \rho x/\epsilon_0$. (b) Suppose that an electron of charge $-e$ and mass m_e is placed inside the slab. If it is released from rest at a distance x from the center, show that the electron exhibits simple harmonic motion with a frequency described by the expression

$$f = \frac{1}{2\pi}\sqrt{\frac{\rho e}{m_e \epsilon_0}}$$

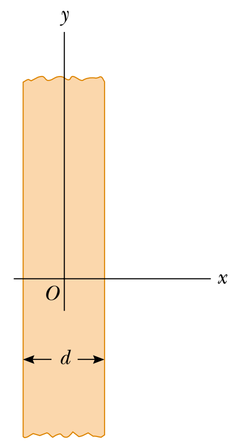


Figure P24.67 Problems 67 and 68.

68. A slab of insulating material has a nonuniform positive charge density $\rho = Cx^2$, where x is measured from the center of the slab, as shown in Figure P24.67, and C is a constant. The slab is infinite in the y and z directions. Derive expressions for the electric field in (a) the exterior regions and (b) the interior region of the slab $(-d/2 < x < d/2)$.

69. (a) Using the mathematical similarity between Coulomb's law and Newton's law of universal gravitation, show that Gauss's law for gravitation can be written as

$$\oint \mathbf{g} \cdot d\mathbf{A} = -4\pi G m_{\text{in}}$$

where m_{in} is the mass inside the gaussian surface and $\mathbf{g} = \mathbf{F}_g/m$ represents the gravitational field at any point on the gaussian surface. (b) Determine the gravitational field at a distance r from the center of the Earth where $r < R_E$, assuming that the Earth's mass density is uniform.

Answers to Quick Quizzes

24.1 Zero, because there is no net charge within the surface.

24.2 (b) and (d). Statement (a) is not necessarily true because an equal number of positive and negative charges could be present inside the surface. Statement (c) is not necessarily true, as can be seen from Figure 24.8: A nonzero electric field exists everywhere on the surface, but the charge is not enclosed within the surface; thus, the net flux is zero.

24.3 Any gaussian surface surrounding the system encloses the same amount of charge, regardless of how the components of the system are moved. Thus, the flux through the gaussian surface would be the same as it is when the sphere and shell are concentric.

PUZZLER

Jennifer is holding on to an electrically charged sphere that reaches an electric potential of about 100 000 V. The device that generates this high electric potential is called a *Van de Graaff generator.* What causes Jennifer's hair to stand on end like the needles of a porcupine? Why is she safe in this situation in view of the fact that 110 V from a wall outlet can kill you? *(Henry Leap and Jim Lehman)*

chapter

25 Electric Potential

Chapter Outline

The concept of potential energy was introduced in Chapter 8 in connection with such conservative forces as the force of gravity and the elastic force exerted by a spring. By using the law of conservation of energy, we were able to avoid working directly with forces when solving various problems in mechanics. In this chapter we see that the concept of potential energy is also of great value in the study of electricity. Because the electrostatic force given by Coulomb's law is conservative, electrostatic phenomena can be conveniently described in terms of an electric potential energy. This idea enables us to define a scalar quantity known as *electric potential.* Because the electric potential at any point in an electric field is a scalar function, we can use it to describe electrostatic phenomena more simply than if we were to rely only on the concepts of the electric field and electric forces. In later chapters we shall see that the concept of electric potential is of great practical value.

25.1 POTENTIAL DIFFERENCE AND ELECTRIC POTENTIAL

11.8 When a test charge q_0 is placed in an electric field $\mathbf{E}$ created by some other charged object, the electric force acting on the test charge is $q_0\mathbf{E}$. (If the field is produced by more than one charged object, this force acting on the test charge is the vector sum of the individual forces exerted on it by the various other charged objects.) The force $q_0\mathbf{E}$ is conservative because the individual forces described by Coulomb's law are conservative. When the test charge is moved in the field by some external agent, the work done by the field on the charge is equal to the negative of the work done by the external agent causing the displacement. For an infinitesimal displacement $d\mathbf{s}$, the work done by the electric field on the charge is $\mathbf{F}\cdot d\mathbf{s} = q_0\mathbf{E}\cdot d\mathbf{s}$. As this amount of work is done by the field, the potential energy of the charge–field system is decreased by an amount $dU = -q_0\mathbf{E}\cdot d\mathbf{s}$. For a finite displacement of the charge from a point A to a point B, the change in potential energy of the system $\Delta U = U_B - U_A$ is

$$\Delta U = -q_0\int_A^B \mathbf{E}\cdot d\mathbf{s} \qquad \textbf{(25.1)}$$

Change in potential energy

The integration is performed along the path that q_0 follows as it moves from A to B, and the integral is called either a *path integral* or a *line integral* (the two terms are synonymous). Because the force $q_0\mathbf{E}$ is conservative, **this line integral does not depend on the path taken from A to B.**

Quick Quiz 25.1

If the path between A and B does not make any difference in Equation 25.1, why don't we just use the expression $\Delta U = -q_0 Ed$, where d is the straight-line distance between A and B?

The potential energy per unit charge U/q_0 is independent of the value of q_0 and has a unique value at every point in an electric field. This quantity U/q_0 is called the **electric potential** (or simply the **potential**) V. Thus, the electric potential at any point in an electric field is

$$V = \frac{U}{q_0} \qquad \textbf{(25.2)}$$

The fact that potential energy is a scalar quantity means that electric potential also is a scalar quantity.

The **potential difference** $\Delta V = V_B - V_A$ between any two points A and B in an electric field is defined as the change in potential energy of the system divided by the test charge q_0:

Potential difference

$$\Delta V = \frac{\Delta U}{q_0} = -\int_A^B \mathbf{E} \cdot d\mathbf{s} \tag{25.3}$$

Potential difference should not be confused with difference in potential energy. The potential difference is proportional to the change in potential energy, and we see from Equation 25.3 that the two are related by $\Delta U = q_0 \Delta V$.

Electric potential is a scalar characteristic of an electric field, independent of the charges that may be placed in the field. However, when we speak of potential energy, we are referring to the charge–field system. Because we are usually interested in knowing the electric potential at the location of a charge and the potential energy resulting from the interaction of the charge with the field, we follow the common convention of speaking of the potential energy as if it belonged to the charge.

Because the change in potential energy of a charge is the negative of the work done by the electric field on the charge (as noted in Equation 25.1), the potential difference ΔV between points A and B equals the work per unit charge that an external agent must perform to move a test charge from A to B without changing the kinetic energy of the test charge.

Just as with potential energy, only *differences* in electric potential are meaningful. To avoid having to work with potential differences, however, we often take the value of the electric potential to be zero at some convenient point in an electric field. This is what we do here: arbitrarily establish the electric potential to be zero at a point that is infinitely remote from the charges producing the field. Having made this choice, we can state that the **electric potential at an arbitrary point in an electric field equals the work required per unit charge to bring a positive test charge from infinity to that point.** Thus, if we take point A in Equation 25.3 to be at infinity, the electric potential at any point P is

$$V_P = -\int_\infty^P \mathbf{E} \cdot d\mathbf{s} \tag{25.4}$$

In reality, V_P represents the potential difference ΔV between the point P and a point at infinity. (Eq. 25.4 is a special case of Eq. 25.3.)

Because electric potential is a measure of potential energy per unit charge, the SI unit of both electric potential and potential difference is joules per coulomb, which is defined as a **volt** (V):

Definition of volt

$$1\ \mathrm{V} \equiv 1\ \frac{\mathrm{J}}{\mathrm{C}}$$

That is, 1 J of work must be done to move a 1-C charge through a potential difference of 1 V.

Equation 25.3 shows that potential difference also has units of electric field times distance. From this, it follows that the SI unit of electric field (N/C) can also be expressed in volts per meter:

$$1\ \frac{\mathrm{N}}{\mathrm{C}} = 1\ \frac{\mathrm{V}}{\mathrm{m}}$$

A unit of energy commonly used in atomic and nuclear physics is the **electron volt** (eV), which is defined as **the energy an electron (or proton) gains or loses by moving through a potential difference of 1 V.** Because 1 V = 1 J/C and because the fundamental charge is approximately 1.60×10^{-19} C, the electron volt is related to the joule as follows:

The electron volt

$$1 \text{ eV} = 1.60 \times 10^{-19} \text{ C}\cdot\text{V} = 1.60 \times 10^{-19} \text{ J} \tag{25.5}$$

For instance, an electron in the beam of a typical television picture tube may have a speed of 3.5×10^7 m/s. This corresponds to a kinetic energy of 5.6×10^{-16} J, which is equivalent to 3.5×10^3 eV. Such an electron has to be accelerated from rest through a potential difference of 3.5 kV to reach this speed.

25.2 POTENTIAL DIFFERENCES IN A UNIFORM ELECTRIC FIELD

Equations 25.1 and 25.3 hold in all electric fields, whether uniform or varying, but they can be simplified for a uniform field. First, consider a uniform electric field directed along the negative y axis, as shown in Figure 25.1a. Let us calculate the potential difference between two points A and B separated by a distance d, where d is measured parallel to the field lines. Equation 25.3 gives

$$V_B - V_A = \Delta V = -\int_A^B \mathbf{E}\cdot d\mathbf{s} = -\int_A^B E\cos 0°\, ds = -\int_A^B E\, ds$$

Because E is constant, we can remove it from the integral sign; this gives

Potential difference in a uniform electric field

$$\Delta V = -E\int_A^B ds = -Ed \tag{25.6}$$

The minus sign indicates that point B is at a lower electric potential than point A; that is, $V_B < V_A$. **Electric field lines always point in the direction of decreasing electric potential,** as shown in Figure 25.1a.

Now suppose that a test charge q_0 moves from A to B. We can calculate the change in its potential energy from Equations 25.3 and 25.6:

$$\Delta U = q_0\,\Delta V = -q_0 Ed \tag{25.7}$$

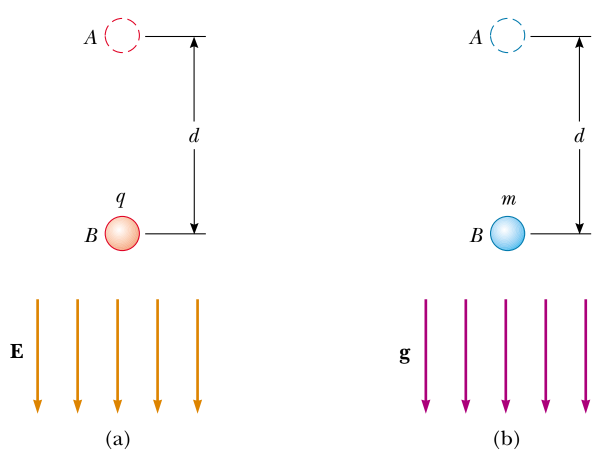


Figure 25.1 (a) When the electric field **E** is directed downward, point B is at a lower electric potential than point A. A positive test charge that moves from point A to point B loses electric potential energy. (b) A mass m moving downward in the direction of the gravitational field **g** loses gravitational potential energy.

QuickLab

It takes an electric field of about 30 000 V/cm to cause a spark in dry air. Shuffle across a rug and reach toward a doorknob. By estimating the length of the spark, determine the electric potential difference between your finger and the doorknob after shuffling your feet but before touching the knob. (If it is very humid on the day you attempt this, it may not work. Why?)

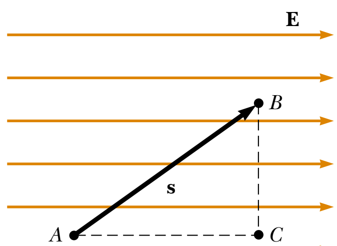


Figure 25.2 A uniform electric field directed along the positive x axis. Point B is at a lower electric potential than point A. Points B and C are at the *same* electric potential.

An equipotential surface

From this result, we see that if q_0 is positive, then ΔU is negative. We conclude that **a positive charge loses electric potential energy when it moves in the direction of the electric field.** This means that an electric field does work on a positive charge when the charge moves in the direction of the electric field. (This is analogous to the work done by the gravitational field on a falling mass, as shown in Figure 25.1b.) If a positive test charge is released from rest in this electric field, it experiences an electric force $q_0\mathbf{E}$ in the direction of $\mathbf{E}$ (downward in Fig. 25.1a). Therefore, it accelerates downward, gaining kinetic energy. **As the charged particle gains kinetic energy, it loses an equal amount of potential energy.**

If q_0 is negative, then ΔU is positive and the situation is reversed: **A negative charge gains electric potential energy when it moves in the direction of the electric field.** If a negative charge is released from rest in the field $\mathbf{E}$, it accelerates in a direction opposite the direction of the field.

Now consider the more general case of a charged particle that is free to move between any two points in a uniform electric field directed along the x axis, as shown in Figure 25.2. (In this situation, the charge is not being moved by an external agent as before.) If $\mathbf{s}$ represents the displacement vector between points A and B, Equation 25.3 gives

$$\Delta V = -\int_A^B \mathbf{E} \cdot d\mathbf{s} = -\mathbf{E} \cdot \int_A^B d\mathbf{s} = -\mathbf{E} \cdot \mathbf{s} \tag{25.8}$$

where again we are able to remove $\mathbf{E}$ from the integral because it is constant. The change in potential energy of the charge is

$$\Delta U = q_0\, \Delta V = -q_0\, \mathbf{E} \cdot \mathbf{s} \tag{25.9}$$

11.9 Finally, we conclude from Equation 25.8 that all points in a plane perpendicular to a uniform electric field are at the same electric potential. We can see this in Figure 25.2, where the potential difference $V_B - V_A$ is equal to the potential difference $V_C - V_A$. (Prove this to yourself by working out the dot product $\mathbf{E} \cdot \mathbf{s}$ for $\mathbf{s}_{A\to B}$, where the angle θ between $\mathbf{E}$ and $\mathbf{s}$ is arbitrary as shown in Figure 25.2, and the dot product for $\mathbf{s}_{A\to C}$, where $\theta = 0$.) Therefore, $V_B = V_C$. **The name equipotential surface is given to any surface consisting of a continuous distribution of points having the same electric potential.**

Note that because $\Delta U = q_0 \Delta V$, no work is done in moving a test charge between any two points on an equipotential surface. The equipotential surfaces of a uniform electric field consist of a family of planes that are all perpendicular to the field. Equipotential surfaces for fields with other symmetries are described in later sections.

Quick Quiz 25.2

The labeled points in Figure 25.3 are on a series of equipotential surfaces associated with an electric field. Rank (from greatest to least) the work done by the electric field on a positively charged particle that moves from A to B; from B to C; from C to D; from D to E.

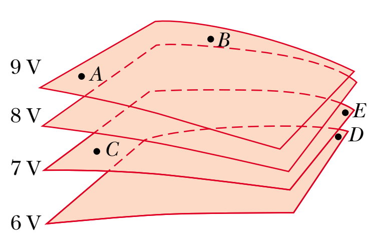


Figure 25.3 Four equipotential surfaces.

EXAMPLE 25.1 The Electric Field Between Two Parallel Plates of Opposite Charge

A battery produces a specified potential difference between conductors attached to the battery terminals. A 12-V battery is connected between two parallel plates, as shown in Figure 25.4. The separation between the plates is $d = 0.30$ cm, and we assume the electric field between the plates to be uniform. (This assumption is reasonable if the plate separation is small relative to the plate dimensions and if we do not consider points near the plate edges.) Find the magnitude of the electric field between the plates.

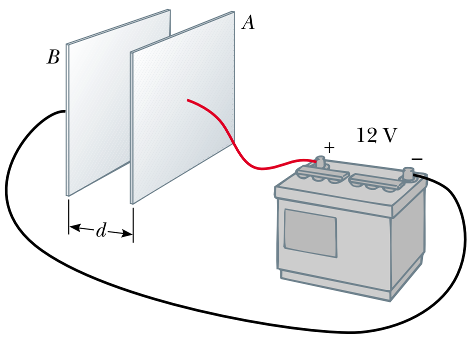


Figure 25.4 A 12-V battery connected to two parallel plates. The electric field between the plates has a magnitude given by the potential difference ΔV divided by the plate separation d.

Solution The electric field is directed from the positive plate (A) to the negative one (B), and the positive plate is at a higher electric potential than the negative plate is. The potential difference between the plates must equal the potential difference between the battery terminals. We can understand this by noting that all points on a conductor in equilibrium are at the same electric potential[1]; no potential difference exists between a terminal and any portion of the plate to which it is connected. Therefore, the magnitude of the electric field between the plates is, from Equation 25.6,

$$E = \frac{|V_B - V_A|}{d} = \frac{12\ \text{V}}{0.30 \times 10^{-2}\ \text{m}} = \boxed{4.0 \times 10^3\ \text{V/m}}$$

This configuration, which is shown in Figure 25.4 and called a *parallel-plate capacitor,* is examined in greater detail in Chapter 26.

EXAMPLE 25.2 Motion of a Proton in a Uniform Electric Field

A proton is released from rest in a uniform electric field that has a magnitude of 8.0×10^4 V/m and is directed along the positive x axis (Fig. 25.5). The proton undergoes a displacement of 0.50 m in the direction of $\mathbf{E}$. (a) Find the change in electric potential between points A and B.

Solution Because the proton (which, as you remember, carries a positive charge) moves in the direction of the field, we expect it to move to a position of lower electric potential.

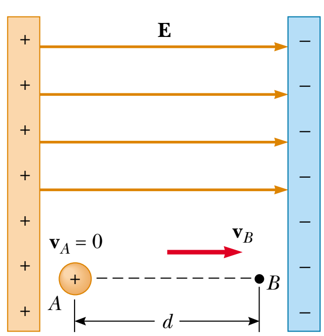


Figure 25.5 A proton accelerates from A to B in the direction of the electric field.

From Equation 25.6, we have

$$\Delta V = -Ed = -(8.0 \times 10^4\ \text{V/m})(0.50\ \text{m})$$
$$= \boxed{-4.0 \times 10^4\ \text{V}}$$

(b) Find the change in potential energy of the proton for this displacement.

Solution

$$\Delta U = q_0\,\Delta V = e\,\Delta V$$
$$= (1.6 \times 10^{-19}\ \text{C})(-4.0 \times 10^4\ \text{V})$$
$$= \boxed{-6.4 \times 10^{-15}\ \text{J}}$$

The negative sign means the potential energy of the proton decreases as it moves in the direction of the electric field. As the proton accelerates in the direction of the field, it gains kinetic energy and at the same time loses electric potential energy (because energy is conserved).

Exercise Use the concept of conservation of energy to find the speed of the proton at point B.

Answer 2.77×10^6 m/s.

[1] The electric field vanishes within a conductor in electrostatic equilibrium; thus, the path integral $\int \mathbf{E} \cdot d\mathbf{s}$ between any two points in the conductor must be zero. A more complete discussion of this point is given in Section 25.6.

25.3 ELECTRIC POTENTIAL AND POTENTIAL ENERGY DUE TO POINT CHARGES

Consider an isolated positive point charge q. Recall that such a charge produces an electric field that is directed radially outward from the charge. To find the electric potential at a point located a distance r from the charge, we begin with the general expression for potential difference:

$$V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{s}$$

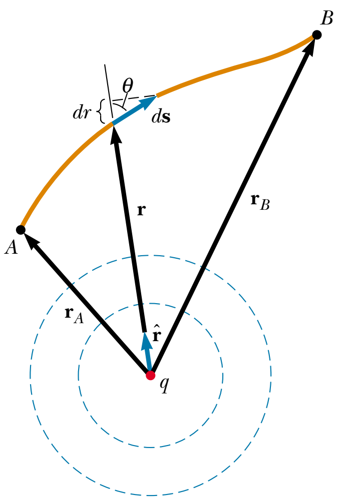


Figure 25.6 The potential difference between points A and B due to a point charge q depends *only* on the initial and final radial coordinates r_A and r_B. The two dashed circles represent cross-sections of spherical equipotential surfaces.

where A and B are the two arbitrary points shown in Figure 25.6. At any field point, the electric field due to the point charge is $\mathbf{E} = k_e q\hat{\mathbf{r}}/r^2$ (Eq. 23.4), where $\hat{\mathbf{r}}$ is a unit vector directed from the charge toward the field point. The quantity $\mathbf{E} \cdot d\mathbf{s}$ can be expressed as

$$\mathbf{E} \cdot d\mathbf{s} = k_e \frac{q}{r^2} \hat{\mathbf{r}} \cdot d\mathbf{s}$$

Because the magnitude of $\hat{\mathbf{r}}$ is 1, the dot product $\hat{\mathbf{r}} \cdot d\mathbf{s} = ds \cos\theta$, where θ is the angle between $\hat{\mathbf{r}}$ and $d\mathbf{s}$. Furthermore, $ds \cos \theta$ is the projection of $d\mathbf{s}$ onto $\mathbf{r}$; thus, $ds \cos \theta = dr$. That is, any displacement $d\mathbf{s}$ along the path from point A to point B produces a change dr in the magnitude of $\mathbf{r}$, the radial distance to the charge creating the field. Making these substitutions, we find that $\mathbf{E} \cdot d\mathbf{s} = (k_e q/r^2)\,dr$; hence, the expression for the potential difference becomes

$$V_B - V_A = -\int E_r\,dr = -k_e q \int_{r_A}^{r_B} \frac{dr}{r^2} = \frac{k_e q}{r}\Bigg]_{r_A}^{r_B}$$

$$V_B - V_A = k_e q \left[\frac{1}{r_B} - \frac{1}{r_A}\right] \tag{25.10}$$

The integral of $\mathbf{E} \cdot d\mathbf{s}$ is *independent* of the path between points A and B—as it must be because the electric field of a point charge is conservative. Furthermore, Equation 25.10 expresses the important result that the potential difference between any two points A and B in a field created by a point charge depends only on the radial coordinates r_A and r_B. It is customary to choose the reference of electric potential to be zero at $r_A = \infty$. With this reference, the electric potential created by a point charge at any distance r from the charge is

Electric potential created by a point charge

$$V = k_e \frac{q}{r} \tag{25.11}$$

Electric potential is graphed in Figure 25.7 as a function of r, the radial distance from a positive charge in the xy plane. Consider the following analogy to gravitational potential: Imagine trying to roll a marble toward the top of a hill shaped like Figure 25.7a. The gravitational force experienced by the marble is analogous to the repulsive force experienced by a positively charged object as it approaches another positively charged object. Similarly, the electric potential graph of the region surrounding a negative charge is analogous to a "hole" with respect to any approaching positively charged objects. A charged object must be infinitely distant from another charge before the surface is "flat" and has an electric potential of zero.

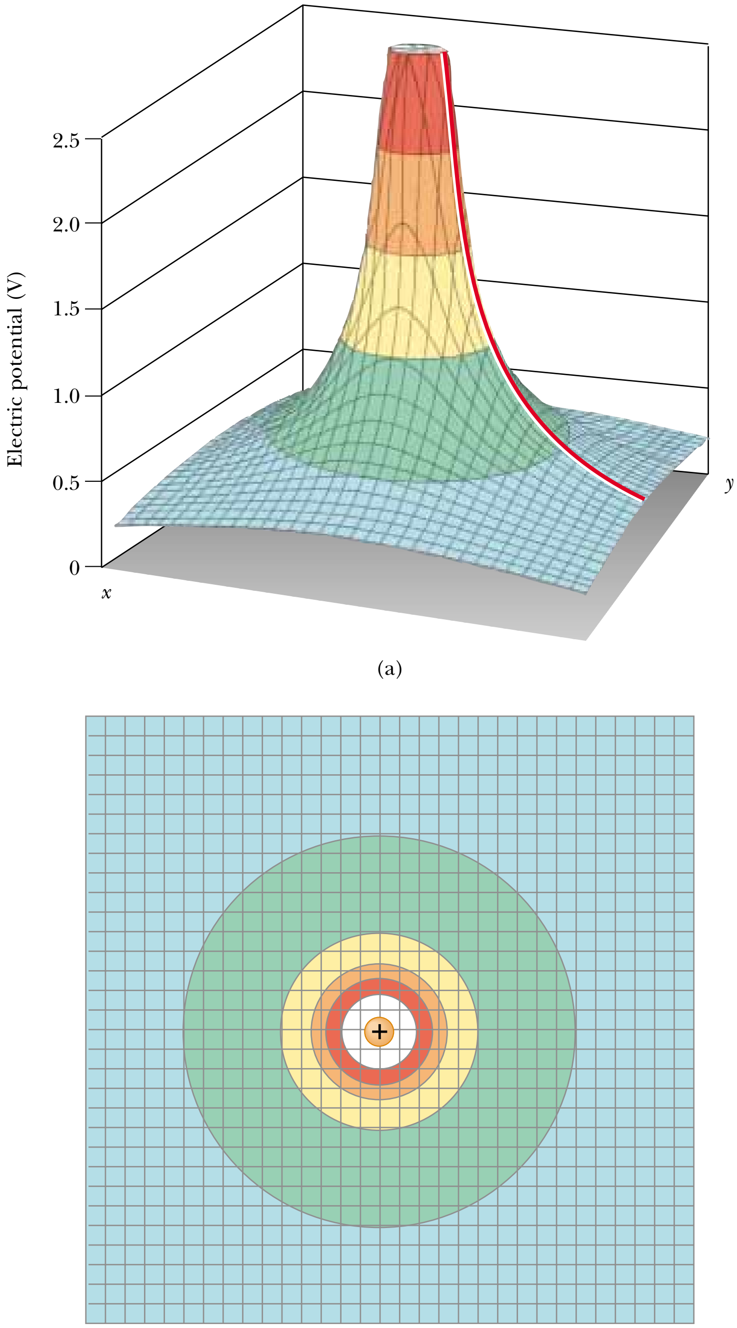


Figure 25.7 (a) The electric potential in the plane around a single positive charge is plotted on the vertical axis. (The electric potential function for a negative charge would look like a hole instead of a hill.) The red line shows the $1/r$ nature of the electric potential, as given by Equation 25.11. (b) View looking straight down the vertical axis of the graph in part (a), showing concentric circles where the electric potential is constant. These circles are cross sections of equipotential spheres having the charge at the center.

Quick Quiz 25.3

A spherical balloon contains a positively charged object at its center. As the balloon is inflated to a greater volume while the charged object remains at the center, does the electric potential at the surface of the balloon increase, decrease, or remain the same? How about the magnitude of the electric field? The electric flux?

We obtain the electric potential resulting from two or more point charges by applying the superposition principle. That is, the total electric potential at some point P due to several point charges is the sum of the potentials due to the individual charges. For a group of point charges, we can write the total electric potential at P in the form

Electric potential due to several point charges

$$V = k_e \sum_i \frac{q_i}{r_i} \tag{25.12}$$

where the potential is again taken to be zero at infinity and r_i is the distance from the point P to the charge q_i. Note that the sum in Equation 25.12 is an algebraic sum of scalars rather than a vector sum (which we use to calculate the electric field of a group of charges). Thus, it is often much easier to evaluate V than to evaluate $\mathbf{E}$. The electric potential around a dipole is illustrated in Figure 25.8.

We now consider the potential energy of a system of two charged particles. If V_1 is the electric potential at a point P due to charge q_1, then the work an external agent must do to bring a second charge q_2 from infinity to P without acceleration is q_2V_1. By definition, this work equals the potential energy U of the two-particle system when the particles are separated by a distance r_{12} (Fig. 25.9). Therefore, we can express the potential energy as[2]

Electric potential energy due to two charges

$$U = k_e \frac{q_1 q_2}{r_{12}} \tag{25.13}$$

Note that if the charges are of the same sign, U is positive. This is consistent with the fact that positive work must be done by an external agent on the system to bring the two charges near one another (because like charges repel). If the charges are of opposite sign, U is negative; this means that negative work must be done against the attractive force between the unlike charges for them to be brought near each other.

If more than two charged particles are in the system, we can obtain the total potential energy by calculating U for every pair of charges and summing the terms algebraically. As an example, the total potential energy of the system of three charges shown in Figure 25.10 is

$$U = k_e \left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right) \tag{25.14}$$

Physically, we can interpret this as follows: Imagine that q_1 is fixed at the position shown in Figure 25.10 but that q_2 and q_3 are at infinity. The work an external agent must do to bring q_2 from infinity to its position near q_1 is $k_e q_1 q_2/r_{12}$, which is the first term in Equation 25.14. The last two terms represent the work required to bring q_3 from infinity to its position near q_1 and q_2. (The result is independent of the order in which the charges are transported.)

[2] The expression for the electric potential energy of a system made up of two point charges, Equation 25.13, is of the *same* form as the equation for the gravitational potential energy of a system made up of two point masses, Gm_1m_2/r (see Chapter 14). The similarity is not surprising in view of the fact that both expressions are derived from an inverse-square force law.

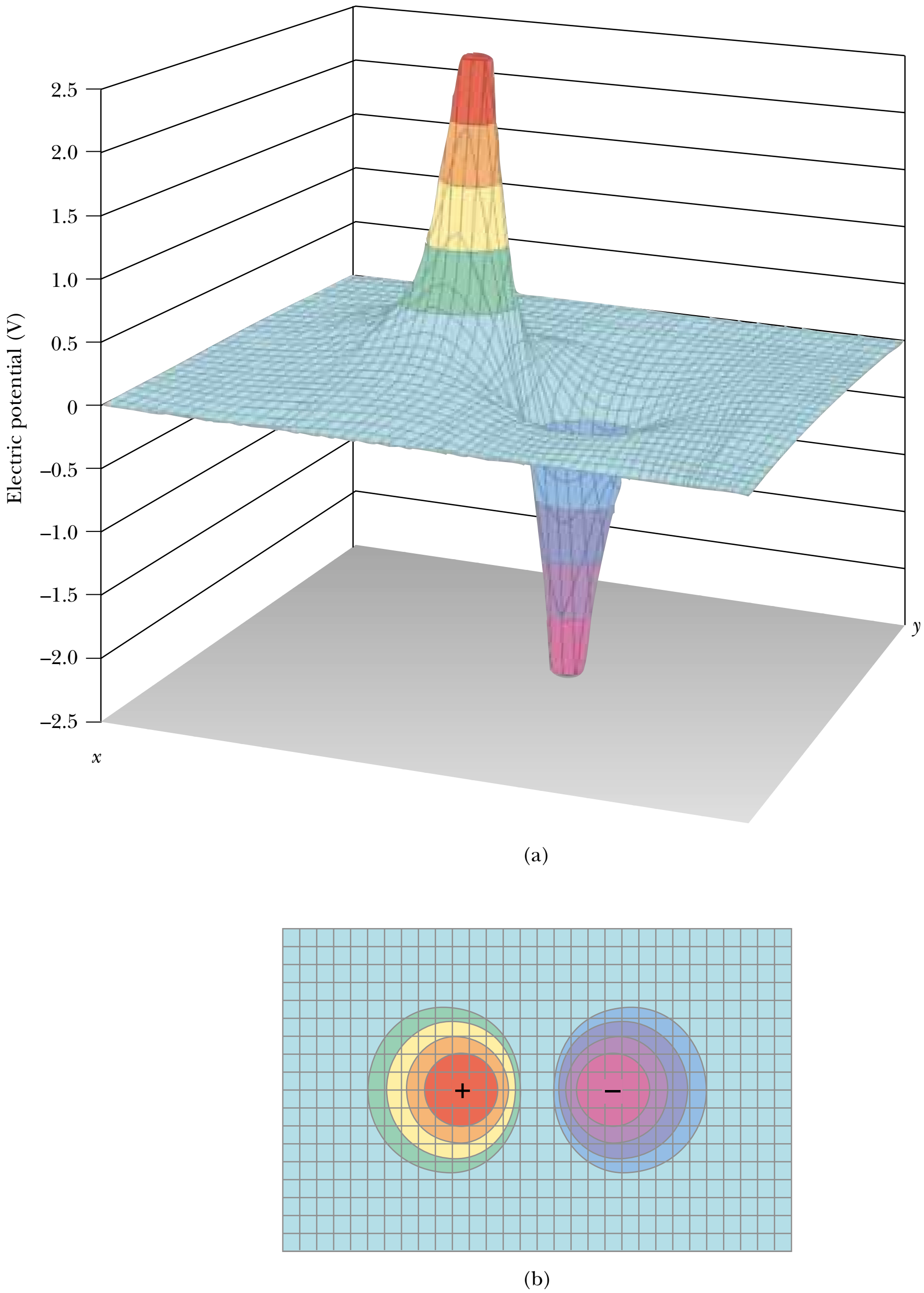


Figure 25.8 (a) The electric potential in the plane containing a dipole. (b) Top view of the function graphed in part (a).

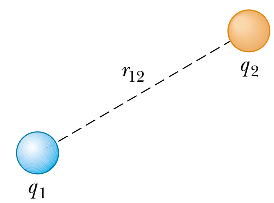


Figure 25.9 If two point charges are separated by a distance r_{12}, the potential energy of the pair of charges is given by $k_e q_1 q_2 / r_{12}$.

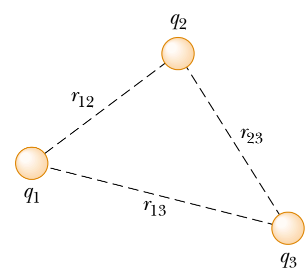


Figure 25.10 Three point charges are fixed at the positions shown. The potential energy of this system of charges is given by Equation 25.14.

EXAMPLE 25.3 The Electric Potential Due to Two Point Charges

A charge $q_1 = 2.00\ \mu C$ is located at the origin, and a charge $q_2 = -6.00\ \mu C$ is located at (0, 3.00) m, as shown in Figure 25.11a. (a) Find the total electric potential due to these charges at the point P, whose coordinates are (4.00, 0) m.

Solution For two charges, the sum in Equation 25.12 gives

$$V_P = k_e\left(\frac{q_1}{r_1} + \frac{q_2}{r_2}\right)$$

$$= 8.99 \times 10^9\ \frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\left(\frac{2.00 \times 10^{-6}\ \text{C}}{4.00\ \text{m}} + \frac{-6.00 \times 10^{-6}\ \text{C}}{5.00\ \text{m}}\right)$$

$$= \boxed{-6.29 \times 10^3\ \text{V}}$$

(b) Find the change in potential energy of a 3.00-μC charge as it moves from infinity to point P (Fig. 25.11b).

Solution When the charge is at infinity, $U_i = 0$, and when the charge is at P, $U_f = q_3 V_P$; therefore,

$$\Delta U = q_3 V_P - 0 = (3.00 \times 10^{-6}\ \text{C})(-6.29 \times 10^3\ \text{V})$$

$$= \boxed{-18.9 \times 10^{-3}\ \text{J}}$$

Therefore, because $W = -\Delta U$, positive work would have to be done by an external agent to remove the charge from point P back to infinity.

Exercise Find the total potential energy of the system illustrated in Figure 25.11b.

Answer -5.48×10^{-2} J.

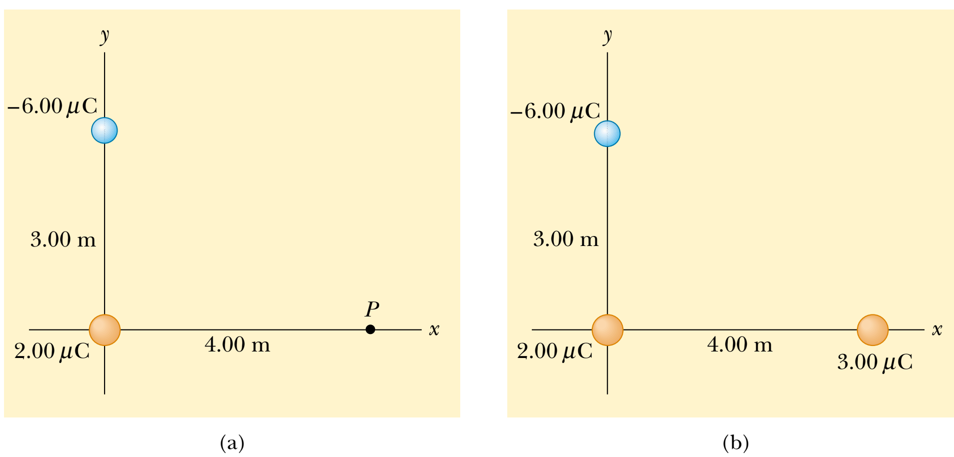


Figure 25.11 (a) The electric potential at P due to the two charges is the algebraic sum of the potentials due to the individual charges. (b) What is the potential energy of the three-charge system?

25.4 OBTAINING THE VALUE OF THE ELECTRIC FIELD FROM THE ELECTRIC POTENTIAL

The electric field **E** and the electric potential V are related as shown in Equation 25.3. We now show how to calculate the value of the electric field if the electric potential is known in a certain region.

From Equation 25.3 we can express the potential difference dV between two points a distance ds apart as

$$dV = -\mathbf{E}\cdot d\mathbf{s} \tag{25.15}$$

If the electric field has only one component E_x, then $\mathbf{E}\cdot d\mathbf{s} = E_x\,dx$. Therefore, Equation 25.15 becomes $dV = -E_x\,dx$, or

$$E_x = -\frac{dV}{dx} \tag{25.16}$$

That is, the magnitude of the electric field in the direction of some coordinate is equal to the negative of the derivative of the electric potential with respect to that coordinate. Recall from the discussion following Equation 25.8 that the electric potential does not change for any displacement perpendicular to an electric field. This is consistent with the notion, developed in Section 25.2, that equipotential surfaces are perpendicular to the field, as shown in Figure 25.12. A small positive charge placed at rest on an electric field line begins to move along the direction of $\mathbf{E}$ because that is the direction of the force exerted on the charge by the charge distribution creating the electric field (and hence is the direction of $\mathbf{a}$). Because the charge starts with zero velocity, it moves in the direction of the change in velocity—that is, in the direction of $\mathbf{a}$. In Figures 25.12a and 25.12b, a charge placed at rest in the field will move in a straight line because its acceleration vector is always parallel to its velocity vector. The magnitude of $\mathbf{v}$ increases, but its direction does not change. The situation is different in Figure 25.12c. A positive charge placed at some point near the dipole first moves in a direction parallel to $\mathbf{E}$ at that point. Because the direction of the electric field is different at different locations, however, the force acting on the charge changes direction, and $\mathbf{a}$ is no longer parallel to $\mathbf{v}$. This causes the moving charge to change direction and speed, but it does not necessarily follow the electric field lines. Recall that it is not the velocity vector but rather the acceleration vector that is proportional to force.

If the charge distribution creating an electric field has spherical symmetry such that the volume charge density depends only on the radial distance r, then the electric field is radial. In this case, $\mathbf{E} \cdot d\mathbf{s} = E_r\, dr$, and thus we can express dV in the form $dV = -E_r\, dr$. Therefore,

$$E_r = -\frac{dV}{dr} \qquad \textbf{(25.17)}$$

For example, the electric potential of a point charge is $V = k_e q/r$. Because V is a function of r only, the potential function has spherical symmetry. Applying Equation 25.17, we find that the electric field due to the point charge is $E_r = k_e q/r^2$, a familiar result. Note that the potential changes only in the radial direction, not in

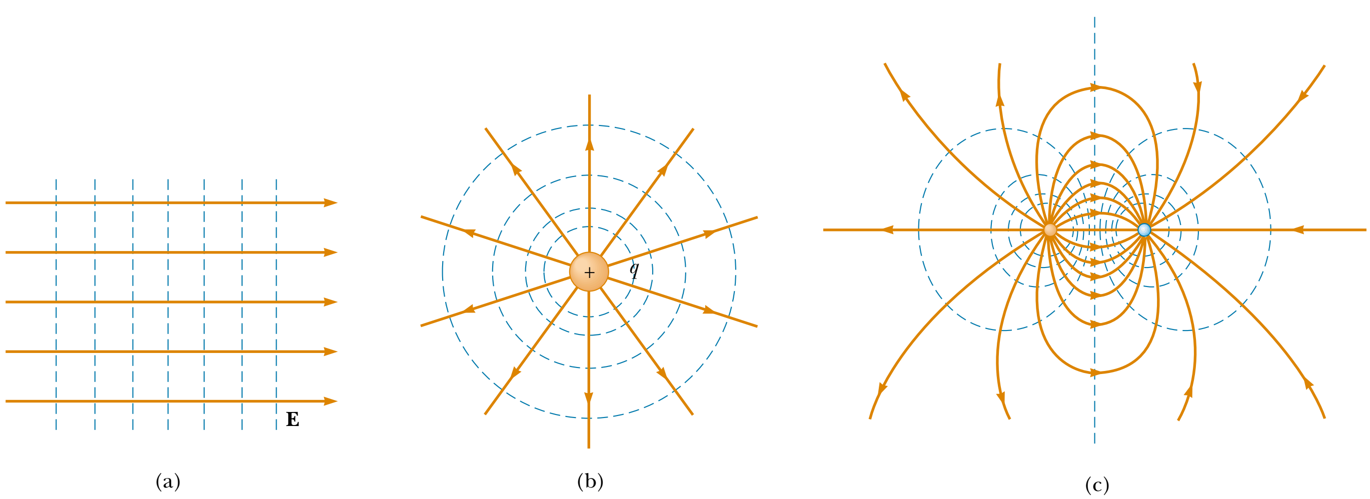


Figure 25.12 Equipotential surfaces (dashed blue lines) and electric field lines (red lines) for (a) a uniform electric field produced by an infinite sheet of charge, (b) a point charge, and (c) an electric dipole. In all cases, the equipotential surfaces are *perpendicular* to the electric field lines at every point. Compare these drawings with Figures 25.2, 25.7b, and 25.8b.

Equipotential surfaces are perpendicular to the electric field lines

any direction perpendicular to r. Thus, V (like E_r) is a function only of r. Again, this is consistent with the idea that **equipotential surfaces are perpendicular to field lines.** In this case the equipotential surfaces are a family of spheres concentric with the spherically symmetric charge distribution (Fig. 25.12b).

The equipotential surfaces for an electric dipole are sketched in Figure 25.12c. When a test charge undergoes a displacement $d\mathbf{s}$ along an equipotential surface, then $dV = 0$ because the potential is constant along an equipotential surface. From Equation 25.15, then, $dV = -\mathbf{E}\cdot d\mathbf{s} = 0$; thus, $\mathbf{E}$ must be perpendicular to the displacement along the equipotential surface. This shows that the equipotential surfaces must *always* be *perpendicular* to the electric field lines.

In general, the electric potential is a function of all three spatial coordinates. If $V(r)$ is given in terms of the cartesian coordinates, the electric field components E_x, E_y, and E_z can readily be found from $V(x, y, z)$ as the partial derivatives[3]

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

For example, if $V = 3x^2y + y^2 + yz$, then

$$\frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(3x^2y + y^2 + yz) = \frac{\partial}{\partial x}(3x^2y) = 3y\frac{d}{dx}(x^2) = 6xy$$

EXAMPLE 25.4 The Electric Potential Due to a Dipole

An electric dipole consists of two charges of equal magnitude and opposite sign separated by a distance $2a$, as shown in Figure 25.13. The dipole is along the x axis and is centered at the origin. (a) Calculate the electric potential at point P.

Solution For point P in Figure 25.13,

$$V = k_e\sum\frac{q_i}{r_i} = k_e\left(\frac{q}{x-a} - \frac{q}{x+a}\right) = \boxed{\frac{2k_eqa}{x^2-a^2}}$$

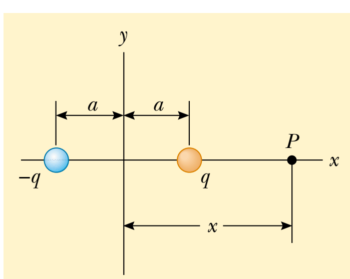


Figure 25.13 An electric dipole located on the x axis.

(How would this result change if point P happened to be located to the left of the negative charge?)

(b) Calculate V and E_x at a point far from the dipole.

Solution If point P is far from the dipole, such that $x \gg a$, then a^2 can be neglected in the term $x^2 - a^2$, and V becomes

$$V \approx \boxed{\frac{2k_eqa}{x^2}} \qquad (x \gg a)$$

Using Equation 25.16 and this result, we can calculate the electric field at a point far from the dipole:

$$E_x = -\frac{dV}{dx} = \boxed{\frac{4k_eqa}{x^3}} \qquad (x \gg a)$$

(c) Calculate V and E_x if point P is located anywhere between the two charges.

Solution

$$V = k_e\sum\frac{q_i}{r_i} = k_e\left(\frac{q}{a-x} - \frac{q}{x+a}\right) = -\frac{2k_eqx}{x^2-a^2}$$

$$E_x = -\frac{dV}{dx} = -\frac{d}{dx}\left(-\frac{2k_eqx}{x^2-a^2}\right) = 2k_eq\left(\frac{-x^2-a^2}{(x^2-a^2)^2}\right)$$

[3] In vector notation, $\mathbf{E}$ is often written

$$\mathbf{E} = -\nabla V = -\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)V$$

where ∇ is called the *gradient operator.*

We can check these results by considering the situation at the center of the dipole, where $x = 0$, $V = 0$, and $E_x = -2k_e q/a^2$.

Exercise Verify the electric field result in part (c) by calculating the sum of the individual electric field vectors at the origin due to the two charges.

25.5 ELECTRIC POTENTIAL DUE TO CONTINUOUS CHARGE DISTRIBUTIONS

We can calculate the electric potential due to a continuous charge distribution in two ways. If the charge distribution is known, we can start with Equation 25.11 for the electric potential of a point charge. We then consider the potential due to a small charge element dq, treating this element as a point charge (Fig. 25.14). The electric potential dV at some point P due to the charge element dq is

$$dV = k_e \frac{dq}{r} \tag{25.18}$$

where r is the distance from the charge element to point P. To obtain the total potential at point P, we integrate Equation 25.18 to include contributions from all elements of the charge distribution. Because each element is, in general, a different distance from point P and because k_e is constant, we can express V as

$$V = k_e \int \frac{dq}{r} \tag{25.19}$$

In effect, we have replaced the sum in Equation 25.12 with an integral. Note that this expression for V uses a particular reference: The electric potential is taken to be zero when point P is infinitely far from the charge distribution.

If the electric field is already known from other considerations, such as Gauss's law, we can calculate the electric potential due to a continuous charge distribution using Equation 25.3. If the charge distribution is highly symmetric, we first evaluate $\mathbf{E}$ at any point using Gauss's law and then substitute the value obtained into Equation 25.3 to determine the potential difference ΔV between any two points. We then choose the electric potential V to be zero at some convenient point.

We illustrate both methods with several examples.

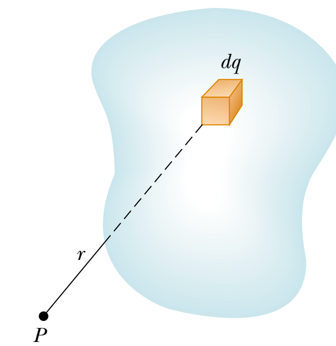


Figure 25.14 The electric potential at the point P due to a continuous charge distribution can be calculated by dividing the charged body into segments of charge dq and summing the electric potential contributions over all segments.

EXAMPLE 25.5 Electric Potential Due to a Uniformly Charged Ring

(a) Find an expression for the electric potential at a point P located on the perpendicular central axis of a uniformly charged ring of radius a and total charge Q.

Solution Let us orient the ring so that its plane is perpendicular to an x axis and its center is at the origin. We can then take point P to be at a distance x from the center of the ring, as shown in Figure 25.15. The charge element dq is at a distance $\sqrt{x^2 + a^2}$ from point P. Hence, we can express V as

$$V = k_e \int \frac{dq}{r} = k_e \int \frac{dq}{\sqrt{x^2 + a^2}}$$

Because each element dq is at the same distance from point P, we can remove $\sqrt{x^2 + a^2}$ from the integral, and V reduces to

$$V = \frac{k_e}{\sqrt{x^2 + a^2}} \int dq = \boxed{\frac{k_e Q}{\sqrt{x^2 + a^2}}} \tag{25.20}$$

The only variable in this expression for V is x. This is not surprising because our calculation is valid only for points along the x axis, where y and z are both zero.

(b) Find an expression for the magnitude of the electric field at point P.

Solution From symmetry, we see that along the x axis $\mathbf{E}$ can have only an x component. Therefore, we can use Equa-

tion 25.16:

$$E_x = -\frac{dV}{dx} = -k_eQ\frac{d}{dx}(x^2 + a^2)^{-1/2}$$
$$= -k_eQ(-\tfrac{1}{2})(x^2 + a^2)^{-3/2}(2x)$$
$$= \boxed{\frac{k_eQx}{(x^2 + a^2)^{3/2}}} \qquad \textbf{(25.21)}$$

This result agrees with that obtained by direct integration (see Example 23.8). Note that $E_x = 0$ at $x = 0$ (the center of the ring). Could you have guessed this from Coulomb's law?

Exercise What is the electric potential at the center of the ring? What does the value of the field at the center tell you about the value of V at the center?

Answer $V = k_eQ/a$. Because $E_x = -dV/dx = 0$ at the center, V has either a maximum or minimum value; it is, in fact, a maximum.

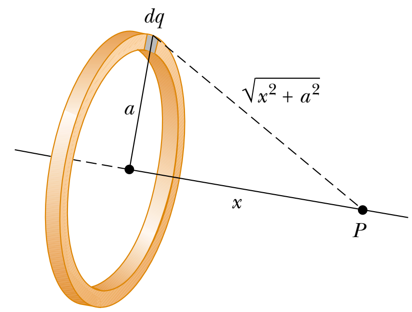


Figure 25.15 A uniformly charged ring of radius a lies in a plane perpendicular to the x axis. All segments dq of the ring are the same distance from any point P lying on the x axis.

EXAMPLE 25.6 Electric Potential Due to a Uniformly Charged Disk

Find (a) the electric potential and (b) the magnitude of the electric field along the perpendicular central axis of a uniformly charged disk of radius a and surface charge density σ.

Solution (a) Again, we choose the point P to be at a distance x from the center of the disk and take the plane of the disk to be perpendicular to the x axis. We can simplify the problem by dividing the disk into a series of charged rings. The electric potential of each ring is given by Equation 25.20. Consider one such ring of radius r and width dr, as indicated in Figure 25.16. The surface area of the ring is $dA = 2\pi r\,dr$; from the definition of surface charge density (see Section 23.5), we know that the charge on the ring is $dq = \sigma\,dA = \sigma 2\pi r\,dr$. Hence, the potential at the point P due to this ring is

$$dV = \frac{k_e\,dq}{\sqrt{r^2 + x^2}} = \frac{k_e\sigma 2\pi r\,dr}{\sqrt{r^2 + x^2}}$$

To find the *total* electric potential at P, we sum over all rings making up the disk. That is, we integrate dV from $r = 0$ to $r = a$:

$$V = \pi k_e\sigma\int_0^a \frac{2r\,dr}{\sqrt{r^2 + x^2}} = \pi k_e\sigma\int_0^a (r^2 + x^2)^{-1/2}\,2r\,dr$$

This integral is of the form $u^n\,du$ and has the value $u^{n+1}/(n+1)$, where $n = -\frac{1}{2}$ and $u = r^2 + x^2$. This gives

$$V = \boxed{2\pi k_e\sigma[(x^2 + a^2)^{1/2} - x]} \qquad \textbf{(25.22)}$$

(b) As in Example 25.5, we can find the electric field at any axial point from

$$E_x = -\frac{dV}{dx} = \boxed{2\pi k_e\sigma\left(1 - \frac{x}{\sqrt{x^2 + a^2}}\right)} \qquad \textbf{(25.23)}$$

The calculation of V and **E** for an arbitrary point off the axis is more difficult to perform, and we do not treat this situation in this text.

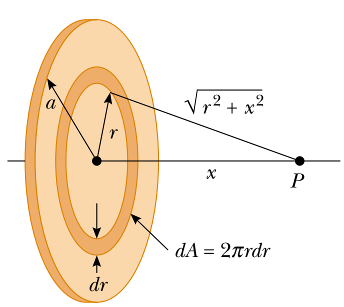


Figure 25.16 A uniformly charged disk of radius a lies in a plane perpendicular to the x axis. The calculation of the electric potential at any point P on the x axis is simplified by dividing the disk into many rings each of area $2\pi r\,dr$.

EXAMPLE 25.7 Electric Potential Due to a Finite Line of Charge

A rod of length ℓ located along the x axis has a total charge Q and a uniform linear charge density $\lambda = Q/\ell$. Find the electric potential at a point P located on the y axis a distance a from the origin (Fig. 25.17).

Solution The length element dx has a charge $dq = \lambda\,dx$. Because this element is a distance $r = \sqrt{x^2 + a^2}$ from point P, we can express the potential at point P due to this element as

$$dV = k_e \frac{dq}{r} = k_e \frac{\lambda\,dx}{\sqrt{x^2 + a^2}}$$

To obtain the total potential at P, we integrate this expression over the limits $x = 0$ to $x = \ell$. Noting that k_e and λ are constants, we find that

$$V = k_e \lambda \int_0^\ell \frac{dx}{\sqrt{x^2 + a^2}} = k_e \frac{Q}{\ell} \int_0^\ell \frac{dx}{\sqrt{x^2 + a^2}}$$

This integral has the following value (see Appendix B):

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \ln(x + \sqrt{x^2 + a^2})$$

Evaluating V, we find that

$$\boxed{V = \frac{k_e Q}{\ell} \ln\left(\frac{\ell + \sqrt{\ell^2 + a^2}}{a}\right)} \qquad \textbf{(25.24)}$$

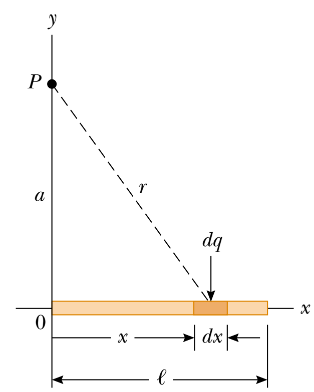


Figure 25.17 A uniform line charge of length ℓ located along the x axis. To calculate the electric potential at P, the line charge is divided into segments each of length dx and each carrying a charge $dq = \lambda\,dx$.

EXAMPLE 25.8 Electric Potential Due to a Uniformly Charged Sphere

An insulating solid sphere of radius R has a uniform positive volume charge density and total charge Q. (a) Find the electric potential at a point outside the sphere, that is, for $r > R$. Take the potential to be zero at $r = \infty$.

Solution In Example 24.5, we found that the magnitude of the electric field outside a uniformly charged sphere of radius R is

$$E_r = k_e \frac{Q}{r^2} \qquad (\text{for } r > R)$$

where the field is directed radially outward when Q is positive. In this case, to obtain the electric potential at an exterior point, such as B in Figure 25.18, we use Equation 25.4 and the expression for E_r given above:

$$V_B = -\int_\infty^r E_r\,dr = -k_e Q \int_\infty^r \frac{dr}{r^2}$$

$$\boxed{V_B = k_e \frac{Q}{r}} \qquad (\text{for } r > R)$$

Note that the result is identical to the expression for the electric potential due to a point charge (Eq. 25.11).

Because the potential must be continuous at $r = R$, we can use this expression to obtain the potential at the surface of the sphere. That is, the potential at a point such as C shown in Figure 25.18 is

$$V_C = k_e \frac{Q}{R} \qquad (\text{for } r = R)$$

(b) Find the potential at a point inside the sphere, that is, for $r < R$.

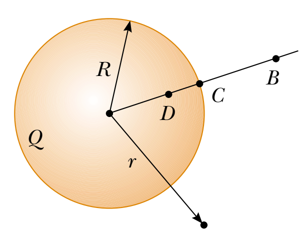


Figure 25.18 A uniformly charged insulating sphere of radius R and total charge Q. The electric potentials at points B and C are equivalent to those produced by a point charge Q located at the center of the sphere, but this is not true for point D.

Solution In Example 24.5 we found that the electric field inside an insulating uniformly charged sphere is

$$E_r = \frac{k_eQ}{R^3} r \qquad \text{(for } r < R\text{)}$$

We can use this result and Equation 25.3 to evaluate the potential difference $V_D - V_C$ at some interior point D:

$$V_D - V_C = -\int_R^r E_r \, dr = -\frac{k_eQ}{R^3}\int_R^r r \, dr = \frac{k_eQ}{2R^3}(R^2 - r^2)$$

Substituting $V_C = k_eQ/R$ into this expression and solving for V_D, we obtain

$$\boxed{V_D = \frac{k_eQ}{2R}\left(3 - \frac{r^2}{R^2}\right)} \qquad \text{(for } r < R\text{)} \qquad \textbf{(25.25)}$$

At $r = R$, this expression gives a result that agrees with that for the potential at the surface, that is, V_C. A plot of V versus r for this charge distribution is given in Figure 25.19.

Exercise What are the magnitude of the electric field and the electric potential at the center of the sphere?

Answer $E = 0$; $V_0 = 3k_eQ/2R$.

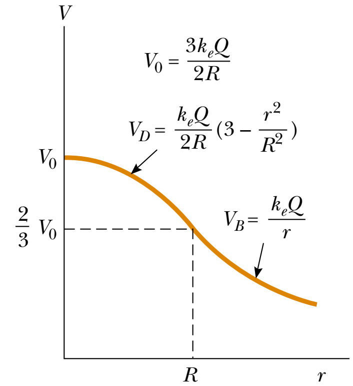


Figure 25.19 A plot of electric potential V versus distance r from the center of a uniformly charged insulating sphere of radius R. The curve for V_D inside the sphere is parabolic and joins smoothly with the curve for V_B outside the sphere, which is a hyperbola. The potential has a maximum value V_0 at the center of the sphere. We could make this graph three dimensional (similar to Figures 25.7a and 25.8a) by spinning it around the vertical axis.

25.6 ELECTRIC POTENTIAL DUE TO A CHARGED CONDUCTOR

In Section 24.4 we found that when a solid conductor in equilibrium carries a net charge, the charge resides on the outer surface of the conductor. Furthermore, we showed that the electric field just outside the conductor is perpendicular to the surface and that the field inside is zero.

We now show that **every point on the surface of a charged conductor in equilibrium is at the same electric potential.** Consider two points A and B on the surface of a charged conductor, as shown in Figure 25.20. Along a surface path connecting these points, $\mathbf{E}$ is always perpendicular to the displacement $d\mathbf{s}$; there-

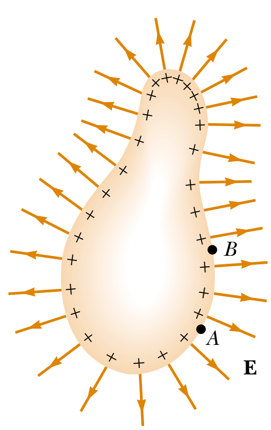


Figure 25.20 An arbitrarily shaped conductor carrying a positive charge. When the conductor is in electrostatic equilibrium, all of the charge resides at the surface, $\mathbf{E} = 0$ inside the conductor, and the direction of $\mathbf{E}$ just outside the conductor is perpendicular to the surface. The electric potential is constant inside the conductor and is equal to the potential at the surface. Note from the spacing of the plus signs that the surface charge density is nonuniform.

fore $\mathbf{E} \cdot d\mathbf{s} = 0$. Using this result and Equation 25.3, we conclude that the potential difference between A and B is necessarily zero:

$$V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{s} = 0$$

This result applies to any two points on the surface. Therefore, V is constant everywhere on the surface of a charged conductor in equilibrium. That is,

the surface of any charged conductor in electrostatic equilibrium is an equipotential surface. Furthermore, because the electric field is zero inside the conductor, we conclude from the relationship $E_r = -dV/dr$ that the electric potential is constant everywhere inside the conductor and equal to its value at the surface.

The surface of a charged conductor is an equipotential surface

Because this is true about the electric potential, no work is required to move a test charge from the interior of a charged conductor to its surface.

Consider a solid metal conducting sphere of radius R and total positive charge Q, as shown in Figure 25.21a. The electric field outside the sphere is k_eQ/r^2 and points radially outward. From Example 25.8, we know that the electric potential at the interior and surface of the sphere must be k_eQ/R relative to infinity. The potential outside the sphere is k_eQ/r. Figure 25.21b is a plot of the electric potential as a function of r, and Figure 25.21c shows how the electric field varies with r.

When a net charge is placed on a spherical conductor, the surface charge density is uniform, as indicated in Figure 25.21a. However, if the conductor is nonspherical, as in Figure 25.20, the surface charge density is high where the radius of curvature is small and the surface is convex (as noted in Section 24.4), and it is low where the radius of curvature is small and the surface is concave. Because the electric field just outside the conductor is proportional to the surface charge density, we see that **the electric field is large near convex points having small radii of curvature and reaches very high values at sharp points.**

Figure 25.22 shows the electric field lines around two spherical conductors: one carrying a net charge Q, and a larger one carrying zero net charge. In this case, the surface charge density is not uniform on either conductor. The sphere having zero net charge has negative charges induced on its side that faces the

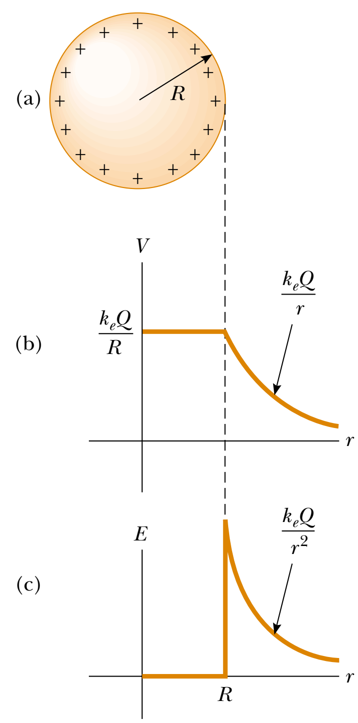
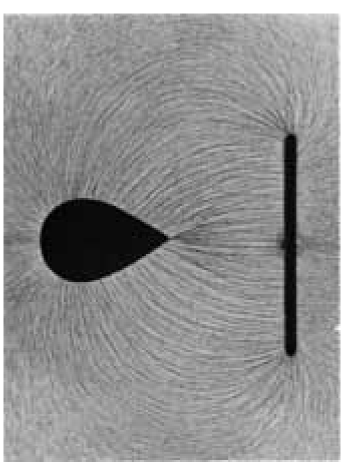


Figure 25.21 (a) The excess charge on a conducting sphere of radius R is uniformly distributed on its surface. (b) Electric potential versus distance r from the center of the charged conducting sphere. (c) Electric field magnitude versus distance r from the center of the charged conducting sphere.



Electric field pattern of a charged conducting plate placed near an oppositely charged pointed conductor. Small pieces of thread suspended in oil align with the electric field lines. The field surrounding the pointed conductor is most intense near the pointed end and at other places where the radius of curvature is small.

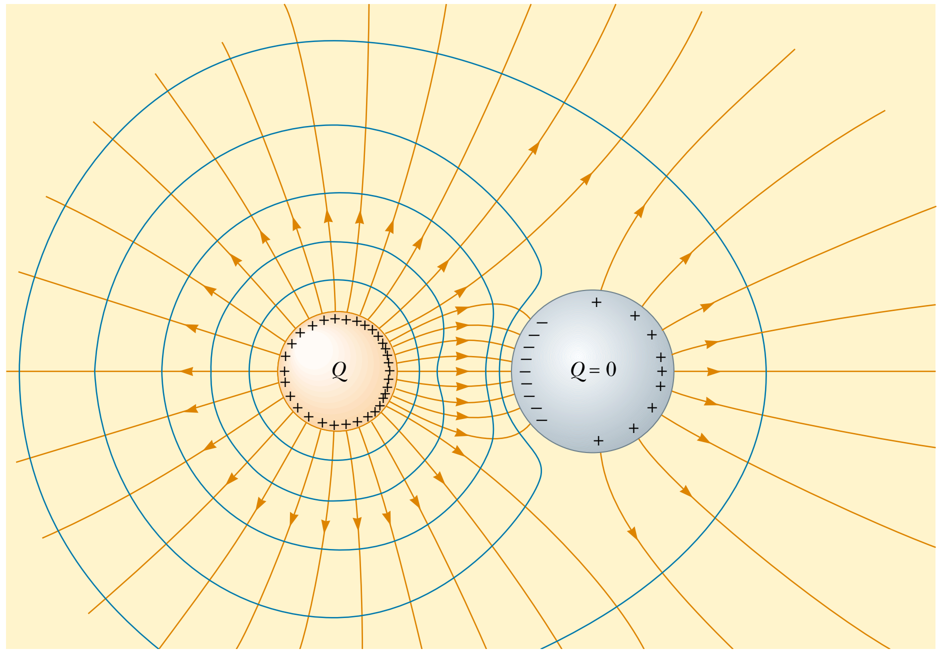


Figure 25.22 The electric field lines (in red) around two spherical conductors. The smaller sphere has a net charge Q, and the larger one has zero net charge. The blue curves are cross-sections of equipotential surfaces.

charged sphere and positive charges induced on its side opposite the charged sphere. The blue curves in the figure represent the cross-sections of the equipotential surfaces for this charge configuration. As usual, the field lines are perpendicular to the conducting surfaces at all points, and the equipotential surfaces are perpendicular to the field lines everywhere. Trying to move a positive charge in the region of these conductors would be like moving a marble on a hill that is flat on top (representing the conductor on the left) and has another flat area partway down the side of the hill (representing the conductor on the right).

Example 25.9 Two Connected Charged Spheres

Two spherical conductors of radii r_1 and r_2 are separated by a distance much greater than the radius of either sphere. The spheres are connected by a conducting wire, as shown in Figure 25.23. The charges on the spheres in equilibrium are q_1 and q_2, respectively, and they are uniformly charged. Find the ratio of the magnitudes of the electric fields at the surfaces of the spheres.

Solution Because the spheres are connected by a conducting wire, they must both be at the same electric potential:

$$V = k_e \frac{q_1}{r_1} = k_e \frac{q_2}{r_2}$$

Therefore, the ratio of charges is

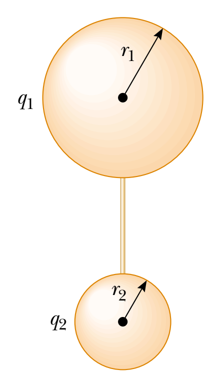


Figure 25.23 Two charged spherical conductors connected by a conducting wire. The spheres are at the *same* electric potential V.

$$(1) \qquad \frac{q_1}{q_2} = \frac{r_1}{r_2}$$

Because the spheres are very far apart and their surfaces uniformly charged, we can express the magnitude of the electric fields at their surfaces as

$$E_1 = k_e \frac{q_1}{r_1^2} \qquad \text{and} \qquad E_2 = k_e \frac{q_2}{r_2^2}$$

Taking the ratio of these two fields and making use of Equation (1), we find that

$$\boxed{\frac{E_1}{E_2} = \frac{r_2}{r_1}}$$

Hence, the field is more intense in the vicinity of the smaller sphere even though the electric potentials of both spheres are the same.

A Cavity Within a Conductor

Now consider a conductor of arbitrary shape containing a cavity as shown in Figure 25.24. Let us assume that no charges are inside the cavity. **In this case, the electric field inside the cavity must be zero** regardless of the charge distribution on the outside surface of the conductor. Furthermore, the field in the cavity is zero even if an electric field exists outside the conductor.

To prove this point, we use the fact that every point on the conductor is at the same electric potential, and therefore any two points A and B on the surface of the cavity must be at the same potential. Now imagine that a field $\mathbf{E}$ exists in the cavity and evaluate the potential difference $V_B - V_A$ defined by Equation 25.3:

$$V_B - V_A = -\int_A^B \mathbf{E} \cdot d\mathbf{s}$$

If $\mathbf{E}$ is nonzero, we can always find a path between A and B for which $\mathbf{E} \cdot d\mathbf{s}$ is a positive number; thus, the integral must be positive. However, because $V_B - V_A = 0$, the integral of $\mathbf{E} \cdot d\mathbf{s}$ must be zero for all paths between any two points on the conductor, which implies that $\mathbf{E}$ is zero everywhere. This contradiction can be reconciled only if $\mathbf{E}$ is zero inside the cavity. Thus, we conclude that a cavity surrounded by conducting walls is a field-free region as long as no charges are inside the cavity.

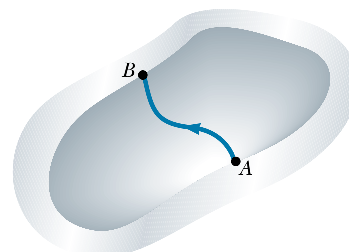


Figure 25.24 A conductor in electrostatic equilibrium containing a cavity. The electric field in the cavity is zero, regardless of the charge on the conductor.

Corona Discharge

A phenomenon known as **corona discharge** is often observed near a conductor such as a high-voltage power line. When the electric field in the vicinity of the conductor is sufficiently strong, electrons are stripped from air molecules. This causes the molecules to be ionized, thereby increasing the air's ability to conduct. The observed glow (or corona discharge) results from the recombination of free electrons with the ionized air molecules. If a conductor has an irregular shape, the electric field can be very high near sharp points or edges of the conductor; consequently, the ionization process and corona discharge are most likely to occur around such points.

Quick Quiz 25.4

(a) Is it possible for the magnitude of the electric field to be zero at a location where the electric potential is not zero? (b) Can the electric potential be zero where the electric field is nonzero?

Optional Section

25.7 THE MILLIKAN OIL-DROP EXPERIMENT

During the period from 1909 to 1913, Robert Millikan performed a brilliant set of experiments in which he measured e, the elementary charge on an electron, and demonstrated the quantized nature of this charge. His apparatus, diagrammed in Figure 25.25, contains two parallel metallic plates. Charged oil droplets from an atomizer are allowed to pass through a small hole in the upper plate. A horizontally directed light beam (not shown in the diagram) is used to illuminate the oil droplets, which are viewed through a telescope whose long axis is at right angles to the light beam. When the droplets are viewed in this manner, they appear as shining stars against a dark background, and the rate at which individual drops fall can be determined.[4]

Let us assume that a single drop having a mass m and carrying a charge q is being viewed and that its charge is negative. If no electric field is present between the plates, the two forces acting on the charge are the force of gravity $m\mathbf{g}$ acting downward and a viscous drag force $\mathbf{F}_D$ acting upward as indicated in Figure 25.26a. The drag force is proportional to the drop's speed. When the drop reaches its terminal speed v, the two forces balance each other ($mg = F_D$).

Now suppose that a battery connected to the plates sets up an electric field between the plates such that the upper plate is at the higher electric potential. In this case, a third force $q\mathbf{E}$ acts on the charged drop. Because q is negative and $\mathbf{E}$ is directed downward, this electric force is directed upward, as shown in Figure 25.26b. If this force is sufficiently great, the drop moves upward and the drag force $\mathbf{F}'_D$ acts downward. When the upward electric force $q\mathbf{E}$ balances the sum of the gravitational force and the downward drag force $\mathbf{F}'_D$, the drop reaches a new terminal speed v' in the upward direction.

With the field turned on, a drop moves slowly upward, typically at rates of hundredths of a centimeter per second. The rate of fall in the absence of a field is comparable. Hence, one can follow a single droplet for hours, alternately rising and falling, by simply turning the electric field on and off.

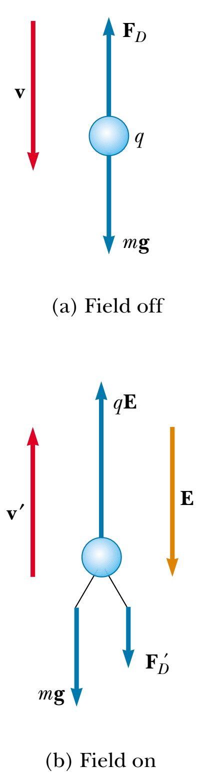


Figure 25.26 The forces acting on a negatively charged oil droplet in the Millikan experiment.

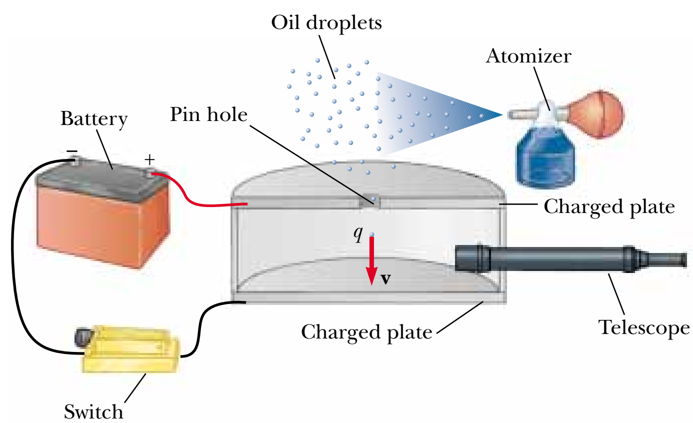


Figure 25.25 Schematic drawing of the Millikan oil-drop apparatus.

[4] At one time, the oil droplets were termed "Millikan's Shining Stars." Perhaps this description has lost its popularity because of the generations of physics students who have experienced hallucinations, near blindness, migraine headaches, and so forth, while repeating Millikan's experiment!

After recording measurements on thousands of droplets, Millikan and his co-workers found that all droplets, to within about 1% precision, had a charge equal to some integer multiple of the elementary charge e:

$$q = ne \qquad n = 0, -1, -2, -3, \ldots$$

where $e = 1.60 \times 10^{-19}$ C. Millikan's experiment yields conclusive evidence that charge is quantized. For this work, he was awarded the Nobel Prize in Physics in 1923.

Optional Section

25.8 APPLICATIONS OF ELECTROSTATICS

The practical application of electrostatics is represented by such devices as lightning rods and electrostatic precipitators and by such processes as xerography and the painting of automobiles. Scientific devices based on the principles of electrostatics include electrostatic generators, the field-ion microscope, and ion-drive rocket engines.

The Van de Graaff Generator

11.10 In Section 24.5 we described an experiment that demonstrates a method for transferring charge to a hollow conductor (the Faraday ice-pail experiment). When a charged conductor is placed in contact with the inside of a hollow conductor, all of the charge of the charged conductor is transferred to the hollow conductor. In principle, the charge on the hollow conductor and its electric potential can be increased without limit by repetition of the process.

In 1929 Robert J. Van de Graaff (1901–1967) used this principle to design and build an electrostatic generator. This type of generator is used extensively in nuclear physics research. A schematic representation of the generator is given in Figure 25.27. Charge is delivered continuously to a high-potential electrode by means of a moving belt of insulating material. The high-voltage electrode is a hollow conductor mounted on an insulating column. The belt is charged at point A by means of a corona discharge between comb-like metallic needles and a grounded grid. The needles are maintained at a positive electric potential of typically 10^4 V. The positive charge on the moving belt is transferred to the hollow conductor by a second comb of needles at point B. Because the electric field inside the hollow conductor is negligible, the positive charge on the belt is easily transferred to the conductor regardless of its potential. In practice, it is possible to increase the electric potential of the hollow conductor until electrical discharge occurs through the air. Because the "breakdown" electric field in air is about 3×10^6 V/m, a sphere 1 m in radius can be raised to a maximum potential of 3×10^6 V. The potential can be increased further by increasing the radius of the hollow conductor and by placing the entire system in a container filled with high-pressure gas.

Van de Graaff generators can produce potential differences as large as 20 million volts. Protons accelerated through such large potential differences receive enough energy to initiate nuclear reactions between themselves and various target nuclei. Smaller generators are often seen in science classrooms and museums. If a person insulated from the ground touches the sphere of a Van de Graaff generator, his or her body can be brought to a high electric potential. The hair acquires a net positive charge, and each strand is repelled by all the others. The result is a

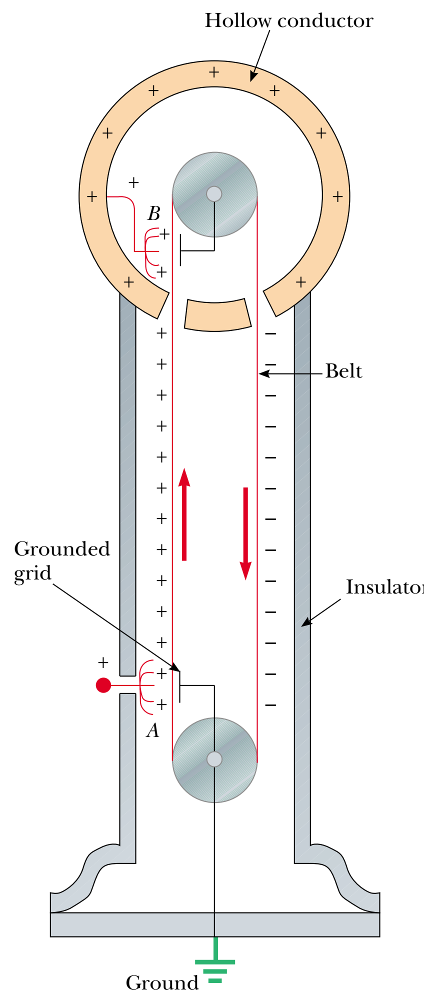


Figure 25.27 Schematic diagram of a Van de Graaff generator. Charge is transferred to the hollow conductor at the top by means of a moving belt. The charge is deposited on the belt at point A and transferred to the hollow conductor at point B.

scene such as that depicted in the photograph at the beginning of this chapter. In addition to being insulated from ground, the person holding the sphere is safe in this demonstration because the total charge on the sphere is very small (on the order of 1 μC). If this amount of charge accidentally passed from the sphere through the person to ground, the corresponding current would do no harm.

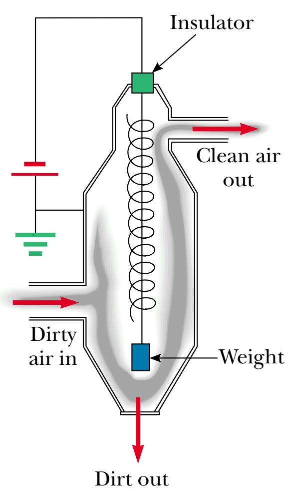
The Electrostatic Precipitator

QuickLab

Sprinkle some salt and pepper on an open dish and mix the two together. Now pull a comb through your hair several times and bring the comb to within 1 cm of the salt and pepper. What happens? How is what happens here related to the operation of an electrostatic precipitator?

One important application of electrical discharge in gases is the *electrostatic precipitator.* This device removes particulate matter from combustion gases, thereby reducing air pollution. Precipitators are especially useful in coal-burning power plants and in industrial operations that generate large quantities of smoke. Current systems are able to eliminate more than 99% of the ash from smoke.

Figure 25.28a shows a schematic diagram of an electrostatic precipitator. A high potential difference (typically 40 to 100 kV) is maintained between a wire running down the center of a duct and the walls of the duct, which are grounded. The wire is maintained at a negative electric potential with respect to the walls, so the electric field is directed toward the wire. The values of the field near the wire become high enough to cause a corona discharge around the wire; the discharge ionizes some air molecules to form positive ions, electrons, and such negative ions as O_2^-. The air to be cleaned enters the duct and moves near the wire. As the electrons and negative ions created by the discharge are accelerated toward the outer wall by the electric field, the dirt particles in the air become charged by collisions and ion capture. Because most of the charged dirt particles are negative, they too are drawn to the duct walls by the electric field. When the duct is periodically shaken, the particles break loose and are collected at the bottom.



(a)

(b)

(c)

Figure 25.28 (a) Schematic diagram of an electrostatic precipitator. The high negative electric potential maintained on the central coiled wire creates an electrical discharge in the vicinity of the wire. Compare the air pollution when the electrostatic precipitator is (b) operating and (c) turned off.

In addition to reducing the level of particulate matter in the atmosphere (compare Figs. 25.28b and c), the electrostatic precipitator recovers valuable materials in the form of metal oxides.

Xerography and Laser Printers

The basic idea of xerography[5] was developed by Chester Carlson, who was granted a patent for the xerographic process in 1940. The one feature of this process that makes it unique is the use of a photoconductive material to form an image. (A *photoconductor* is a material that is a poor electrical conductor in the dark but that becomes a good electrical conductor when exposed to light.)

The xerographic process is illustrated in Figure 25.29a to d. First, the surface of a plate or drum that has been coated with a thin film of photoconductive material (usually selenium or some compound of selenium) is given a positive electrostatic charge in the dark. An image of the page to be copied is then focused by a lens onto the charged surface. The photoconducting surface becomes conducting only in areas where light strikes it. In these areas, the light produces charge carriers in the photoconductor that move the positive charge off the drum. However, positive

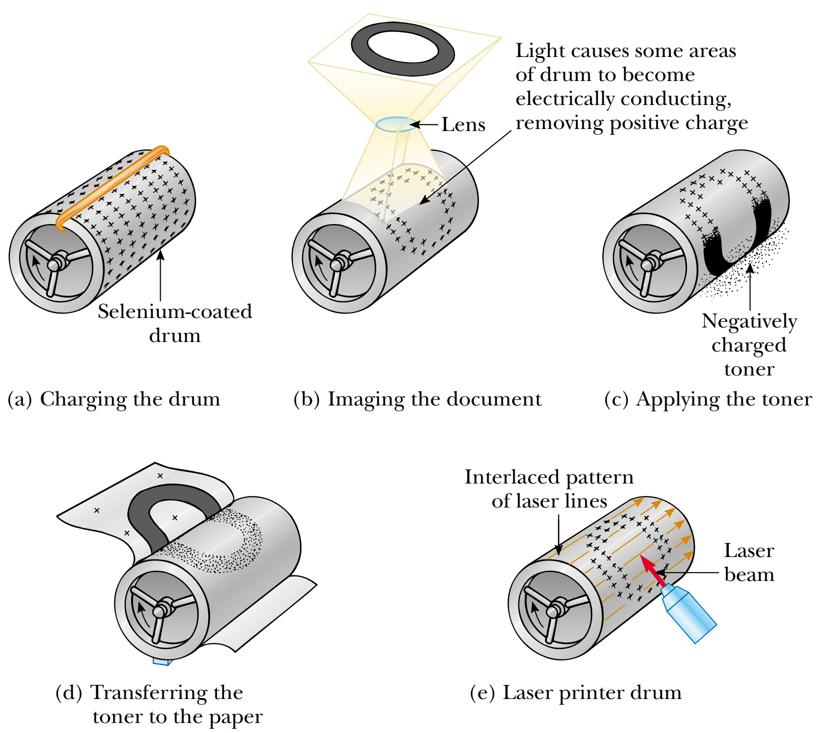


Figure 25.29 The xerographic process: (a) The photoconductive surface of the drum is positively charged. (b) Through the use of a light source and lens, an image is formed on the surface in the form of positive charges. (c) The surface containing the image is covered with a negatively charged powder, which adheres only to the image area. (d) A piece of paper is placed over the surface and given a positive charge. This transfers the image to the paper as the negatively charged powder particles migrate to the paper. The paper is then heat-treated to "fix" the powder. (e) A laser printer operates similarly except the image is produced by turning a laser beam on and off as it sweeps across the selenium-coated drum.

[5] The prefix *xero-* is from the Greek word meaning "dry." Note that no liquid ink is used anywhere in xerography.

charges remain on those areas of the photoconductor not exposed to light, leaving a latent image of the object in the form of a positive surface charge distribution.

Next, a negatively charged powder called a *toner* is dusted onto the photoconducting surface. The charged powder adheres only to those areas of the surface that contain the positively charged image. At this point, the image becomes visible. The toner (and hence the image) are then transferred to the surface of a sheet of positively charged paper.

Finally, the toner is "fixed" to the surface of the paper as the toner melts while passing through high-temperature rollers. This results in a permanent copy of the original.

A laser printer (Fig. 25.29e) operates by the same principle, with the exception that a computer-directed laser beam is used to illuminate the photoconductor instead of a lens.

SUMMARY

When a positive test charge q_0 is moved between points A and B in an electric field $\mathbf{E}$, the **change in the potential energy** is

$$\Delta U = -q_0 \int_A^B \mathbf{E} \cdot d\mathbf{s} \tag{25.1}$$

The **electric potential** $V = U/q_0$ is a scalar quantity and has units of joules per coulomb (J/C), where $1 \text{ J/C} \equiv 1 \text{ V}$.

The **potential difference** ΔV between points A and B in an electric field $\mathbf{E}$ is defined as

$$\Delta V = \frac{\Delta U}{q_0} = -\int_A^B \mathbf{E} \cdot d\mathbf{s} \tag{25.3}$$

The potential difference between two points A and B in a uniform electric field $\mathbf{E}$ is

$$\Delta V = -Ed \tag{25.6}$$

where d is the magnitude of the displacement in the direction parallel to $\mathbf{E}$.

An **equipotential surface** is one on which all points are at the same electric potential. Equipotential surfaces are perpendicular to electric field lines.

If we define $V = 0$ at $r_A = \infty$, the electric potential due to a point charge at any distance r from the charge is

$$V = k_e \frac{q}{r} \tag{25.11}$$

We can obtain the electric potential associated with a group of point charges by summing the potentials due to the individual charges.

The **potential energy associated with a pair of point charges** separated by a distance r_{12} is

$$U = k_e \frac{q_1 q_2}{r_{12}} \tag{25.13}$$

This energy represents the work required to bring the charges from an infinite separation to the separation r_{12}. We obtain the potential energy of a distribution of point charges by summing terms like Equation 25.13 over all pairs of particles.

TABLE 25.1 Electric Potential Due to Various Charge Distributions

Charge Distribution	Electric Potential	Location
Uniformly charged ring of radius a	$V = k_e \frac{Q}{\sqrt{x^2 + a^2}}$	Along perpendicular central axis of ring, distance x from ring center
Uniformly charged disk of radius a	$V = 2\pi k_e \sigma[(x^2 + a^2)^{1/2} - x]$	Along perpendicular central axis of disk, distance x from disk center
Uniformly charged, *insulating* solid sphere of radius R and total charge Q	$V = k_e \frac{Q}{r}$	$r \geq R$
	$V = \frac{k_e Q}{2R}\left(3 - \frac{r^2}{R^2}\right)$	$r < R$
Isolated *conducting* sphere of radius R and total charge Q	$V = k_e \frac{Q}{r}$	$r > R$
	$V = k_e \frac{Q}{R}$	$r \leq R$

If we know the electric potential as a function of coordinates x, y, z, we can obtain the components of the electric field by taking the negative derivative of the electric potential with respect to the coordinates. For example, the x component of the electric field is

$$E_x = -\frac{dV}{dx} \tag{25.16}$$

The **electric potential due to a continuous charge distribution** is

$$V = k_e \int \frac{dq}{r} \tag{25.19}$$

Every point on the surface of a charged conductor in electrostatic equilibrium is at the same electric potential. The potential is constant everywhere inside the conductor and equal to its value at the surface.

Table 25.1 lists electric potentials due to several charge distributions.

Problem-Solving Hints

Calculating Electric Potential

- Remember that electric potential is a scalar quantity, so components need not be considered. Therefore, when using the superposition principle to evaluate the electric potential at a point due to a system of point charges, simply take the algebraic sum of the potentials due to the various charges. However, you must keep track of signs. The potential is positive for positive charges, and it is negative for negative charges.
- Just as with gravitational potential energy in mechanics, only *changes* in electric potential are significant; hence, the point where you choose the poten-

tial to be zero is arbitrary. When dealing with point charges or a charge distribution of finite size, we usually define $V = 0$ to be at a point infinitely far from the charges.

- You can evaluate the electric potential at some point P due to a continuous distribution of charge by dividing the charge distribution into infinitesimal elements of charge dq located at a distance r from P. Then, treat one charge element as a point charge, such that the potential at P due to the element is $dV = k_e dq/r$. Obtain the total potential at P by integrating dV over the entire charge distribution. In performing the integration for most problems, you must express dq and r in terms of a single variable. To simplify the integration, consider the geometry involved in the problem carefully. Review Examples 25.5 through 25.7 for guidance.
- Another method that you can use to obtain the electric potential due to a finite continuous charge distribution is to start with the definition of potential difference given by Equation 25.3. If you know or can easily obtain $\mathbf{E}$ (from Gauss's law), then you can evaluate the line integral of $\mathbf{E} \cdot d\mathbf{s}$. An example of this method is given in Example 25.8.
- Once you know the electric potential at a point, you can obtain the electric field at that point by remembering that the electric field component in a specified direction is equal to the negative of the derivative of the electric potential in that direction. Example 25.4 illustrates this procedure.

QUESTIONS

1. Distinguish between electric potential and electric potential energy.
2. A negative charge moves in the direction of a uniform electric field. Does the potential energy of the charge increase or decrease? Does it move to a position of higher or lower potential?
3. Give a physical explanation of the fact that the potential energy of a pair of like charges is positive whereas the potential energy of a pair of unlike charges is negative.
4. A uniform electric field is parallel to the x axis. In what direction can a charge be displaced in this field without any external work being done on the charge?
5. Explain why equipotential surfaces are always perpendicular to electric field lines.
6. Describe the equipotential surfaces for (a) an infinite line of charge and (b) a uniformly charged sphere.
7. Explain why, under static conditions, all points in a conductor must be at the same electric potential.
8. The electric field inside a hollow, uniformly charged sphere is zero. Does this imply that the potential is zero inside the sphere? Explain.
9. The potential of a point charge is defined to be zero at an infinite distance. Why can we not define the potential of an infinite line of charge to be zero at $r = \infty$?
10. Two charged conducting spheres of different radii are connected by a conducting wire, as shown in Figure 25.23. Which sphere has the greater charge density?
11. What determines the maximum potential to which the dome of a Van de Graaff generator can be raised?
12. Explain the origin of the glow sometimes observed around the cables of a high-voltage power line.
13. Why is it important to avoid sharp edges or points on conductors used in high-voltage equipment?
14. How would you shield an electronic circuit or laboratory from stray electric fields? Why does this work?
15. Why is it relatively safe to stay in an automobile with a metal body during a severe thunderstorm?
16. Walking across a carpet and then touching someone can result in a shock. Explain why this occurs.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 25.1 Potential Difference and Electric Potential

1. How much work is done (by a battery, generator, or some other source of electrical energy) in moving Avogadro's number of electrons from an initial point where the electric potential is 9.00 V to a point where the potential is -5.00 V? (The potential in each case is measured relative to a common reference point.)
2. An ion accelerated through a potential difference of 115 V experiences an increase in kinetic energy of 7.37×10^{-17} J. Calculate the charge on the ion.
3. (a) Calculate the speed of a proton that is accelerated from rest through a potential difference of 120 V. (b) Calculate the speed of an electron that is accelerated through the same potential difference.
4. **Review Problem.** Through what potential difference would an electron need to be accelerated for it to achieve a speed of 40.0% of the speed of light, starting from rest? The speed of light is $c = 3.00 \times 10^8$ m/s; review Section 7.7.
5. What potential difference is needed to stop an electron having an initial speed of 4.20×10^5 m/s?

Section 25.2 Potential Differences in a Uniform Electric Field

6. A uniform electric field of magnitude 250 V/m is directed in the positive x direction. A $+12.0$-μC charge moves from the origin to the point $(x, y) =$ (20.0 cm, 50.0 cm). (a) What was the change in the potential energy of this charge? (b) Through what potential difference did the charge move?
7. The difference in potential between the accelerating plates of a TV set is about 25 000 V. If the distance between these plates is 1.50 cm, find the magnitude of the uniform electric field in this region.
8. Suppose an electron is released from rest in a uniform electric field whose magnitude is 5.90×10^3 V/m. (a) Through what potential difference will it have passed after moving 1.00 cm? (b) How fast will the electron be moving after it has traveled 1.00 cm?

WEB 9. An electron moving parallel to the x axis has an initial speed of 3.70×10^6 m/s at the origin. Its speed is reduced to 1.40×10^5 m/s at the point $x = 2.00$ cm. Calculate the potential difference between the origin and that point. Which point is at the higher potential?

10. A uniform electric field of magnitude 325 V/m is directed in the *negative* y direction as shown in Figure P25.10. The coordinates of point A are $(-0.200, -0.300)$ m, and those of point B are (0.400, 0.500) m. Calculate the potential difference $V_B - V_A$, using the blue path.

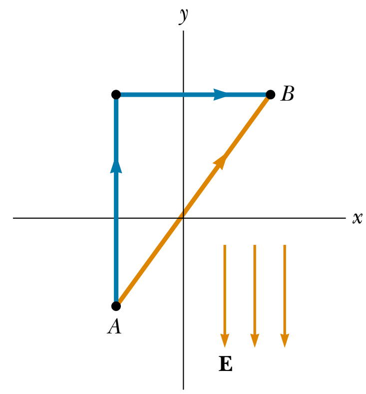


Figure P25.10

11. A 4.00-kg block carrying a charge $Q = 50.0\ \mu$C is connected to a spring for which $k = 100$ N/m. The block lies on a frictionless horizontal track, and the system is immersed in a uniform electric field of magnitude $E = 5.00 \times 10^5$ V/m, directed as shown in Figure P25.11. If the block is released from rest when the spring is unstretched (at $x = 0$), (a) by what maximum amount does the spring expand? (b) What is the equilibrium position of the block? (c) Show that the block's motion is simple harmonic, and determine its period. (d) Repeat part (a) if the coefficient of kinetic friction between block and surface is 0.200.
12. A block having mass m and charge Q is connected to a spring having constant k. The block lies on a frictionless horizontal track, and the system is immersed in a uniform electric field of magnitude E, directed as shown in Figure P25.11. If the block is released from rest when the spring is unstretched (at $x = 0$), (a) by what maximum amount does the spring expand? (b) What is the equilibrium position of the block? (c) Show that the block's motion is simple harmonic, and determine its period. (d) Repeat part (a) if the coefficient of kinetic friction between block and surface is μ_k.

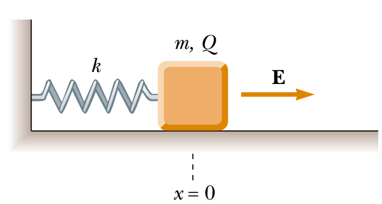


Figure P25.11 Problems 11 and 12.

13. On planet Tehar, the acceleration due to gravity is the same as that on Earth but there is also a strong downward electric field with the field being uniform close to the planet's surface. A 2.00-kg ball having a charge of 5.00 μC is thrown upward at a speed of 20.1 m/s and it hits the ground after an interval of 4.10 s. What is the potential difference between the starting point and the top point of the trajectory?

14. An insulating rod having linear charge density λ = 40.0 μC/m and linear mass density μ = 0.100 kg/m is released from rest in a uniform electric field E = 100 V/m directed perpendicular to the rod (Fig. P25.14). (a) Determine the speed of the rod after it has traveled 2.00 m. (b) How does your answer to part (a) change if the electric field is not perpendicular to the rod? Explain.

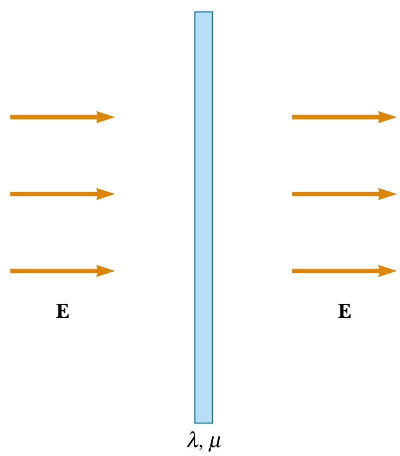


Figure P25.14

15. A particle having charge q = +2.00 μC and mass m = 0.010 0 kg is connected to a string that is L = 1.50 m long and is tied to the pivot point P in Figure P25.15. The particle, string, and pivot point all lie on a horizontal table. The particle is released from rest when the string makes an angle θ = 60.0° with a uniform electric field of magnitude E = 300 V/m. Determine the speed of the particle when the string is parallel to the electric field (point a in Fig. P25.15).

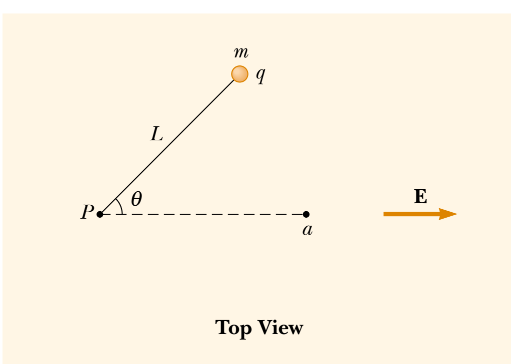


Figure P25.15

Section 25.3 Electric Potential and Potential Energy Due to Point Charges

Note: Unless stated otherwise, assume a reference level of potential V = 0 at $r = \infty$.

16. (a) Find the potential at a distance of 1.00 cm from a proton. (b) What is the potential difference between two points that are 1.00 cm and 2.00 cm from a proton? (c) Repeat parts (a) and (b) for an electron.

17. Given two 2.00-μC charges, as shown in Figure P25.17, and a positive test charge $q = 1.28 \times 10^{-18}$ C at the origin, (a) what is the net force exerted on q by the two 2.00-μC charges? (b) What is the electric field at the origin due to the two 2.00-μC charges? (c) What is the electric potential at the origin due to the two 2.00-μC charges?

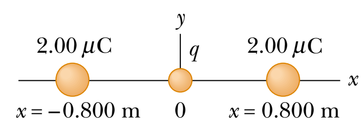


Figure P25.17

18. A charge $+q$ is at the origin. A charge $-2q$ is at x = 2.00 m on the x axis. For what finite value(s) of x is (a) the electric field zero? (b) the electric potential zero?

19. The Bohr model of the hydrogen atom states that the single electron can exist only in certain allowed orbits around the proton. The radius of each Bohr orbit is $r = n^2$ (0.052 9 nm) where n = 1, 2, 3, ... Calculate the electric potential energy of a hydrogen atom when the electron is in the (a) first allowed orbit, n = 1; (b) second allowed orbit, n = 2; and (c) when the electron has escaped from the atom ($r = \infty$). Express your answers in electron volts.

20. Two point charges Q_1 = +5.00 nC and Q_2 = −3.00 nC are separated by 35.0 cm. (a) What is the potential energy of the pair? What is the significance of the algebraic sign of your answer? (b) What is the electric potential at a point midway between the charges?

21. The three charges in Figure P25.21 are at the vertices of an isosceles triangle. Calculate the electric potential at the midpoint of the base, taking q = 7.00 μC.

22. *Compare this problem with Problem 55 in Chapter 23.* Four identical point charges (q = +10.0 μC) are located on the corners of a rectangle, as shown in Figure P23.55. The dimensions of the rectangle are L = 60.0 cm and W = 15.0 cm. Calculate the electric potential energy of the charge at the lower left corner due to the other three charges.

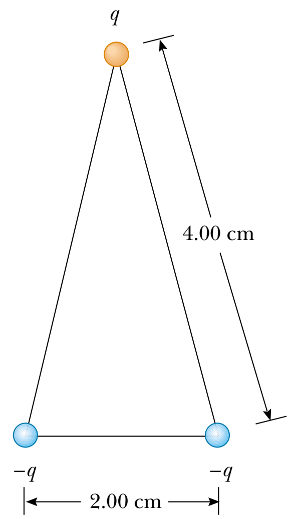


Figure P25.21

WEB 23. Show that the amount of work required to assemble four identical point charges of magnitude Q at the corners of a square of side s is $5.41\, k_e Q^2/s$.

24. *Compare this problem with Problem 18 in Chapter 23.* Two point charges each of magnitude 2.00 μC are located on the x axis. One is at $x = 1.00$ m, and the other is at $x = -1.00$ m. (a) Determine the electric potential on the y axis at $y = 0.500$ m. (b) Calculate the electric potential energy of a third charge, of $-3.00\ \mu$C, placed on the y axis at $y = 0.500$ m.

25. *Compare this problem with Problem 22 in Chapter 23.* Five equal negative point charges $-q$ are placed symmetrically around a circle of radius R. Calculate the electric potential at the center of the circle.

26. *Compare this problem with Problem 17 in Chapter 23.* Three equal positive charges q are at the corners of an equilateral triangle of side a, as shown in Figure P23.17. (a) At what point, if any, in the plane of the charges is the electric potential zero? (b) What is the electric potential at the point P due to the two charges at the base of the triangle?

27. **Review Problem.** Two insulating spheres having radii 0.300 cm and 0.500 cm, masses 0.100 kg and 0.700 kg, and charges $-2.00\ \mu$C and 3.00 μC are released from rest when their centers are separated by 1.00 m. (a) How fast will each be moving when they collide? (*Hint:* Consider conservation of energy and linear momentum.) (b) If the spheres were conductors would the speeds be larger or smaller than those calculated in part (a)? Explain.

28. **Review Problem.** Two insulating spheres having radii r_1 and r_2, masses m_1 and m_2, and charges $-q_1$ and q_2 are released from rest when their centers are separated by a distance d. (a) How fast is each moving when they collide? (*Hint:* Consider conservation of energy and conservation of linear momentum.) (b) If the spheres were conductors, would the speeds be greater or less than those calculated in part (a)?

29. A small spherical object carries a charge of 8.00 nC. At what distance from the center of the object is the potential equal to 100 V? 50.0 V? 25.0 V? Is the spacing of the equipotentials proportional to the change in potential?

30. Two point charges of equal magnitude are located along the y axis equal distances above and below the x axis, as shown in Figure P25.30. (a) Plot a graph of the potential at points along the x axis over the interval $-3a < x < 3a$. You should plot the potential in units of $k_e Q/a$. (b) Let the charge located at $-a$ be negative and plot the potential along the y axis over the interval $-4a < y < 4a$.

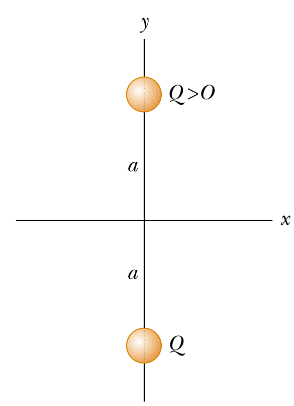


Figure P25.30

31. In Rutherford's famous scattering experiments that led to the planetary model of the atom, alpha particles (charge $+2e$, mass $= 6.64 \times 10^{-27}$ kg) were fired at a gold nucleus (charge $+79e$). An alpha particle, initially very far from the gold nucleus, is fired with a velocity of 2.00×10^7 m/s directly toward the center of the nucleus. How close does the alpha particle get to this center before turning around? Assume the gold nucleus remains stationary.

32. An electron starts from rest 3.00 cm from the center of a uniformly charged insulating sphere of radius 2.00 cm and total charge 1.00 nC. What is the speed of the electron when it reaches the surface of the sphere?

33. Calculate the energy required to assemble the array of charges shown in Figure P25.33, where $a = 0.200$ m, $b = 0.400$ m, and $q = 6.00\ \mu$C.

34. Four identical particles each have charge q and mass m. They are released from rest at the vertices of a square of side L. How fast is each charge moving when their distance from the center of the square doubles?

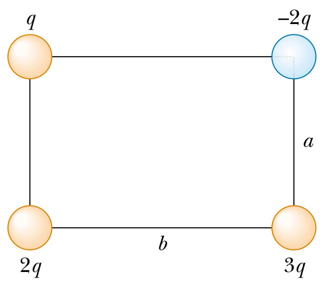


Figure P25.33

35. How much work is required to assemble eight identical point charges, each of magnitude q, at the corners of a cube of side s?

Section 25.4 Obtaining the Value of the Electric Field from the Electric Potential

36. The potential in a region between $x = 0$ and $x = 6.00$ m is $V = a + bx$ where $a = 10.0$ V and $b = -7.00$ V/m. Determine (a) the potential at $x = 0$, 3.00 m, and 6.00 m and (b) the magnitude and direction of the electric field at $x = 0$, 3.00 m, and 6.00 m.

WEB 37. Over a certain region of space, the electric potential is $V = 5x - 3x^2y + 2yz^2$. Find the expressions for the x, y, and z components of the electric field over this region. What is the magnitude of the field at the point P, which has coordinates $(1, 0, -2)$ m?

38. The electric potential inside a charged spherical conductor of radius R is given by $V = k_eQ/R$ and outside the conductor is given by $V = k_eQ/r$. Using $E_r = -dV/dr$, derive the electric field (a) inside and (b) outside this charge distribution.

39. It is shown in Example 25.7 that the potential at a point P a distance a above one end of a uniformly charged rod of length ℓ lying along the x axis is

$$V = \frac{k_eQ}{\ell}\ln\left(\frac{\ell + \sqrt{\ell^2 + a^2}}{a}\right)$$

Use this result to derive an expression for the y component of the electric field at P. (*Hint:* Replace a with y.)

40. When an uncharged conducting sphere of radius a is placed at the origin of an xyz coordinate system that lies in an initially uniform electric field $\mathbf{E} = E_0\mathbf{k}$, the resulting electric potential is

$$V(x, y, z) = V_0 - E_0z + \frac{E_0a^3z}{(x^2 + y^2 + z^2)^{3/2}}$$

for points outside the sphere, where V_0 is the (constant) electric potential on the conductor. Use this equation to determine the x, y, and z components of the resulting electric field.

Section 25.5 Electric Potential Due to Continuous Charge Distributions

41. Consider a ring of radius R with the total charge Q spread uniformly over its perimeter. What is the potential difference between the point at the center of the ring and a point on its axis a distance $2R$ from the center?

42. *Compare this problem with Problem 33 in Chapter 23.* A uniformly charged insulating rod of length 14.0 cm is bent into the shape of a semicircle, as shown in Figure P23.33. If the rod has a total charge of -7.50 μC, find the electric potential at O, the center of the semicircle.

43. A rod of length L (Fig. P25.43) lies along the x axis with its left end at the origin and has a nonuniform charge density $\lambda = \alpha x$ (where α is a positive constant). (a) What are the units of α? (b) Calculate the electric potential at A.

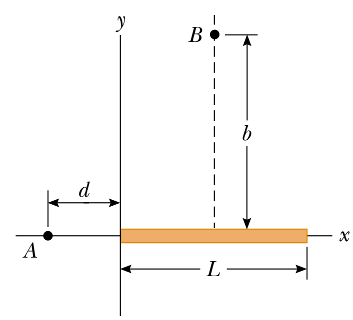


Figure P25.43 Problems 43 and 44.

44. For the arrangement described in the previous problem, calculate the electric potential at point B that lies on the perpendicular bisector of the rod a distance b above the x axis.

45. Calculate the electric potential at point P on the axis of the annulus shown in Figure P25.45, which has a uniform charge density σ.

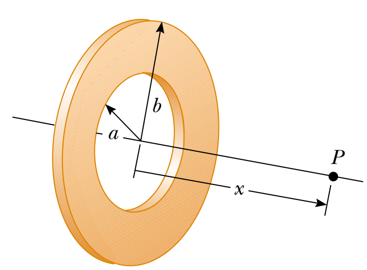


Figure P25.45

46. A wire of finite length that has a uniform linear charge density λ is bent into the shape shown in Figure P25.46. Find the electric potential at point O.

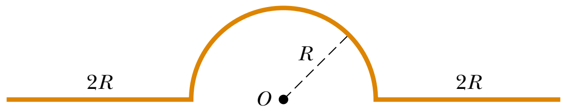


Figure P25.46

Section 25.6 Electric Potential Due to a Charged Conductor

47. How many electrons should be removed from an initially uncharged spherical conductor of radius 0.300 m to produce a potential of 7.50 kV at the surface?

48. Two charged spherical conductors are connected by a long conducting wire, and a charge of 20.0 μC is placed on the combination. (a) If one sphere has a radius of 4.00 cm and the other has a radius of 6.00 cm, what is the electric field near the surface of each sphere? (b) What is the electric potential of each sphere?

WEB **49.** A spherical conductor has a radius of 14.0 cm and charge of 26.0 μC. Calculate the electric field and the electric potential at (a) $r = 10.0$ cm, (b) $r = 20.0$ cm, and (c) $r = 14.0$ cm from the center.

50. Two concentric spherical conducting shells of radii $a = 0.400$ m and $b = 0.500$ m are connected by a thin wire, as shown in Figure P25.50. If a total charge $Q = 10.0$ μC is placed on the system, how much charge settles on each sphere?

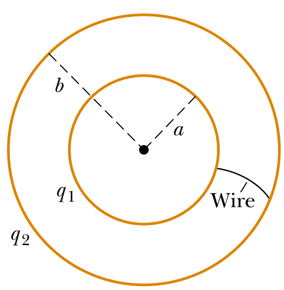


Figure P25.50

(Optional)
Section 25.7 The Millikan Oil-Drop Experiment

(Optional)
Section 25.8 Applications of Electrostatics

51. Consider a Van de Graaff generator with a 30.0-cm-diameter dome operating in dry air. (a) What is the maximum potential of the dome? (b) What is the maximum charge on the dome?

52. The spherical dome of a Van de Graaff generator can be raised to a maximum potential of 600 kV; then additional charge leaks off in sparks, by producing breakdown of the surrounding dry air. Determine (a) the charge on the dome and (b) the radius of the dome.

ADDITIONAL PROBLEMS

53. The liquid-drop model of the nucleus suggests that high-energy oscillations of certain nuclei can split the nucleus into two unequal fragments plus a few neutrons. The fragments acquire kinetic energy from their mutual Coulomb repulsion. Calculate the electric potential energy (in electron volts) of two spherical fragments from a uranium nucleus having the following charges and radii: $38e$ and 5.50×10^{-15} m; $54e$ and 6.20×10^{-15} m. Assume that the charge is distributed uniformly throughout the volume of each spherical fragment and that their surfaces are initially in contact at rest. (The electrons surrounding the nucleus can be neglected.)

54. On a dry winter day you scuff your leather-soled shoes across a carpet and get a shock when you extend the tip of one finger toward a metal doorknob. In a dark room you see a spark perhaps 5 mm long. Make order-of-magnitude estimates of (a) your electric potential and (b) the charge on your body before you touch the doorknob. Explain your reasoning.

55. The charge distribution shown in Figure P25.55 is referred to as a linear quadrupole. (a) Show that the potential at a point on the x axis where $x > a$ is

$$V = \frac{2k_eQa^2}{x^3 - xa^2}$$

(b) Show that the expression obtained in part (a) when $x \gg a$ reduces to

$$V = \frac{2k_eQa^2}{x^3}$$

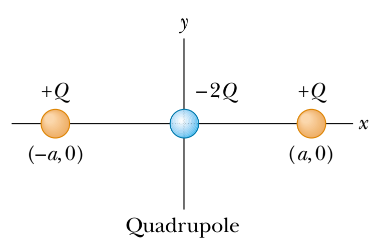


Figure P25.55

56. (a) Use the exact result from Problem 55 to find the electric field at any point along the axis of the linear quadrupole for $x > a$. (b) Evaluate E at $x = 3a$ if $a = 2.00$ mm and $Q = 3.00$ μC.

57. At a certain distance from a point charge, the magnitude of the electric field is 500 V/m and the electric potential is -3.00 kV. (a) What is the distance to the charge? (b) What is the magnitude of the charge?

58. An electron is released from rest on the axis of a uniform positively charged ring, 0.100 m from the ring's

center. If the linear charge density of the ring is $+0.100\ \mu C/m$ and the radius of the ring is 0.200 m, how fast will the electron be moving when it reaches the center of the ring?

59. (a) Consider a uniformly charged cylindrical shell having total charge Q, radius R, and height h. Determine the electrostatic potential at a point a distance d from the right side of the cylinder, as shown in Figure P25.59. (*Hint:* Use the result of Example 25.5 by treating the cylinder as a collection of ring charges.) (b) Use the result of Example 25.6 to solve the same problem for a solid cylinder.

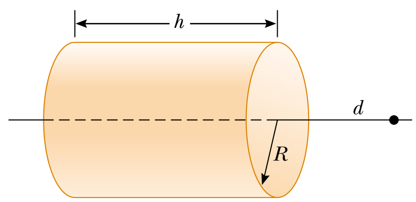


Figure P25.59

60. Two parallel plates having charges of equal magnitude but opposite sign are separated by 12.0 cm. Each plate has a surface charge density of 36.0 nC/m^2. A proton is released from rest at the positive plate. Determine (a) the potential difference between the plates, (b) the energy of the proton when it reaches the negative plate, (c) the speed of the proton just before it strikes the negative plate, (d) the acceleration of the proton, and (e) the force on the proton. (f) From the force, find the magnitude of the electric field and show that it is equal to that found from the charge densities on the plates.

61. Calculate the work that must be done to charge a spherical shell of radius R to a total charge Q.

62. A Geiger–Müller counter is a radiation detector that essentially consists of a hollow cylinder (the cathode) of inner radius r_a and a coaxial cylindrical wire (the anode) of radius r_b (Fig. P25.62). The charge per unit length on the anode is λ, while the charge per unit length on the cathode is $-\lambda$. (a) Show that the magnitude of the potential difference between the wire and the cylinder in the sensitive region of the detector is

$$\Delta V = 2k_e \lambda \ln\left(\frac{r_a}{r_b}\right)$$

(b) Show that the magnitude of the electric field over that region is given by

$$E = \frac{\Delta V}{\ln(r_a/r_b)}\left(\frac{1}{r}\right)$$

where r is the distance from the center of the anode to the point where the field is to be calculated.

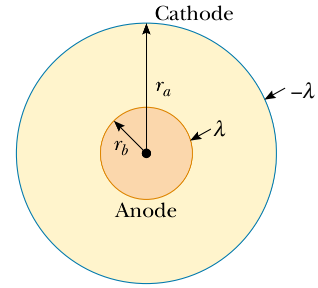


Figure P25.62

WEB **63.** From Gauss's law, the electric field set up by a uniform line of charge is

$$\mathbf{E} = \left(\frac{\lambda}{2\pi\epsilon_0 r}\right)\hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is a unit vector pointing radially away from the line and λ is the charge per unit length along the line. Derive an expression for the potential difference between $r = r_1$ and $r = r_2$.

64. A point charge q is located at $x = -R$, and a point charge $-2q$ is located at the origin. Prove that the equipotential surface that has zero potential is a sphere centered at $(-4R/3, 0, 0)$ and having a radius $r = 2R/3$.

65. Consider two thin, conducting, spherical shells as shown in cross-section in Figure P25.65. The inner shell has a radius $r_1 = 15.0$ cm and a charge of 10.0 nC. The outer shell has a radius $r_2 = 30.0$ cm and a charge of -15.0 nC. Find (a) the electric field $\mathbf{E}$ and (b) the electric potential V in regions A, B, and C, with $V = 0$ at $r = \infty$.

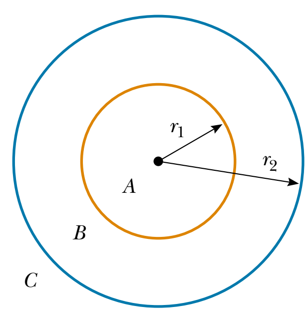


Figure P25.65

66. The x axis is the symmetry axis of a uniformly charged ring of radius R and charge Q (Fig. P25.66). A point charge Q of mass M is located at the center of the ring. When it is displaced slightly, the point charge acceler-

ates along the x axis to infinity. Show that the ultimate speed of the point charge is

$$v = \left(\frac{2k_eQ^2}{MR}\right)^{1/2}$$

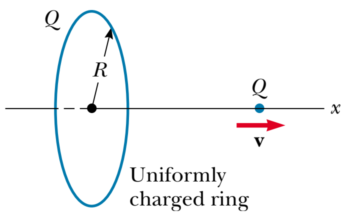


Figure P25.66

67. An infinite sheet of charge that has a surface charge density of 25.0 nC/m^2 lies in the yz plane, passes through the origin, and is at a potential of 1.00 kV at the point $y = 0$, $z = 0$. A long wire having a linear charge density of 80.0 nC/m lies parallel to the y axis and intersects the x axis at $x = 3.00$ m. (a) Determine, as a function of x, the potential along the x axis between wire and sheet. (b) What is the potential energy of a 2.00-nC charge placed at $x = 0.800$ m?

68. The thin, uniformly charged rod shown in Figure P25.68 has a linear charge density λ. Find an expression for the electric potential at P.

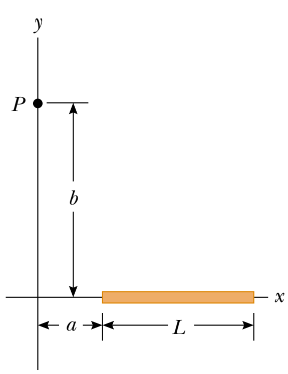


Figure P25.68

69. A dipole is located along the y axis as shown in Figure P25.69. (a) At a point P, which is far from the dipole ($r \gg a$), the electric potential is

$$V = k_e \frac{p\cos\theta}{r^2}$$

where $p = 2qa$. Calculate the radial component E_r and the perpendicular component E_θ of the associated electric field. Note that $E_\theta = -(1/r)(\partial V/\partial\theta)$. Do these results seem reasonable for $\theta = 90°$ and $0°$? for $r = 0$?

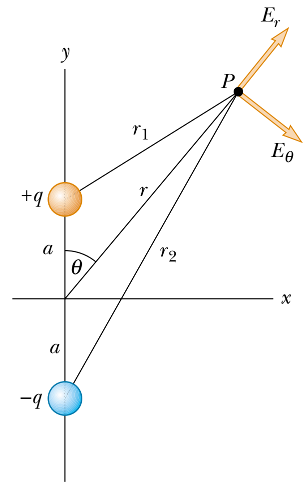


Figure P25.69

(b) For the dipole arrangement shown, express V in terms of cartesian coordinates using $r = (x^2 + y^2)^{1/2}$ and

$$\cos\theta = \frac{y}{(x^2 + y^2)^{1/2}}$$

Using these results and taking $r \gg a$, calculate the field components E_x and E_y.

70. Figure P25.70 shows several equipotential lines each labeled by its potential in volts. The distance between the lines of the square grid represents 1.00 cm. (a) Is the magnitude of the field bigger at A or at B? Why? (b) What is $\mathbf{E}$ at B? (c) Represent what the field looks like by drawing at least eight field lines.

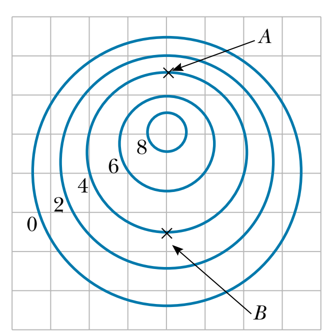


Figure P25.70

71. A disk of radius R has a nonuniform surface charge density $\sigma = Cr$, where C is a constant and r is measured from the center of the disk (Fig. P25.71). Find (by direct integration) the potential at P.

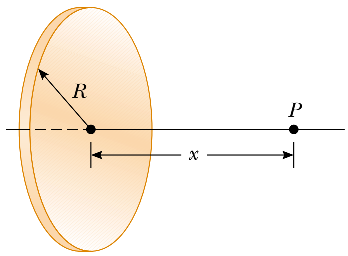


Figure P25.71

72. A solid sphere of radius R has a uniform charge density ρ and total charge Q. Derive an expression for its total electric potential energy. (*Hint:* Imagine that the sphere is constructed by adding successive layers of concentric shells of charge $dq = (4\pi r^2\, dr)\rho$ and use $dU = V\,dq$.)

73. The results of Problem 62 apply also to an electrostatic precipitator (see Figs. 25.28a and P25.62). An applied voltage $\Delta V = V_a - V_b = 50.0$ kV is to produce an electric field of magnitude 5.50 MV/m at the surface of the central wire. The outer cylindrical wall has uniform radius $r_a = 0.850$ m. (a) What should be the radius r_b of the central wire? You will need to solve a transcendental equation. (b) What is the magnitude of the electric field at the outer wall?

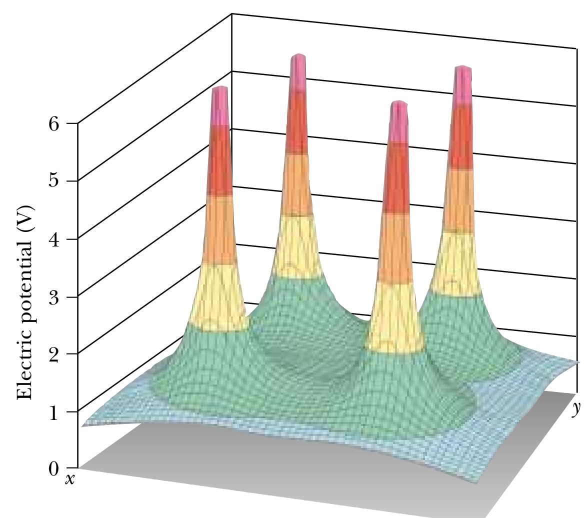
Answers to Quick Quizzes

25.1 We do if the electric field is uniform. (This is precisely what we do in the next section.) In general, however, an electric field changes from one place to another.

25.2 $B \rightarrow C$, $C \rightarrow D$, $A \rightarrow B$, $D \rightarrow E$. Moving from B to C decreases the electric potential by 2 V, so the electric field performs 2 J of work on each coulomb of charge that moves. Moving from C to D decreases the electric potential by 1 V, so 1 J of work is done by the field. It takes no work to move the charge from A to B because the electric potential does not change. Moving from D to E increases the electric potential by 1 V, and thus the field does -1 J of work, just as raising a mass to a higher elevation causes the gravitational field to do negative work on the mass.

25.3 The electric potential decreases in inverse proportion to the radius (see Eq. 25.11). The electric field magnitude decreases as the reciprocal of the radius squared (see Eq. 23.4). Because the surface area increases as r^2 while the electric field magnitude decreases as $1/r^2$, the electric flux through the surface remains constant (see Eq. 24.1).

25.4 (a) Yes. Consider four equal charges placed at the corners of a square. The electric potential graph for this situation is shown in the figure. At the center of the square, the electric field is zero because the individual fields from the four charges cancel, but the potential is not zero. This is also the situation inside a charged conductor. (b) Yes again. In Figure 25.8, for instance, the electric potential is zero at the center of the dipole, but the magnitude of the field at that point is not zero. (The two charges in a dipole are by definition of opposite sign; thus, the electric field lines created by the two charges extend from the positive to the negative charge and do not cancel anywhere.) This is the situation we presented in Example 25.4c, in which the equations we obtained give $V = 0$ and $E_x \neq 0$.



PUZZLER

Many electronic components carry a warning label like this one. What is there inside these devices that makes them so dangerous? Why wouldn't you be safe if you unplugged the equipment before opening the case? *(George Semple)*

chapter

26

Capacitance and Dielectrics

Chapter Outline

In this chapter, we discuss *capacitors*—devices that store electric charge. Capacitors are commonly used in a variety of electric circuits. For instance, they are used to tune the frequency of radio receivers, as filters in power supplies, to eliminate sparking in automobile ignition systems, and as energy-storing devices in electronic flash units.

A capacitor consists of two conductors separated by an insulator. We shall see that the capacitance of a given capacitor depends on its geometry and on the material—called a *dielectric*—that separates the conductors.

26.1 DEFINITION OF CAPACITANCE

13.5

Consider two conductors carrying charges of equal magnitude but of opposite sign, as shown in Figure 26.1. Such a combination of two conductors is called a **capacitor.** The conductors are called *plates.* A potential difference ΔV exists between the conductors due to the presence of the charges. Because the unit of potential difference is the volt, a potential difference is often called a **voltage.** We shall use this term to describe the potential difference across a circuit element or between two points in space.

What determines how much charge is on the plates of a capacitor for a given voltage? In other words, what is the *capacity* of the device for storing charge at a particular value of ΔV? Experiments show that the quantity of charge Q on a capacitor[1] is linearly proportional to the potential difference between the conductors; that is, $Q \propto \Delta V$. The proportionality constant depends on the shape and separation of the conductors.[2] We can write this relationship as $Q = C\,\Delta V$ if we define capacitance as follows:

Definition of capacitance

The **capacitance** C of a capacitor is the ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between them:

$$C \equiv \frac{Q}{\Delta V} \tag{26.1}$$

Note that by definition *capacitance is always a positive quantity.* Furthermore, the potential difference ΔV is always expressed in Equation 26.1 as a positive quantity. Because the potential difference increases linearly with the stored charge, the ratio $Q/\Delta V$ is constant for a given capacitor. Therefore, capacitance is a measure of a capacitor's ability to store charge and electric potential energy.

From Equation 26.1, we see that capacitance has SI units of coulombs per volt. The SI unit of capacitance is the **farad** (F), which was named in honor of Michael Faraday:

$$1 \text{ F} = 1 \text{ C/V}$$

The farad is a very large unit of capacitance. In practice, typical devices have capacitances ranging from microfarads (10^{-6} F) to picofarads (10^{-12} F). For practical purposes, capacitors often are labeled "mF" for microfarads and "mmF" for micromicrofarads or, equivalently, "pF" for picofarads.

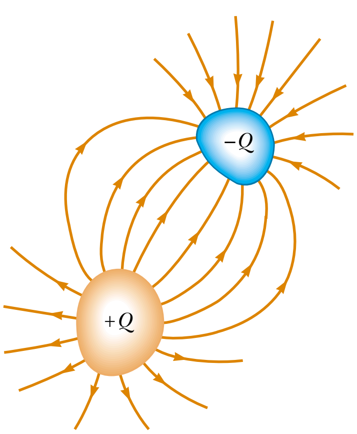


Figure 26.1 A capacitor consists of two conductors carrying charges of equal magnitude but opposite sign.

[1] Although the total charge on the capacitor is zero (because there is as much excess positive charge on one conductor as there is excess negative charge on the other), it is common practice to refer to the magnitude of the charge on either conductor as "the charge on the capacitor."

[2] The proportionality between ΔV and Q can be proved from Coulomb's law or by experiment.

A collection of capacitors used in a variety of applications.

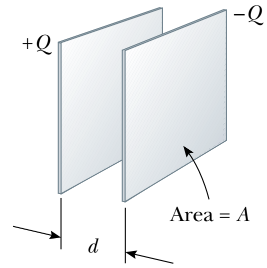


Figure 26.2 A parallel-plate capacitor consists of two parallel conducting plates, each of area A, separated by a distance d. When the capacitor is charged, the plates carry equal amounts of charge. One plate carries positive charge, and the other carries negative charge.

Let us consider a capacitor formed from a pair of parallel plates, as shown in Figure 26.2. Each plate is connected to one terminal of a battery (not shown in Fig. 26.2), which acts as a source of potential difference. If the capacitor is initially uncharged, the battery establishes an electric field in the connecting wires when the connections are made. Let us focus on the plate connected to the negative terminal of the battery. The electric field applies a force on electrons in the wire just outside this plate; this force causes the electrons to move onto the plate. This movement continues until the plate, the wire, and the terminal are all at the same electric potential. Once this equilibrium point is attained, a potential difference no longer exists between the terminal and the plate, and as a result no electric field is present in the wire, and the movement of electrons stops. The plate now carries a negative charge. A similar process occurs at the other capacitor plate, with electrons moving from the plate to the wire, leaving the plate positively charged. In this final configuration, the potential difference across the capacitor plates is the same as that between the terminals of the battery.

Suppose that we have a capacitor rated at 4 pF. This rating means that the capacitor can store 4 pC of charge for each volt of potential difference between the two conductors. If a 9-V battery is connected across this capacitor, one of the conductors ends up with a net charge of -36 pC and the other ends up with a net charge of $+36$ pC.

26.2 CALCULATING CAPACITANCE

We can calculate the capacitance of a pair of oppositely charged conductors in the following manner: We assume a charge of magnitude Q, and we calculate the potential difference using the techniques described in the preceding chapter. We then use the expression $C = Q/\Delta V$ to evaluate the capacitance. As we might expect, we can perform this calculation relatively easily if the geometry of the capacitor is simple.

We can calculate the capacitance of an isolated spherical conductor of radius R and charge Q if we assume that the second conductor making up the capacitor is a concentric hollow sphere of infinite radius. The electric potential of the sphere of radius R is simply k_eQ/R, and setting $V = 0$ at infinity as usual, we have

$$C = \frac{Q}{\Delta V} = \frac{Q}{k_eQ/R} = \frac{R}{k_e} = 4\pi\epsilon_0 R \qquad \textbf{(26.2)}$$

This expression shows that the capacitance of an isolated charged sphere is proportional to its radius and is independent of both the charge on the sphere and the potential difference.

QuickLab

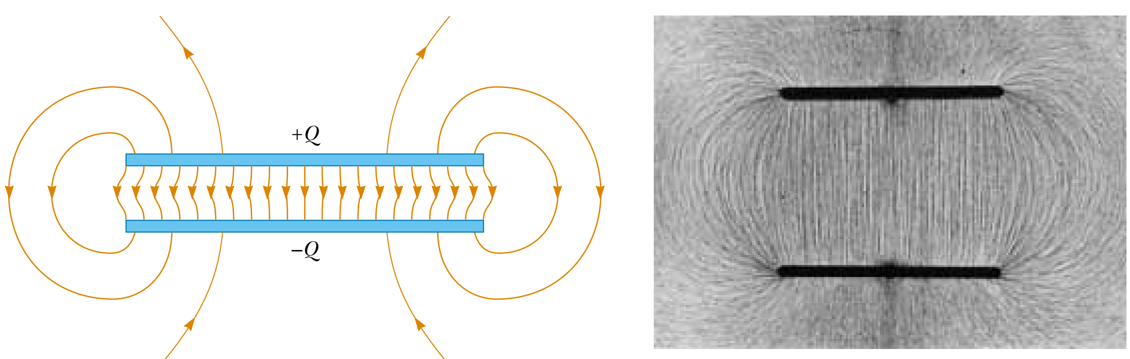
Roll some socks into balls and stuff them into a shoebox. What determines how many socks fit in the box? Relate how hard you push on the socks to ΔV for a capacitor. How does the size of the box influence its "sock capacity"?

The capacitance of a pair of conductors depends on the geometry of the conductors. Let us illustrate this with three familiar geometries, namely, parallel plates, concentric cylinders, and concentric spheres. In these examples, we assume that the charged conductors are separated by a vacuum. The effect of a dielectric material placed between the conductors is treated in Section 26.5.

Parallel-Plate Capacitors

Two parallel metallic plates of equal area A are separated by a distance d, as shown in Figure 26.2. One plate carries a charge Q, and the other carries a charge $-Q$. Let us consider how the geometry of these conductors influences the capacity of the combination to store charge. Recall that charges of like sign repel one another. As a capacitor is being charged by a battery, electrons flow into the negative plate and out of the positive plate. If the capacitor plates are large, the accumulated charges are able to distribute themselves over a substantial area, and the amount of charge that can be stored on a plate for a given potential difference increases as the plate area is increased. Thus, we expect the capacitance to be proportional to the plate area A.

Now let us consider the region that separates the plates. If the battery has a constant potential difference between its terminals, then the electric field between the plates must increase as d is decreased. Let us imagine that we move the plates closer together and consider the situation before any charges have had a chance to move in response to this change. Because no charges have moved, the electric field between the plates has the same value but extends over a shorter distance. Thus, the magnitude of the potential difference between the plates $\Delta V = Ed$ (Eq. 25.6) is now smaller. The difference between this new capacitor voltage and the terminal voltage of the battery now exists as a potential difference across the wires connecting the battery to the capacitor. This potential difference results in an electric field in the wires that drives more charge onto the plates, increasing the potential difference between the plates. When the potential difference between the plates again matches that of the battery, the potential difference across the wires falls back to zero, and the flow of charge stops. Thus, moving the plates closer together causes the charge on the capacitor to increase. If d is increased, the charge decreases. As a result, we expect the device's capacitance to be inversely proportional to d.



(a) (b)

Figure 26.3 (a) The electric field between the plates of a parallel-plate capacitor is uniform near the center but nonuniform near the edges. (b) Electric field pattern of two oppositely charged conducting parallel plates. Small pieces of thread on an oil surface align with the electric field.

We can verify these physical arguments with the following derivation. The surface charge density on either plate is $\sigma = Q/A$. If the plates are very close together (in comparison with their length and width), we can assume that the electric field is uniform between the plates and is zero elsewhere. According to the last paragraph of Example 24.8, the value of the electric field between the plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Because the field between the plates is uniform, the magnitude of the potential difference between the plates equals Ed (see Eq. 25.6); therefore,

$$\Delta V = Ed = \frac{Qd}{\epsilon_0 A}$$

Substituting this result into Equation 26.1, we find that the capacitance is

$$C = \frac{Q}{\Delta V} = \frac{Q}{Qd/\epsilon_0 A}$$

$$\boxed{C = \frac{\epsilon_0 A}{d}} \qquad \textbf{(26.3)}$$

That is, **the capacitance of a parallel-plate capacitor is proportional to the area of its plates and inversely proportional to the plate separation,** just as we expect from our conceptual argument.

A careful inspection of the electric field lines for a parallel-plate capacitor reveals that the field is uniform in the central region between the plates, as shown in Figure 26.3a. However, the field is nonuniform at the edges of the plates. Figure 26.3b is a photograph of the electric field pattern of a parallel-plate capacitor. Note the nonuniform nature of the electric field at the ends of the plates. Such end effects can be neglected if the plate separation is small compared with the length of the plates.

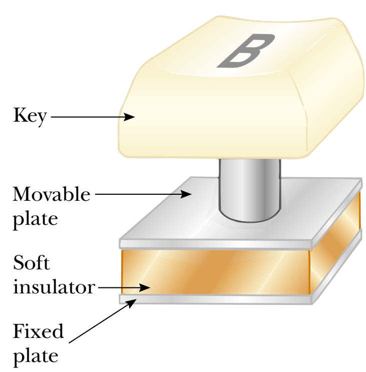


Figure 26.4 One type of computer keyboard button.

Quick Quiz 26.1

Many computer keyboard buttons are constructed of capacitors, as shown in Figure 26.4. When a key is pushed down, the soft insulator between the movable plate and the fixed plate is compressed. When the key is pressed, the capacitance (a) increases, (b) decreases, or (c) changes in a way that we cannot determine because the complicated electric circuit connected to the keyboard button may cause a change in ΔV.

EXAMPLE 26.1 Parallel-Plate Capacitor

A parallel-plate capacitor has an area $A = 2.00 \times 10^{-4}\ \text{m}^2$ and a plate separation $d = 1.00$ mm. Find its capacitance.

Solution From Equation 26.3, we find that

$$C = \epsilon_0 \frac{A}{d} = (8.85 \times 10^{-12}\ \text{C}^2/\text{N}\cdot\text{m}^2)\left(\frac{2.00 \times 10^{-4}\ \text{m}^2}{1.00 \times 10^{-3}\ \text{m}}\right)$$

$$= 1.77 \times 10^{-12}\ \text{F} = \boxed{1.77\ \text{pF}}$$

Exercise What is the capacitance for a plate separation of 3.00 mm?

Answer 0.590 pF.

Cylindrical and Spherical Capacitors

From the definition of capacitance, we can, in principle, find the capacitance of any geometric arrangement of conductors. The following examples demonstrate the use of this definition to calculate the capacitance of the other familiar geometries that we mentioned: cylinders and spheres.

EXAMPLE 26.2 The Cylindrical Capacitor

A solid cylindrical conductor of radius a and charge Q is coaxial with a cylindrical shell of negligible thickness, radius $b > a$, and charge $-Q$ (Fig. 26.5a). Find the capacitance of this cylindrical capacitor if its length is ℓ.

Solution It is difficult to apply physical arguments to this configuration, although we can reasonably expect the capacitance to be proportional to the cylinder length ℓ for the same reason that parallel-plate capacitance is proportional to plate area: Stored charges have more room in which to be distributed. If we assume that ℓ is much greater than a and b, we can neglect end effects. In this case, the electric field is perpendicular to the long axis of the cylinders and is confined to the region between them (Fig. 26.5b). We must first calculate the potential difference between the two cylinders, which is given in general by

$$V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{s}$$

where $\mathbf{E}$ is the electric field in the region $a < r < b$. In Chapter 24, we showed using Gauss's law that the magnitude of the electric field of a cylindrical charge distribution having linear charge density λ is $E_r = 2k_e\lambda/r$ (Eq. 24.7). The same result applies here because, according to Gauss's law, the charge on the outer cylinder does not contribute to the electric field inside it. Using this result and noting from Figure 26.5b that $\mathbf{E}$ is along r, we find that

$$V_b - V_a = -\int_a^b E_r\,dr = -2k_e\lambda \int_a^b \frac{dr}{r} = -2k_e\lambda \ln\left(\frac{b}{a}\right)$$

Substituting this result into Equation 26.1 and using the fact that $\lambda = Q/\ell$, we obtain

$$C = \frac{Q}{\Delta V} = \frac{Q}{\dfrac{2k_eQ}{\ell}\ln\left(\dfrac{b}{a}\right)} = \boxed{\frac{\ell}{2k_e\ln\left(\dfrac{b}{a}\right)}} \qquad \textbf{(26.4)}$$

where ΔV is the magnitude of the potential difference, given by $\Delta V = |V_b - V_a| = 2k_e\lambda \ln\,(b/a)$, a positive quantity. As predicted, the capacitance is proportional to the length of the cylinders. As we might expect, the capacitance also depends on the radii of the two cylindrical conductors. From Equation 26.4, we see that the capacitance per unit length of a combination of concentric cylindrical conductors is

$$\frac{C}{\ell} = \frac{1}{2k_e\ln\left(\dfrac{b}{a}\right)} \qquad \textbf{(26.5)}$$

An example of this type of geometric arrangement is a *coaxial cable,* which consists of two concentric cylindrical conductors separated by an insulator. The cable carries electrical signals in the inner and outer conductors. Such a geometry is especially useful for shielding the signals from any possible external influences.

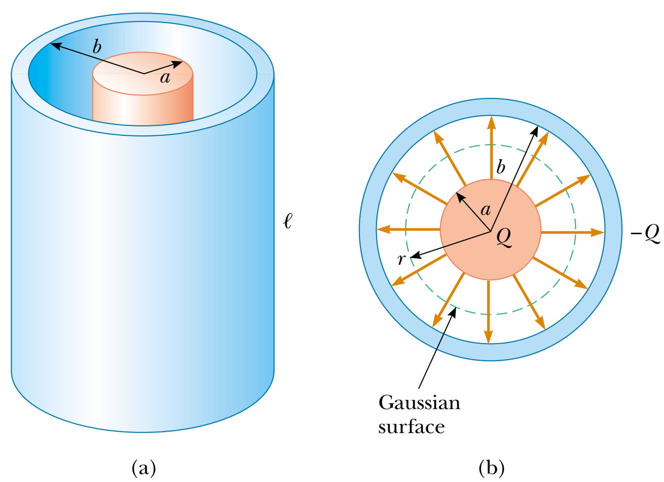


Figure 26.5 (a) A cylindrical capacitor consists of a solid cylindrical conductor of radius a and length ℓ surrounded by a coaxial cylindrical shell of radius b. (b) End view. The dashed line represents the end of the cylindrical gaussian surface of radius r and length ℓ.

EXAMPLE 26.3 The Spherical Capacitor

A spherical capacitor consists of a spherical conducting shell of radius b and charge $-Q$ concentric with a smaller conducting sphere of radius a and charge Q (Fig. 26.6). Find the capacitance of this device.

Solution As we showed in Chapter 24, the field outside a spherically symmetric charge distribution is radial and given by the expression k_eQ/r^2. In this case, this result applies to the field between the spheres ($a < r < b$). From

Gauss's law we see that only the inner sphere contributes to this field. Thus, the potential difference between the spheres is

$$V_b - V_a = -\int_a^b E_r\,dr = -k_eQ\int_a^b \frac{dr}{r^2} = k_eQ\left[\frac{1}{r}\right]_a^b$$
$$= k_eQ\left(\frac{1}{b} - \frac{1}{a}\right)$$

The magnitude of the potential difference is

$$\Delta V = |V_b - V_a| = k_eQ\frac{(b-a)}{ab}$$

Substituting this value for ΔV into Equation 26.1, we obtain

$$C = \frac{Q}{\Delta V} = \boxed{\frac{ab}{k_e(b-a)}} \qquad \textbf{(26.6)}$$

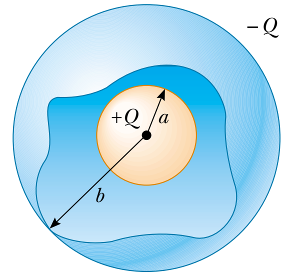


Figure 26.6 A spherical capacitor consists of an inner sphere of radius a surrounded by a concentric spherical shell of radius b. The electric field between the spheres is directed radially outward when the inner sphere is positively charged.

Exercise Show that as the radius b of the outer sphere approaches infinity, the capacitance approaches the value $a/k_e = 4\pi\epsilon_0 a$.

Quick Quiz 26.2

What is the magnitude of the electric field in the region outside the spherical capacitor described in Example 26.3?

26.3 COMBINATIONS OF CAPACITORS

13.5 Two or more capacitors often are combined in electric circuits. We can calculate the equivalent capacitance of certain combinations using methods described in this section. The circuit symbols for capacitors and batteries, as well as the color codes used for them in this text, are given in Figure 26.7. The symbol for the capacitor reflects the geometry of the most common model for a capacitor—a pair of parallel plates. The positive terminal of the battery is at the higher potential and is represented in the circuit symbol by the longer vertical line.

Parallel Combination

Two capacitors connected as shown in Figure 26.8a are known as a *parallel combination* of capacitors. Figure 26.8b shows a circuit diagram for this combination of capacitors. The left plates of the capacitors are connected by a conducting wire to the positive terminal of the battery and are therefore both at the same electric potential as the positive terminal. Likewise, the right plates are connected to the negative terminal and are therefore both at the same potential as the negative terminal. Thus, **the individual potential differences across capacitors connected in parallel are all the same and are equal to the potential difference applied across the combination.**

In a circuit such as that shown in Figure 26.8, the voltage applied across the combination is the terminal voltage of the battery. Situations can occur in which

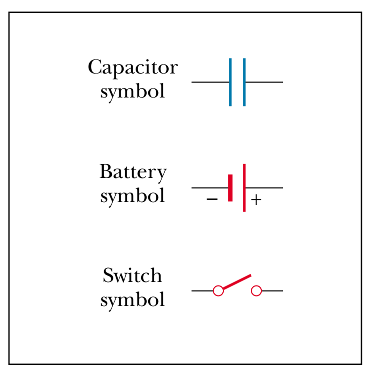


Figure 26.7 Circuit symbols for capacitors, batteries, and switches. Note that capacitors are in blue and batteries and switches are in red.

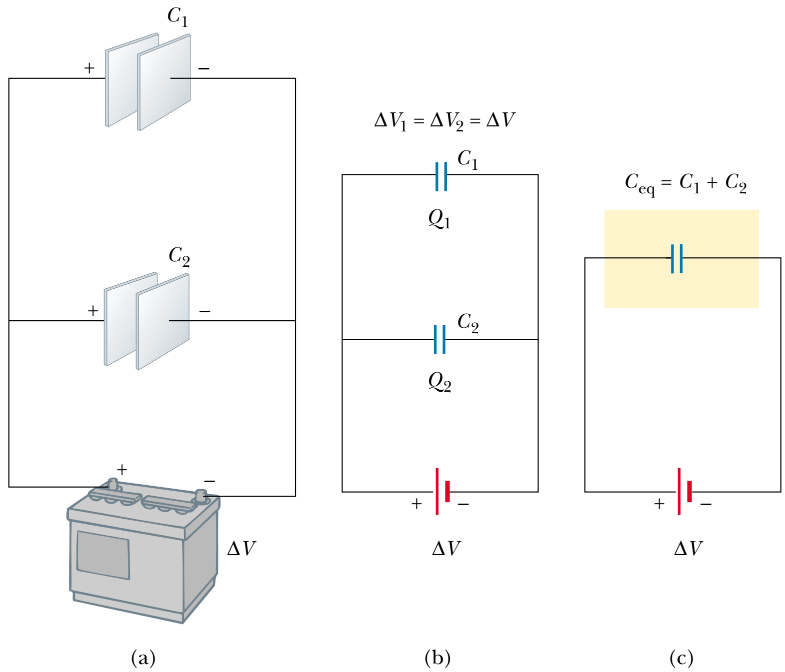


Figure 26.8 (a) A parallel combination of two capacitors in an electric circuit in which the potential difference across the battery terminals is ΔV. (b) The circuit diagram for the parallel combination. (c) The equivalent capacitance is $C_{eq} = C_1 + C_2$.

the parallel combination is in a circuit with other circuit elements; in such situations, we must determine the potential difference across the combination by analyzing the entire circuit.

When the capacitors are first connected in the circuit shown in Figure 26.8, electrons are transferred between the wires and the plates; this transfer leaves the left plates positively charged and the right plates negatively charged. The energy source for this charge transfer is the internal chemical energy stored in the battery, which is converted to electric potential energy associated with the charge separation. The flow of charge ceases when the voltage across the capacitors is equal to that across the battery terminals. The capacitors reach their maximum charge when the flow of charge ceases. Let us call the maximum charges on the two capacitors Q_1 and Q_2. The *total charge* Q stored by the two capacitors is

$$Q = Q_1 + Q_2 \tag{26.7}$$

That is, **the total charge on capacitors connected in parallel is the sum of the charges on the individual capacitors.** Because the voltages across the capacitors are the same, the charges that they carry are

$$Q_1 = C_1\,\Delta V \qquad Q_2 = C_2\,\Delta V$$

Suppose that we wish to replace these two capacitors by one *equivalent capacitor* having a capacitance C_{eq}, as shown in Figure 26.8c. The effect this equivalent capacitor has on the circuit must be exactly the same as the effect of the combination of the two individual capacitors. That is, the equivalent capacitor must store Q units of charge when connected to the battery. We can see from Figure 26.8c that the voltage across the equivalent capacitor also is ΔV because the equivalent capac-

itor is connected directly across the battery terminals. Thus, for the equivalent capacitor,

$$Q = C_{eq}\,\Delta V$$

Substituting these three relationships for charge into Equation 26.7, we have

$$C_{eq}\,\Delta V = C_1\,\Delta V + C_2\,\Delta V$$

$$C_{eq} = C_1 + C_2 \qquad \begin{pmatrix}\text{parallel}\\ \text{combination}\end{pmatrix}$$

If we extend this treatment to three or more capacitors connected in parallel, we find the equivalent capacitance to be

$$C_{eq} = C_1 + C_2 + C_3 + \cdots \qquad \text{(parallel combination)} \tag{26.8}$$

Thus, **the equivalent capacitance of a parallel combination of capacitors is greater than any of the individual capacitances.** This makes sense because we are essentially combining the areas of all the capacitor plates when we connect them with conducting wire.

Series Combination

Two capacitors connected as shown in Figure 26.9a are known as a *series combination* of capacitors. The left plate of capacitor 1 and the right plate of capacitor 2 are connected to the terminals of a battery. The other two plates are connected to each other and to nothing else; hence, they form an isolated conductor that is initially uncharged and must continue to have zero net charge. To analyze this combination, let us begin by considering the uncharged capacitors and follow what happens just after a battery is connected to the circuit. When the battery is con-

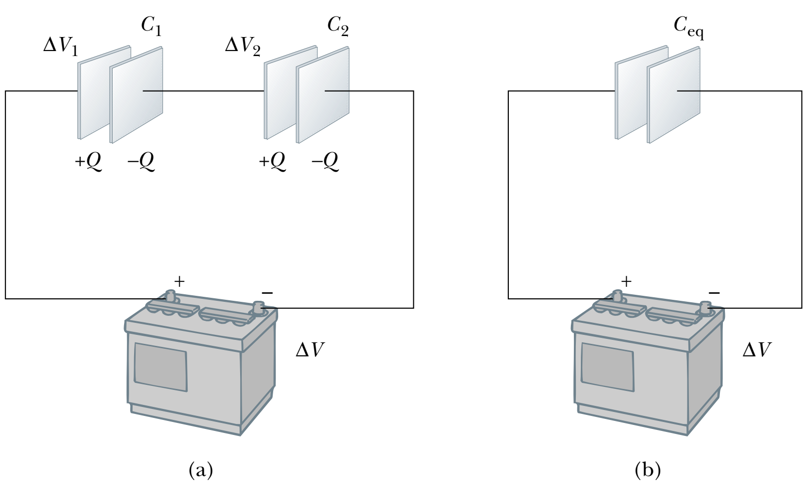


Figure 26.9 (a) A series combination of two capacitors. The charges on the two capacitors are the same. (b) The capacitors replaced by a single equivalent capacitor. The equivalent capacitance can be calculated from the relationship

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

nected, electrons are transferred out of the left plate of C_1 and into the right plate of C_2. As this negative charge accumulates on the right plate of C_2, an equivalent amount of negative charge is forced off the left plate of C_2, and this left plate therefore has an excess positive charge. The negative charge leaving the left plate of C_2 travels through the connecting wire and accumulates on the right plate of C_1. As a result, all the right plates end up with a charge $-Q$, and all the left plates end up with a charge $+Q$. Thus, **the charges on capacitors connected in series are the same.**

From Figure 26.9a, we see that the voltage ΔV across the battery terminals is split between the two capacitors:

$$\Delta V = \Delta V_1 + \Delta V_2 \tag{26.9}$$

where ΔV_1 and ΔV_2 are the potential differences across capacitors C_1 and C_2, respectively. In general, **the total potential difference across any number of capacitors connected in series is the sum of the potential differences across the individual capacitors.**

Suppose that an equivalent capacitor has the same effect on the circuit as the series combination. After it is fully charged, the equivalent capacitor must have a charge of $-Q$ on its right plate and a charge of $+Q$ on its left plate. Applying the definition of capacitance to the circuit in Figure 26.9b, we have

$$\Delta V = \frac{Q}{C_{\text{eq}}}$$

Because we can apply the expression $Q = C\,\Delta V$ to each capacitor shown in Figure 26.9a, the potential difference across each is

$$\Delta V_1 = \frac{Q}{C_1} \qquad \Delta V_2 = \frac{Q}{C_2}$$

Substituting these expressions into Equation 26.9 and noting that $\Delta V = Q/C_{\text{eq}}$, we have

$$\frac{Q}{C_{\text{eq}}} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

Canceling Q, we arrive at the relationship

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} \qquad \left(\begin{array}{l}\text{series}\\ \text{combination}\end{array}\right)$$

When this analysis is applied to three or more capacitors connected in series, the relationship for the equivalent capacitance is

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \qquad \left(\begin{array}{l}\text{series}\\ \text{combination}\end{array}\right) \tag{26.10}$$

This demonstrates that **the equivalent capacitance of a series combination is always less than any individual capacitance in the combination.**

EXAMPLE 26.4 Equivalent Capacitance

Find the equivalent capacitance between a and b for the combination of capacitors shown in Figure 26.10a. All capacitances are in microfarads.

Solution Using Equations 26.8 and 26.10, we reduce the combination step by step as indicated in the figure. The 1.0-μF and 3.0-μF capacitors are in parallel and combine ac-

cording to the expression $C_{eq} = C_1 + C_2 = 4.0\ \mu F$. The 2.0-$\mu$F and 6.0-$\mu$F capacitors also are in parallel and have an equivalent capacitance of 8.0 μF. Thus, the upper branch in Figure 26.10b consists of two 4.0-μF capacitors in series, which combine as follows:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{4.0\ \mu F} + \frac{1}{4.0\ \mu F} = \frac{1}{2.0\ \mu F}$$

$$C_{eq} = \frac{1}{1/2.0\ \mu F} = 2.0\ \mu F$$

The lower branch in Figure 26.10b consists of two 8.0-μF capacitors in series, which combine to yield an equivalent capacitance of 4.0 μF. Finally, the 2.0-μF and 4.0-μF capacitors in Figure 26.10c are in parallel and thus have an equivalent capacitance of 6.0 μF.

Exercise Consider three capacitors having capacitances of 3.0 μF, 6.0 μF, and 12 μF. Find their equivalent capacitance when they are connected (a) in parallel and (b) in series.

Answer (a) 21 μF; (b) 1.7 μF.

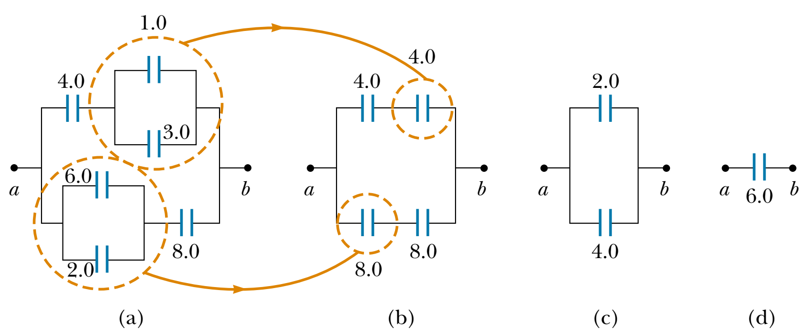


Figure 26.10 To find the equivalent capacitance of the capacitors in part (a), we reduce the various combinations in steps as indicated in parts (b), (c), and (d), using the series and parallel rules described in the text.

26.4 ENERGY STORED IN A CHARGED CAPACITOR

13.5

Almost everyone who works with electronic equipment has at some time verified that a capacitor can store energy. If the plates of a charged capacitor are connected by a conductor, such as a wire, charge moves between the plates and the connecting wire until the capacitor is uncharged. The discharge can often be observed as a visible spark. If you should accidentally touch the opposite plates of a charged capacitor, your fingers act as a pathway for discharge, and the result is an electric shock. The degree of shock you receive depends on the capacitance and on the voltage applied to the capacitor. Such a shock could be fatal if high voltages are present, such as in the power supply of a television set. Because the charges can be stored in a capacitor even when the set is turned off, unplugging the television does not make it safe to open the case and touch the components inside.

Consider a parallel-plate capacitor that is initially uncharged, such that the initial potential difference across the plates is zero. Now imagine that the capacitor is connected to a battery and develops a maximum charge Q. (We assume that the capacitor is charged slowly so that the problem can be considered as an electrostatic system.) When the capacitor is connected to the battery, electrons in the wire just outside the plate connected to the negative terminal move into the plate to give it a negative charge. Electrons in the plate connected to the positive terminal move out of the plate into the wire to give the plate a positive charge. Thus, charges move only a small distance in the wires.

To calculate the energy of the capacitor, we shall assume a different process—one that does not actually occur but gives the same final result. We can make this

QuickLab

Here's how to find out whether your calculator has a capacitor to protect values or programs during battery changes: Store a number in your calculator's memory, remove the calculator battery for a moment, and then quickly replace it. Was the number that you stored preserved while the battery was out of the calculator? (You may want to write down any critical numbers or programs that are stored in the calculator before trying this!)

assumption because the energy in the final configuration does not depend on the actual charge-transfer process. We imagine that we reach in and grab a small amount of positive charge on the plate connected to the negative terminal and apply a force that causes this positive charge to move over to the plate connected to the positive terminal. Thus, we do work on the charge as we transfer it from one plate to the other. At first, no work is required to transfer a small amount of charge dq from one plate to the other.[3] However, once this charge has been transferred, a small potential difference exists between the plates. Therefore, work must be done to move additional charge through this potential difference. As more and more charge is transferred from one plate to the other, the potential difference increases in proportion, and more work is required.

Suppose that q is the charge on the capacitor at some instant during the charging process. At the same instant, the potential difference across the capacitor is $\Delta V = q/C$. From Section 25.2, we know that the work necessary to transfer an increment of charge dq from the plate carrying charge $-q$ to the plate carrying charge q (which is at the higher electric potential) is

$$dW = \Delta V\, dq = \frac{q}{C}\, dq$$

This is illustrated in Figure 26.11. The total work required to charge the capacitor from $q = 0$ to some final charge $q = Q$ is

$$W = \int_0^Q \frac{q}{C}\, dq = \frac{1}{C}\int_0^Q q\, dq = \frac{Q^2}{2C}$$

The work done in charging the capacitor appears as electric potential energy U stored in the capacitor. Therefore, we can express the potential energy stored in a charged capacitor in the following forms:

Energy stored in a charged capacitor

$$U = \frac{Q^2}{2C} = \tfrac{1}{2}Q\,\Delta V = \tfrac{1}{2}C(\Delta V)^2 \qquad \textbf{(26.11)}$$

This result applies to any capacitor, regardless of its geometry. We see that for a given capacitance, the stored energy increases as the charge increases and as the potential difference increases. In practice, there is a limit to the maximum energy

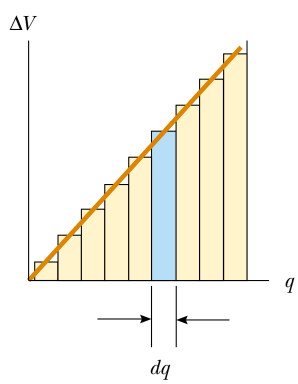


Figure 26.11 A plot of potential difference versus charge for a capacitor is a straight line having a slope $1/C$. The work required to move charge dq through the potential difference ΔV across the capacitor plates is given by the area of the shaded rectangle. The total work required to charge the capacitor to a final charge Q is the triangular area under the straight line, $W = \frac{1}{2}Q\,\Delta V$. (Don't forget that 1 V = 1 J/C; hence, the unit for the area is the joule.)

[3] We shall use lowercase q for the varying charge on the capacitor while it is charging, to distinguish it from uppercase Q, which is the total charge on the capacitor after it is completely charged.

(or charge) that can be stored because, at a sufficiently great value of ΔV, discharge ultimately occurs between the plates. For this reason, capacitors are usually labeled with a maximum operating voltage.

Quick Quiz 26.3

You have three capacitors and a battery. How should you combine the capacitors and the battery in one circuit so that the capacitors will store the maximum possible energy?

We can consider the energy stored in a capacitor as being stored in the electric field created between the plates as the capacitor is charged. This description is reasonable in view of the fact that the electric field is proportional to the charge on the capacitor. For a parallel-plate capacitor, the potential difference is related to the electric field through the relationship $\Delta V = Ed$. Furthermore, its capacitance is $C = \epsilon_0 A/d$ (Eq. 26.3). Substituting these expressions into Equation 26.11, we obtain

$$U = \frac{1}{2}\frac{\epsilon_0 A}{d}(E^2 d^2) = \frac{1}{2}(\epsilon_0 Ad)E^2 \qquad \textbf{(26.12)}$$

Energy stored in a parallel-plate capacitor

Because the volume V (volume, not voltage!) occupied by the electric field is Ad, the *energy per unit volume* $u_E = U/V = U/Ad$, known as the *energy density,* is

$$u_E = \tfrac{1}{2}\epsilon_0 E^2 \qquad \textbf{(26.13)}$$

Energy density in an electric field

Although Equation 26.13 was derived for a parallel-plate capacitor, the expression is generally valid. That is, the **energy density in any electric field is proportional to the square of the magnitude of the electric field at a given point.**

This bank of capacitors stores electrical energy for use in the particle accelerator at FermiLab, located outside Chicago. Because the electric utility company cannot provide a large enough burst of energy to operate the equipment, these capacitors are slowly charged up, and then the energy is rapidly "dumped" into the accelerator. In this sense, the setup is much like a fire-protection water tank on top of a building. The tank collects water and stores it for situations in which a lot of water is needed in a short time.

EXAMPLE 26.5 Rewiring Two Charged Capacitors

Two capacitors C_1 and C_2 (where $C_1 > C_2$) are charged to the same initial potential difference ΔV_i, but with opposite polarity. The charged capacitors are removed from the battery, and their plates are connected as shown in Figure 26.12a. The switches S_1 and S_2 are then closed, as shown in Figure 26.12b. (a) Find the final potential difference ΔV_f between a and b after the switches are closed.

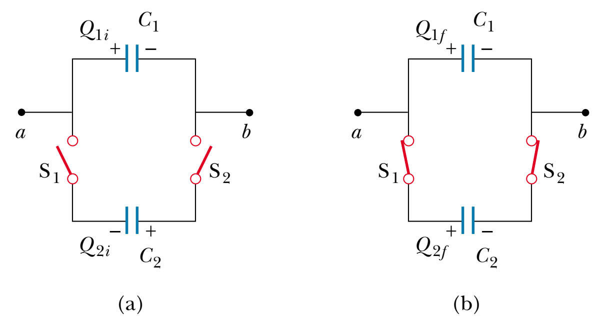


Figure 26.12

Solution Let us identify the left-hand plates of the capacitors as an isolated system because they are not connected to the right-hand plates by conductors. The charges on the left-hand plates before the switches are closed are

$$Q_{1i} = C_1\,\Delta V_i \qquad \text{and} \qquad Q_{2i} = -\,C_2\,\Delta V_i$$

The negative sign for Q_{2i} is necessary because the charge on the left plate of capacitor C_2 is negative. The total charge Q in the system is

$$(1) \qquad Q = Q_{1i} + Q_{2i} = (C_1 - C_2)\,\Delta V_i$$

After the switches are closed, the total charge in the system remains the same:

$$(2) \qquad Q = Q_{1f} + Q_{2f}$$

The charges redistribute until the entire system is at the same potential ΔV_f. Thus, the final potential difference across C_1 must be the same as the final potential difference across C_2. To satisfy this requirement, the charges on the capacitors after the switches are closed are

$$Q_{1f} = C_1\,\Delta V_f \qquad \text{and} \qquad Q_{2f} = C_2\,\Delta V_f$$

Dividing the first equation by the second, we have

$$\frac{Q_{1f}}{Q_{2f}} = \frac{C_1\,\Delta V_f}{C_2\,\Delta V_f} = \frac{C_1}{C_2}$$

$$(3) \qquad Q_{1f} = \frac{C_1}{C_2}\,Q_{2f}$$

Combining Equations (2) and (3), we obtain

$$Q = Q_{1f} + Q_{2f} = \frac{C_1}{C_2}\,Q_{2f} + Q_{2f} = Q_{2f}\left(1 + \frac{C_1}{C_2}\right)$$

$$Q_{2f} = Q\left(\frac{C_2}{C_1 + C_2}\right)$$

Using Equation (3) to find Q_{1f} in terms of Q, we have

$$Q_{1f} = \frac{C_1}{C_2}\,Q_{2f} = \frac{C_1}{C_2}\,Q\left(\frac{C_2}{C_1 + C_2}\right) = Q\left(\frac{C_1}{C_1 + C_2}\right)$$

Finally, using Equation 26.1 to find the voltage across each capacitor, we find that

$$\Delta V_{1f} = \frac{Q_{1f}}{C_1} = \frac{Q\left(\dfrac{C_1}{C_1 + C_2}\right)}{C_1} = \frac{Q}{C_1 + C_2}$$

$$\Delta V_{2f} = \frac{Q_{2f}}{C_2} = \frac{Q\left(\dfrac{C_2}{C_1 + C_2}\right)}{C_2} = \frac{Q}{C_1 + C_2}$$

As noted earlier, $\Delta V_{1f} = \Delta V_{2f} = \Delta V_f$.

To express ΔV_f in terms of the given quantities C_1, C_2, and ΔV_i, we substitute the value of Q from Equation (1) to obtain

$$\boxed{\Delta V_f = \left(\frac{C_1 - C_2}{C_1 + C_2}\right)\Delta V_i}$$

(b) Find the total energy stored in the capacitors before and after the switches are closed and the ratio of the final energy to the initial energy.

Solution Before the switches are closed, the total energy stored in the capacitors is

$$U_i = \tfrac{1}{2}C_1(\Delta V_i)^2 + \tfrac{1}{2}C_2(\Delta V_i)^2 = \boxed{\tfrac{1}{2}\,(C_1 + C_2)(\Delta V_i)^2}$$

After the switches are closed, the total energy stored in the capacitors is

$$U_f = \tfrac{1}{2}C_1(\Delta V_f)^2 + \tfrac{1}{2}C_2(\Delta V_f)^2 = \tfrac{1}{2}\,(C_1 + C_2)(\Delta V_f)^2$$
$$= \frac{1}{2}\,(C_1 + C_2)\left(\frac{Q}{C_1 + C_2}\right)^2 = \frac{1}{2}\,\frac{Q^2}{C_1 + C_2}$$

Using Equation (1), we can express this as

$$U_f = \frac{1}{2}\,\frac{Q^2}{(C_1 + C_2)} = \boxed{\frac{1}{2}\,\frac{(C_1 - C_2)^2(\Delta V_i)^2}{(C_1 + C_2)}}$$

Therefore, the ratio of the final energy stored to the initial energy stored is

$$\frac{U_f}{U_i} = \frac{\dfrac{1}{2}\,\dfrac{(C_1 - C_2)^2(\Delta V_i)^2}{(C_1 + C_2)}}{\dfrac{1}{2}\,(C_1 + C_2)(\Delta V_i)^2} = \boxed{\left(\frac{C_1 - C_2}{C_1 + C_2}\right)^2}$$

This ratio is less than unity, indicating that the final energy is less than the initial energy. At first, you might think that the law of energy conservation has been violated, but this is not the case. The "missing" energy is radiated away in the form of electromagnetic waves, as we shall see in Chapter 34.

Quick Quiz 26.4

You charge a parallel-plate capacitor, remove it from the battery, and prevent the wires connected to the plates from touching each other. When you pull the plates apart, do the following quantities increase, decrease, or stay the same? (a) C; (b) Q; (c) E between the plates; (d) ΔV; (e) energy stored in the capacitor.

Quick Quiz 26.5

Repeat Quick Quiz 26.4, but this time answer the questions for the situation in which the battery remains connected to the capacitor while you pull the plates apart.

One device in which capacitors have an important role is the *defibrillator* (Fig. 26.13). Up to 360 J is stored in the electric field of a large capacitor in a defibrillator when it is fully charged. The defibrillator can deliver all this energy to a patient in about 2 ms. (This is roughly equivalent to 3 000 times the power output of a 60-W lightbulb!) The sudden electric shock stops the fibrillation (random contractions) of the heart that often accompanies heart attacks and helps to restore the correct rhythm.

web

To learn more about defibrillators, visit **www.physiocontrol.com**

A camera's flash unit also uses a capacitor, although the total amount of energy stored is much less than that stored in a defibrillator. After the flash unit's capacitor is charged, tripping the camera's shutter causes the stored energy to be sent through a special lightbulb that briefly illuminates the subject being photographed.

Figure 26.13 In a hospital or at an emergency scene, you might see a patient being revived with a defibrillator. The defibrillator's paddles are applied to the patient's chest, and an electric shock is sent through the chest cavity. The aim of this technique is to restore the heart's normal rhythm pattern.

26.5 CAPACITORS WITH DIELECTRICS

A **dielectric** is a nonconducting material, such as rubber, glass, or waxed paper. When a dielectric is inserted between the plates of a capacitor, the capacitance increases. If the dielectric completely fills the space between the plates, the capacitance increases by a dimensionless factor κ, which is called the **dielectric constant.** The dielectric constant is a property of a material and varies from one material to another. In this section, we analyze this change in capacitance in terms of electrical parameters such as electric charge, electric field, and potential difference; in Section 26.7, we shall discuss the microscopic origin of these changes.

We can perform the following experiment to illustrate the effect of a dielectric in a capacitor: Consider a parallel-plate capacitor that without a dielectric has a charge Q_0 and a capacitance C_0. The potential difference across the capacitor is $\Delta V_0 = Q_0/C_0$. Figure 26.14a illustrates this situation. The potential difference is measured by a *voltmeter,* which we shall study in greater detail in Chapter 28. Note that no battery is shown in the figure; also, we must assume that no charge can flow through an ideal voltmeter, as we shall learn in Section 28.5. Hence, there is no path by which charge can flow and alter the charge on the capacitor. If a dielectric is now inserted between the plates, as shown in Figure 26.14b, the voltmeter indicates that the voltage between the plates decreases to a value ΔV. The voltages with and without the dielectric are related by the factor κ as follows:

$$\Delta V = \frac{\Delta V_0}{\kappa}$$

Because $\Delta V < \Delta V_0$, we see that $\kappa > 1$.

Because the charge Q_0 on the capacitor does not change, we conclude that the capacitance must change to the value

$$C = \frac{Q_0}{\Delta V} = \frac{Q_0}{\Delta V_0/\kappa} = \kappa \frac{Q_0}{\Delta V_0}$$

The capacitance of a filled capacitor is greater than that of an empty one by a factor κ.

$$C = \kappa C_0 \quad \textbf{(26.14)}$$

That is, the capacitance *increases* by the factor κ when the dielectric completely fills the region between the plates.[4] For a parallel-plate capacitor, where $C_0 = \epsilon_0 A/d$ (Eq. 26.3), we can express the capacitance when the capacitor is filled with a dielectric as

$$C = \kappa \frac{\epsilon_0 A}{d} \quad \textbf{(26.15)}$$

From Equations 26.3 and 26.15, it would appear that we could make the capacitance very large by decreasing d, the distance between the plates. In practice, the lowest value of d is limited by the electric discharge that could occur through the dielectric medium separating the plates. For any given separation d, the maximum voltage that can be applied to a capacitor without causing a discharge depends on the **dielectric strength** (maximum electric field) of the dielectric. If the magnitude of the electric field in the dielectric exceeds the dielectric strength, then the insulating properties break down and the dielectric begins to conduct. Insulating materials have values of κ greater than unity and dielectric strengths

[4] If the dielectric is introduced while the potential difference is being maintained constant by a battery, the charge increases to a value $Q = \kappa Q_0$. The additional charge is supplied by the battery, and the capacitance again increases by the factor κ.

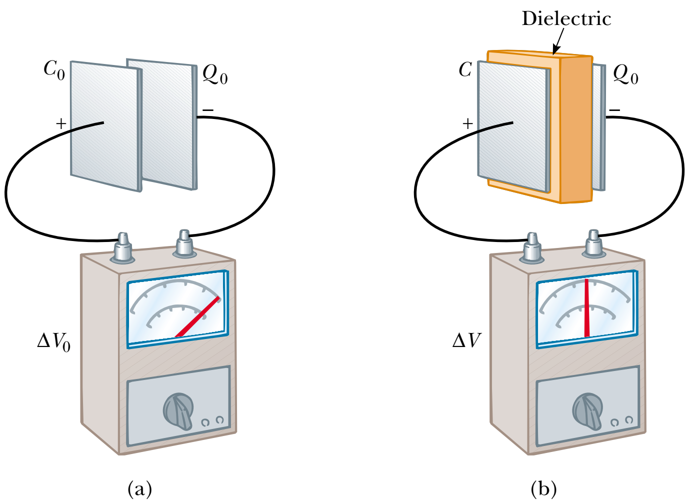


Figure 26.14 A charged capacitor (a) before and (b) after insertion of a dielectric between the plates. The charge on the plates remains unchanged, but the potential difference decreases from ΔV_0 to $\Delta V = \Delta V_0/\kappa$. Thus, the capacitance *increases* from C_0 to κC_0.

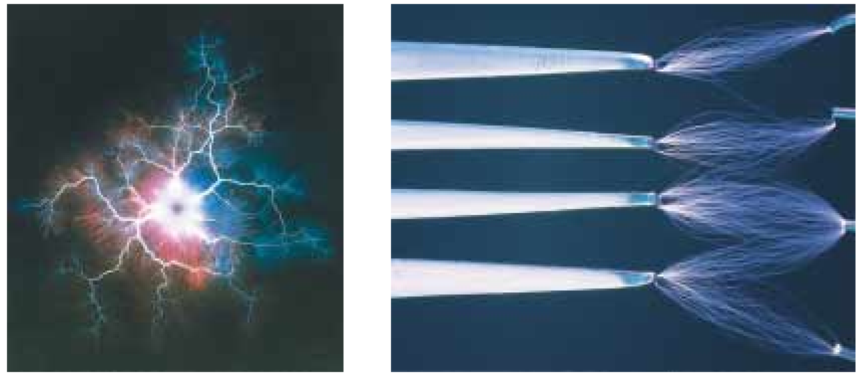
greater than that of air, as Table 26.1 indicates. Thus, we see that a dielectric provides the following advantages:

- Increase in capacitance
- Increase in maximum operating voltage
- Possible mechanical support between the plates, which allows the plates to be close together without touching, thereby decreasing d and increasing C

TABLE 26.1 Dielectric Constants and Dielectric Strengths of Various Materials at Room Temperature

Material	Dielectric Constant κ	Dielectric Strength[a] (V/m)
Air (dry)	1.000 59	3×10^6
Bakelite	4.9	24×10^6
Fused quartz	3.78	8×10^6
Neoprene rubber	6.7	12×10^6
Nylon	3.4	14×10^6
Paper	3.7	16×10^6
Polystyrene	2.56	24×10^6
Polyvinyl chloride	3.4	40×10^6
Porcelain	6	12×10^6
Pyrex glass	5.6	14×10^6
Silicone oil	2.5	15×10^6
Strontium titanate	233	8×10^6
Teflon	2.1	60×10^6
Vacuum	1.000 00	—
Water	80	—

[a] The dielectric strength equals the maximum electric field that can exist in a dielectric without electrical breakdown. Note that these values depend strongly on the presence of impurities and flaws in the materials.



(a) (b)

(a) Kirlian photograph created by dropping a steel ball into a high-energy electric field. Kirlian photography is also known as *electrophotography.* (b) Sparks from static electricity discharge between a fork and four electrodes. Many sparks were used to create this image because only one spark forms for a given discharge. Note that the bottom prong discharges to both electrodes at the bottom right. The light of each spark is created by the excitation of gas atoms along its path.

Types of Capacitors

Commercial capacitors are often made from metallic foil interlaced with thin sheets of either paraffin-impregnated paper or Mylar as the dielectric material. These alternate layers of metallic foil and dielectric are rolled into a cylinder to form a small package (Fig. 26.15a). High-voltage capacitors commonly consist of a number of interwoven metallic plates immersed in silicone oil (Fig. 26.15b). Small capacitors are often constructed from ceramic materials. Variable capacitors (typically 10 to 500 pF) usually consist of two interwoven sets of metallic plates, one fixed and the other movable, and contain air as the dielectric.

Often, an *electrolytic capacitor* is used to store large amounts of charge at relatively low voltages. This device, shown in Figure 26.15c, consists of a metallic foil in contact with an *electrolyte*—a solution that conducts electricity by virtue of the motion of ions contained in the solution. When a voltage is applied between the foil and the electrolyte, a thin layer of metal oxide (an insulator) is formed on the foil,

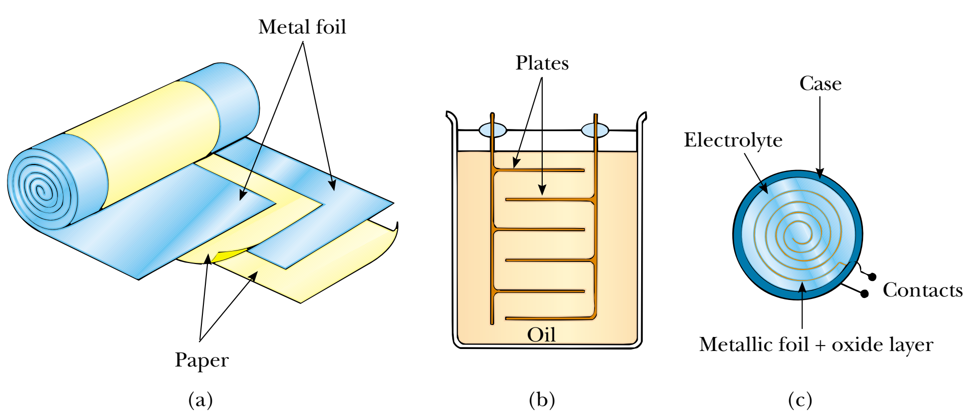


Figure 26.15 Three commercial capacitor designs. (a) A tubular capacitor, whose plates are separated by paper and then rolled into a cylinder. (b) A high-voltage capacitor consisting of many parallel plates separated by insulating oil. (c) An electrolytic capacitor.

and this layer serves as the dielectric. Very large values of capacitance can be obtained in an electrolytic capacitor because the dielectric layer is very thin, and thus the plate separation is very small.

Electrolytic capacitors are not reversible as are many other capacitors—they have a polarity, which is indicated by positive and negative signs marked on the device. When electrolytic capacitors are used in circuits, the polarity must be aligned properly. If the polarity of the applied voltage is opposite that which is intended, the oxide layer is removed and the capacitor conducts electricity instead of storing charge.

Quick Quiz 26.6

If you have ever tried to hang a picture, you know it can be difficult to locate a wooden stud in which to anchor your nail or screw. A carpenter's stud-finder is basically a capacitor with its plates arranged side by side instead of facing one another, as shown in Figure 26.16. When the device is moved over a stud, does the capacitance increase or decrease?

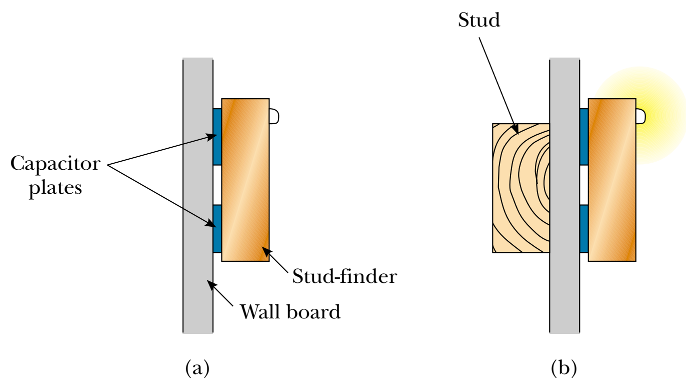


Figure 26.16 A stud-finder. (a) The materials between the plates of the capacitor are the wallboard and air. (b) When the capacitor moves across a stud in the wall, the materials between the plates are the wallboard and the wood. The change in the dielectric constant causes a signal light to illuminate.

Example 26.6 A Paper-Filled Capacitor

A parallel-plate capacitor has plates of dimensions 2.0 cm by 3.0 cm separated by a 1.0-mm thickness of paper. (a) Find its capacitance.

Solution Because $\kappa = 3.7$ for paper (see Table 26.1), we have

$$C = \kappa \frac{\epsilon_0 A}{d} = 3.7(8.85 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2})\left(\frac{6.0 \times 10^{-4}\ \mathrm{m^2}}{1.0 \times 10^{-3}\ \mathrm{m}}\right)$$

$$= 20 \times 10^{-12}\ \mathrm{F} = \boxed{20\ \mathrm{pF}}$$

(b) What is the maximum charge that can be placed on the capacitor?

Solution From Table 26.1 we see that the dielectric strength of paper is 16×10^6 V/m. Because the thickness of the paper is 1.0 mm, the maximum voltage that can be applied before breakdown is

$$\Delta V_{\max} = E_{\max} d = (16 \times 10^6\ \mathrm{V/m})(1.0 \times 10^{-3}\ \mathrm{m})$$
$$= 16 \times 10^3\ \mathrm{V}$$

Hence, the maximum charge is

$$Q_{\max} = C\Delta V_{\max} = (20 \times 10^{-12}\ \mathrm{F})(16 \times 10^3\ \mathrm{V}) = \boxed{0.32\ \mu\mathrm{C}}$$

Exercise What is the maximum energy that can be stored in the capacitor?

Answer 2.6×10^{-3} J.

EXAMPLE 26.7 Energy Stored Before and After

A parallel-plate capacitor is charged with a battery to a charge Q_0, as shown in Figure 26.17a. The battery is then removed, and a slab of material that has a dielectric constant κ is inserted between the plates, as shown in Figure 26.17b. Find the energy stored in the capacitor before and after the dielectric is inserted.

Solution The energy stored in the absence of the dielectric is (see Eq. 26.11):

$$U_0 = \frac{{Q_0}^2}{2C_0}$$

After the battery is removed and the dielectric inserted, the *charge on the capacitor remains the same.* Hence, the energy stored in the presence of the dielectric is

$$U = \frac{{Q_0}^2}{2C}$$

But the capacitance in the presence of the dielectric is $C = \kappa C_0$, so U becomes

$$U = \frac{{Q_0}^2}{2\kappa C_0} = \frac{U_0}{\kappa}$$

Because $\kappa > 1$, the final energy is less than the initial energy. We can account for the "missing" energy by noting that the dielectric, when inserted, gets pulled into the device (see the following discussion and Figure 26.18). An external agent must do negative work to keep the dielectric from accelerating. This work is simply the difference $U - U_0$. (Alternatively, the positive work done by the system on the external agent is $U_0 - U$.)

Exercise Suppose that the capacitance in the absence of a dielectric is 8.50 pF and that the capacitor is charged to a potential difference of 12.0 V. If the battery is disconnected and a slab of polystyrene is inserted between the plates, what is $U_0 - U$?

Answer 373 pJ.

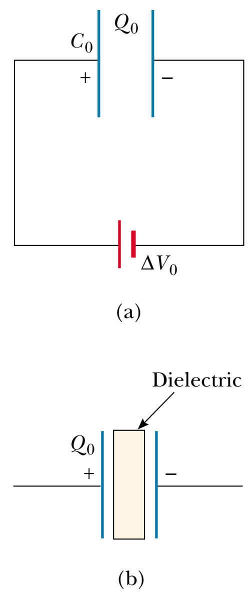


Figure 26.17

As we have seen, the energy of a capacitor not connected to a battery is lowered when a dielectric is inserted between the plates; this means that negative work is done on the dielectric by the external agent inserting the dielectric into the capacitor. This, in turn, implies that a force that draws it into the capacitor must be acting on the dielectric. This force originates from the nonuniform nature of the electric field of the capacitor near its edges, as indicated in Figure 26.18. The horizontal component of this *fringe field* acts on the induced charges on the surface of the dielectric, producing a net horizontal force directed into the space between the capacitor plates.

Quick Quiz 26.7

A fully charged parallel-plate capacitor remains connected to a battery while you slide a dielectric between the plates. Do the following quantities increase, decrease, or stay the same? (a) C; (b) Q; (c) E between the plates; (d) ΔV; (e) energy stored in the capacitor.

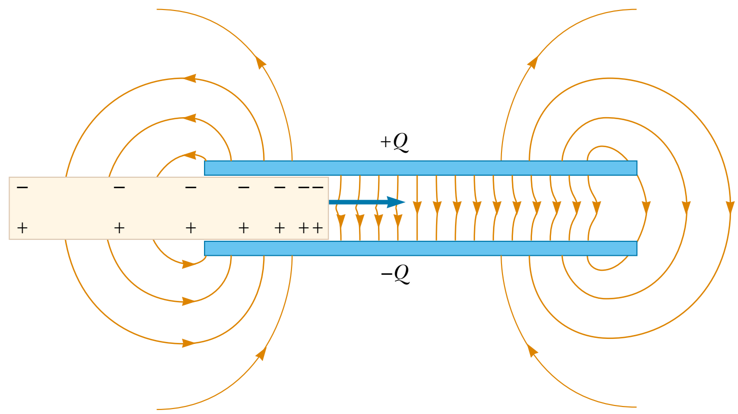


Figure 26.18 The nonuniform electric field near the edges of a parallel-plate capacitor causes a dielectric to be pulled into the capacitor. Note that the field acts on the induced surface charges on the dielectric, which are nonuniformly distributed.

Optional Section

26.6 ELECTRIC DIPOLE IN AN ELECTRIC FIELD

We have discussed the effect on the capacitance of placing a dielectric between the plates of a capacitor. In Section 26.7, we shall describe the microscopic origin of this effect. Before we can do so, however, we need to expand upon the discussion of the electric dipole that we began in Section 23.4 (see Example 23.6). The electric dipole consists of two charges of equal magnitude but opposite sign separated by a distance $2a$, as shown in Figure 26.19. The **electric dipole moment** of this configuration is defined as the vector $\mathbf{p}$ directed from $-q$ to $+q$ along the line joining the charges and having magnitude $2aq$:

$$p \equiv 2aq \tag{26.16}$$

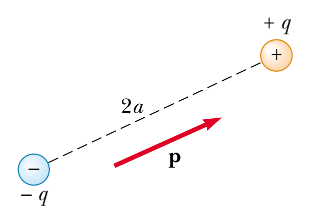


Figure 26.19 An electric dipole consists of two charges of equal magnitude but opposite sign separated by a distance of $2a$. The electric dipole moment $\mathbf{p}$ is directed from $-q$ to $+q$.

Now suppose that an electric dipole is placed in a uniform electric field $\mathbf{E}$, as shown in Figure 26.20. We identify $\mathbf{E}$ as the field *external* to the dipole, distinguishing it from the field *due to* the dipole, which we discussed in Section 23.4. The field $\mathbf{E}$ is established by some other charge distribution, and we place the dipole into this field. Let us imagine that the dipole moment makes an angle θ with the field.

The electric forces acting on the two charges are equal in magnitude but opposite in direction as shown in Figure 26.20 (each has a magnitude $F = qE$). Thus, the net force on the dipole is zero. However, the two forces produce a net torque on the dipole; as a result, the dipole rotates in the direction that brings the dipole moment vector into greater alignment with the field. The torque due to the force on the positive charge about an axis through O in Figure 26.20 is $Fa \sin \theta$, where $a \sin \theta$ is the moment arm of F about O. This force tends to produce a clockwise rotation. The torque about O on the negative charge also is $Fa \sin \theta$; here again, the force tends to produce a clockwise rotation. Thus, the net torque about O is

$$\tau = 2Fa \sin \theta$$

Because $F = qE$ and $p = 2aq$, we can express τ as

$$\tau = 2aqE \sin \theta = pE \sin \theta \tag{26.17}$$

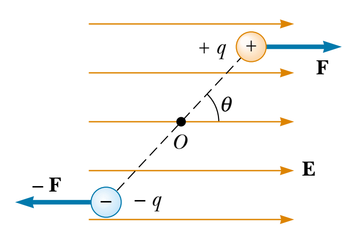


Figure 26.20 An electric dipole in a uniform external electric field. The dipole moment $\mathbf{p}$ is at an angle θ to the field, causing the dipole to experience a torque.

It is convenient to express the torque in vector form as the cross product of the vectors **p** and **E**:

Torque on an electric dipole in an external electric field

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} \tag{26.18}$$

We can determine the potential energy of the system of an electric dipole in an external electric field as a function of the orientation of the dipole with respect to the field. To do this, we recognize that work must be done by an external agent to rotate the dipole through an angle so as to cause the dipole moment vector to become less aligned with the field. The work done is then stored as potential energy in the system of the dipole and the external field. The work dW required to rotate the dipole through an angle $d\theta$ is $dW = \tau\, d\theta$ (Eq. 10.22). Because $\tau = pE \sin\theta$ and because the work is transformed into potential energy U, we find that, for a rotation from θ_i to θ_f, the change in potential energy is

$$U_f - U_i = \int_{\theta_i}^{\theta_f} \tau\, d\theta = \int_{\theta_i}^{\theta_f} pE \sin\theta\, d\theta = pE \int_{\theta_i}^{\theta_f} \sin\theta\, d\theta$$

$$= pE\Big[-\cos\theta\Big]_{\theta_i}^{\theta_f} = pE(\cos\theta_i - \cos\theta_f)$$

The term that contains $\cos\theta_i$ is a constant that depends on the initial orientation of the dipole. It is convenient for us to choose $\theta_i = 90°$, so that $\cos\theta_i = \cos 90° = 0$. Furthermore, let us choose $U_i = 0$ at $\theta_i = 90°$ as our reference of potential energy. Hence, we can express a general value of $U = U_f$ as

$$U = -pE\cos\theta \tag{26.19}$$

We can write this expression for the potential energy of a dipole in an electric field as the dot product of the vectors **p** and **E**:

Potential energy of a dipole in an electric field

$$U = -\mathbf{p}\cdot\mathbf{E} \tag{26.20}$$

To develop a conceptual understanding of Equation 26.19, let us compare this expression with the expression for the potential energy of an object in the gravitational field of the Earth, $U = mgh$ (see Chapter 8). The gravitational expression includes a parameter associated with the object we place in the field—its mass m. Likewise, Equation 26.19 includes a parameter of the object in the electric field—its dipole moment p. The gravitational expression includes the magnitude of the gravitational field g. Similarly, Equation 26.19 includes the magnitude of the electric field E. So far, these two contributions to the potential energy expressions appear analogous. However, the final contribution is somewhat different in the two cases. In the gravitational expression, the potential energy depends on how high we lift the object, measured by h. In Equation 26.19, the potential energy depends on the angle θ through which we rotate the dipole. In both cases, we are making a change in the system. In the gravitational case, the change involves moving an object in a *translational* sense, whereas in the electrical case, the change involves moving an object in a *rotational* sense. In both cases, however, once the change is made, the system tends to return to the original configuration when the object is released: the object of mass m falls back to the ground, and the dipole begins to rotate back toward the configuration in which it was aligned with the field. Thus, apart from the type of motion, the expressions for potential energy in these two cases are similar.

Molecules are said to be *polarized* when a separation exists between the average position of the negative charges and the average position of the positive charges in the molecule. In some molecules, such as water, this condition is always present—such molecules are called **polar molecules.** Molecules that do not possess a permanent polarization are called **nonpolar molecules.**

We can understand the permanent polarization of water by inspecting the geometry of the water molecule. In the water molecule, the oxygen atom is bonded to the hydrogen atoms such that an angle of 105° is formed between the two bonds (Fig. 26.21). The center of the negative charge distribution is near the oxygen atom, and the center of the positive charge distribution lies at a point midway along the line joining the hydrogen atoms (the point labeled × in Fig. 26.21). We can model the water molecule and other polar molecules as dipoles because the average positions of the positive and negative charges act as point charges. As a result, we can apply our discussion of dipoles to the behavior of polar molecules.

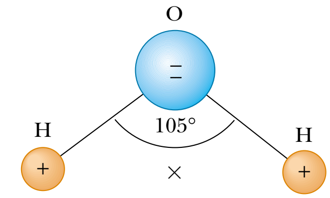


Figure 26.21 The water molecule, H_2O, has a permanent polarization resulting from its bent geometry. The center of the positive charge distribution is at the point ×.

Microwave ovens take advantage of the polar nature of the water molecule. When in operation, microwave ovens generate a rapidly changing electric field that causes the polar molecules to swing back and forth, absorbing energy from the field in the process. Because the jostling molecules collide with each other, the energy they absorb from the field is converted to internal energy, which corresponds to an increase in temperature of the food.

Another household scenario in which the dipole structure of water is exploited is washing with soap and water. Grease and oil are made up of nonpolar molecules, which are generally not attracted to water. Plain water is not very useful for removing this type of grime. Soap contains long molecules called *surfactants.* In a long molecule, the polarity characteristics of one end of the molecule can be different from those at the other end. In a surfactant molecule, one end acts like a nonpolar molecule and the other acts like a polar molecule. The nonpolar end can attach to a grease or oil molecule, and the polar end can attach to a water molecule. Thus, the soap serves as a chain, linking the dirt and water molecules together. When the water is rinsed away, the grease and oil go with it.

A symmetric molecule (Fig. 26.22a) has no permanent polarization, but polarization can be induced by placing the molecule in an electric field. A field directed to the left, as shown in Figure 26.22b, would cause the center of the positive charge distribution to shift to the left from its initial position and the center of the negative charge distribution to shift to the right. This *induced polarization* is the effect that predominates in most materials used as dielectrics in capacitors.

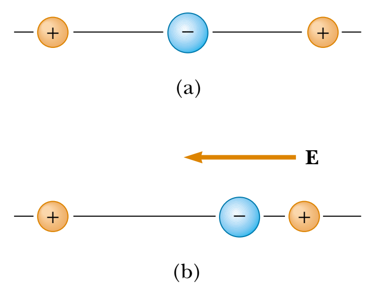


Figure 26.22 (a) A symmetric molecule has no permanent polarization. (b) An external electric field induces a polarization in the molecule.

EXAMPLE 26.8 The H_2O Molecule

The water (H_2O) molecule has an electric dipole moment of 6.3×10^{-30} C·m. A sample contains 10^{21} water molecules, with the dipole moments all oriented in the direction of an electric field of magnitude 2.5×10^5 N/C. How much work is required to rotate the dipoles from this orientation ($\theta = 0°$) to one in which all the dipole moments are perpendicular to the field ($\theta = 90°$)?

Solution The work required to rotate one molecule 90° is equal to the difference in potential energy between the 90° orientation and the 0° orientation. Using Equation 26.19, we obtain

$$\begin{aligned} W &= U_{90} - U_0 = (-pE\cos 90°) - (-pE\cos 0°) \\ &= pE = (6.3 \times 10^{-30}\ \mathrm{C\cdot m})(2.5 \times 10^5\ \mathrm{N/C}) \\ &= 1.6 \times 10^{-24}\ \mathrm{J} \end{aligned}$$

Because there are 10^{21} molecules in the sample, the *total* work required is

$$W_{\text{total}} = (10^{21})(1.6 \times 10^{-24}\ \mathrm{J}) = \boxed{1.6 \times 10^{-3}\ \mathrm{J}}$$

Optional Section

26.7 AN ATOMIC DESCRIPTION OF DIELECTRICS

In Section 26.5 we found that the potential difference ΔV_0 between the plates of a capacitor is reduced to $\Delta V_0/\kappa$ when a dielectric is introduced. Because the potential difference between the plates equals the product of the electric field and the separation d, the electric field is also reduced. Thus, if $\mathbf{E}_0$ is the electric field without the dielectric, the field in the presence of a dielectric is

$$\mathbf{E} = \frac{\mathbf{E}_0}{\kappa} \tag{26.21}$$

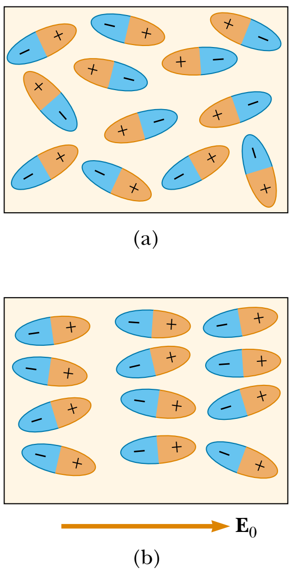


Figure 26.23 (a) Polar molecules are randomly oriented in the absence of an external electric field. (b) When an external field is applied, the molecules partially align with the field.

Let us first consider a dielectric made up of polar molecules placed in the electric field between the plates of a capacitor. The dipoles (that is, the polar molecules making up the dielectric) are randomly oriented in the absence of an electric field, as shown in Figure 26.23a. When an external field $\mathbf{E}_0$ due to charges on the capacitor plates is applied, a torque is exerted on the dipoles, causing them to partially align with the field, as shown in Figure 26.23b. We can now describe the dielectric as being polarized. The degree of alignment of the molecules with the electric field depends on temperature and on the magnitude of the field. In general, the alignment increases with decreasing temperature and with increasing electric field.

If the molecules of the dielectric are nonpolar, then the electric field due to the plates produces some charge separation and an *induced dipole moment.* These induced dipole moments tend to align with the external field, and the dielectric is polarized. Thus, we can polarize a dielectric with an external field regardless of whether the molecules are polar or nonpolar.

With these ideas in mind, consider a slab of dielectric material placed between the plates of a capacitor so that it is in a uniform electric field $\mathbf{E}_0$, as shown in Figure 26.24a. The electric field due to the plates is directed to the right and polarizes the dielectric. The net effect on the dielectric is the formation of an *induced* positive surface charge density σ_{ind} on the right face and an equal negative surface charge density $-\sigma_{\text{ind}}$ on the left face, as shown in Figure 26.24b. These induced surface charges on the dielectric give rise to an induced electric field $\mathbf{E}_{\text{ind}}$ in the direction opposite the external field $\mathbf{E}_0$. Therefore, the net electric field $\mathbf{E}$ in the

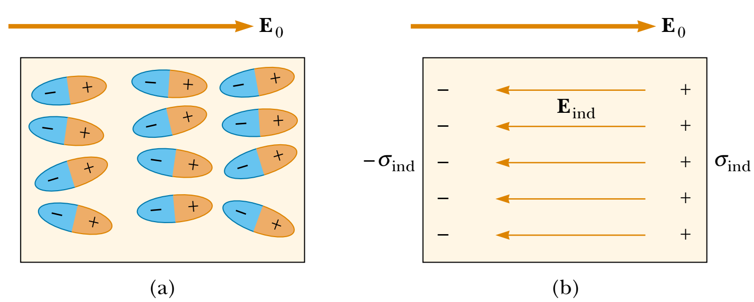


Figure 26.24 (a) When a dielectric is polarized, the dipole moments of the molecules in the dielectric are partially aligned with the external field $\mathbf{E}_0$. (b) This polarization causes an induced negative surface charge on one side of the dielectric and an equal induced positive surface charge on the opposite side. This separation of charge results in a reduction in the net electric field within the dielectric.

dielectric has a magnitude

$$E = E_0 - E_{\text{ind}} \tag{26.22}$$

In the parallel-plate capacitor shown in Figure 26.25, the external field E_0 is related to the charge density σ on the plates through the relationship $E_0 = \sigma/\epsilon_0$. The induced electric field in the dielectric is related to the induced charge density σ_{ind} through the relationship $E_{\text{ind}} = \sigma_{\text{ind}}/\epsilon_0$. Because $E = E_0/\kappa = \sigma/\kappa\epsilon_0$, substitution into Equation 26.22 gives

$$\frac{\sigma}{\kappa\epsilon_0} = \frac{\sigma}{\epsilon_0} - \frac{\sigma_{\text{ind}}}{\epsilon_0}$$

$$\sigma_{\text{ind}} = \left(\frac{\kappa - 1}{\kappa}\right)\sigma \tag{26.23}$$

Because $\kappa > 1$, this expression shows that the charge density σ_{ind} induced on the dielectric is less than the charge density σ on the plates. For instance, if $\kappa = 3$, we see that the induced charge density is two-thirds the charge density on the plates. If no dielectric is present, then $\kappa = 1$ and $\sigma_{\text{ind}} = 0$ as expected. However, if the dielectric is replaced by an electrical conductor, for which $E = 0$, then Equation 26.22 indicates that $E_0 = E_{\text{ind}}$; this corresponds to $\sigma_{\text{ind}} = \sigma$. That is, the surface charge induced on the conductor is equal in magnitude but opposite in sign to that on the plates, resulting in a net electric field of zero in the conductor.

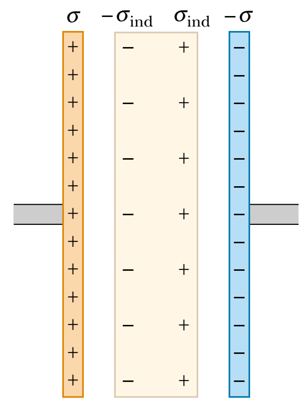


Figure 26.25 Induced charge on a dielectric placed between the plates of a charged capacitor. Note that the induced charge density on the dielectric is *less* than the charge density on the plates.

EXAMPLE 26.9 Effect of a Metallic Slab

A parallel-plate capacitor has a plate separation d and plate area A. An uncharged metallic slab of thickness a is inserted midway between the plates. (a) Find the capacitance of the device.

Solution We can solve this problem by noting that any charge that appears on one plate of the capacitor must induce a charge of equal magnitude but opposite sign on the near side of the slab, as shown in Figure 26.26a. Consequently, the net charge on the slab remains zero, and the electric field inside the slab is zero. Hence, the capacitor is equivalent to two capacitors in series, each having a plate separation $(d - a)/2$, as shown in Figure 26.26b.

Using the rule for adding two capacitors in series (Eq. 26.10), we obtain

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{\dfrac{\epsilon_0 A}{(d-a)/2}} + \frac{1}{\dfrac{\epsilon_0 A}{(d-a)/2}}$$

$$\boxed{C = \frac{\epsilon_0 A}{d - a}}$$

Note that C approaches infinity as a approaches d. Why?

(b) Show that the capacitance is unaffected if the metallic slab is infinitesimally thin.

Solution In the result for part (a), we let $a \rightarrow 0$:

$$C = \lim_{a \to 0} \frac{\epsilon_0 A}{d - a} = \frac{\epsilon_0 A}{d}$$

which is the original capacitance.

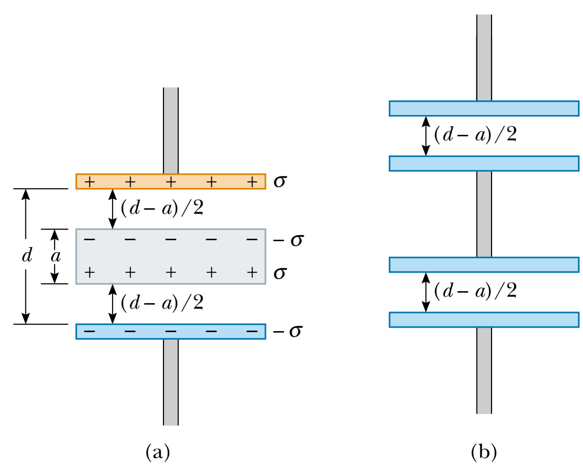


Figure 26.26 (a) A parallel-plate capacitor of plate separation d partially filled with a metallic slab of thickness a. (b) The equivalent circuit of the device in part (a) consists of two capacitors in series, each having a plate separation $(d - a)/2$.

(c) Show that the answer to part (a) does not depend on where the slab is inserted.

Solution Let us imagine that the slab in Figure 26.26a is moved upward so that the distance between the upper edge of the slab and the upper plate is b. Then, the distance between the lower edge of the slab and the lower plate is $d - b - a$. As in part (a), we find the total capacitance of the series combination:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{1}{\frac{\epsilon_0 A}{b}} + \frac{1}{\frac{\epsilon_0 A}{d-b-a}}$$

$$= \frac{b}{\epsilon_0 A} + \frac{d-b-a}{\epsilon_0 A} = \frac{d-a}{\epsilon_0 A}$$

$$C = \frac{\epsilon_0 A}{d-a}$$

This is the same result as in part (a). It is independent of the value of b, so it does not matter where the slab is located.

EXAMPLE 26.10 A Partially Filled Capacitor

A parallel-plate capacitor with a plate separation d has a capacitance C_0 in the absence of a dielectric. What is the capacitance when a slab of dielectric material of dielectric constant κ and thickness $\frac{1}{3}d$ is inserted between the plates (Fig. 26.27a)?

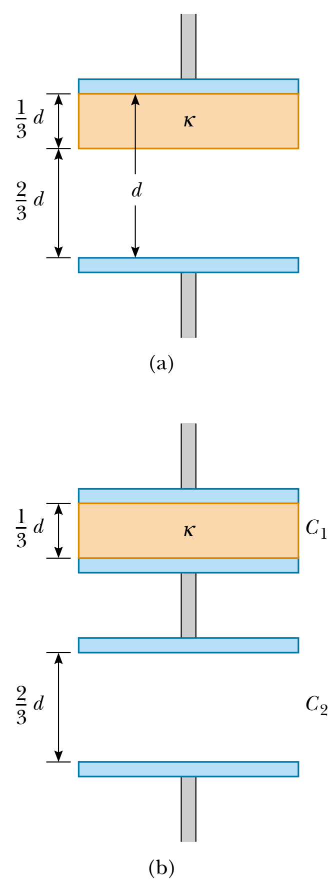


Figure 26.27 (a) A parallel-plate capacitor of plate separation d partially filled with a dielectric of thickness $d/3$. (b) The equivalent circuit of the capacitor consists of two capacitors connected in series.

Solution In Example 26.9, we found that we could insert a metallic slab between the plates of a capacitor and consider the combination as two capacitors in series. The resulting capacitance was independent of the location of the slab. Furthermore, if the thickness of the slab approaches zero, then the capacitance of the system approaches the capacitance when the slab is absent. From this, we conclude that we can insert an infinitesimally thin metallic slab anywhere between the plates of a capacitor without affecting the capacitance. Thus, let us imagine sliding an infinitesimally thin metallic slab along the bottom face of the dielectric shown in Figure 26.27a. We can then consider this system to be the series combination of the two capacitors shown in Figure 26.27b: one having a plate separation $d/3$ and filled with a dielectric, and the other having a plate separation $2d/3$ and air between its plates.

From Equations 26.15 and 26.3, the two capacitances are

$$C_1 = \frac{\kappa\epsilon_0 A}{d/3} \quad \text{and} \quad C_2 = \frac{\epsilon_0 A}{2d/3}$$

Using Equation 26.10 for two capacitors combined in series, we have

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{d/3}{\kappa\epsilon_0 A} + \frac{2d/3}{\epsilon_0 A}$$

$$= \frac{d}{3\epsilon_0 A}\left(\frac{1}{\kappa} + 2\right) = \frac{d}{3\epsilon_0 A}\left(\frac{1+2\kappa}{\kappa}\right)$$

$$C = \left(\frac{3\kappa}{2\kappa+1}\right)\frac{\epsilon_0 A}{d}$$

Because the capacitance without the dielectric is $C_0 = \epsilon_0 A/d$, we see that

$$\boxed{C = \left(\frac{3\kappa}{2\kappa+1}\right) C_0}$$

SUMMARY

A **capacitor** consists of two conductors carrying charges of equal magnitude but opposite sign. The **capacitance** C of any capacitor is the ratio of the charge Q on either conductor to the potential difference ΔV between them:

$$C \equiv \frac{Q}{\Delta V} \tag{26.1}$$

This relationship can be used in situations in which any two of the three variables are known. It is important to remember that this ratio is constant for a given configuration of conductors because the capacitance depends only on the geometry of the conductors and not on an external source of charge or potential difference.

The SI unit of capacitance is coulombs per volt, or the **farad** (F), and 1 F = 1 C/V.

Capacitance expressions for various geometries are summarized in Table 26.2.

If two or more capacitors are connected in parallel, then the potential difference is the same across all of them. The equivalent capacitance of a parallel combination of capacitors is

$$C_{\text{eq}} = C_1 + C_2 + C_3 + \cdots \tag{26.8}$$

If two or more capacitors are connected in series, the charge is the same on all of them, and the equivalent capacitance of the series combination is given by

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots \tag{26.10}$$

These two equations enable you to simplify many electric circuits by replacing multiple capacitors with a single equivalent capacitance.

Work is required to charge a capacitor because the charging process is equivalent to the transfer of charges from one conductor at a lower electric potential to another conductor at a higher potential. The work done in charging the capacitor to a charge Q equals the electric potential energy U stored in the capacitor, where

$$U = \frac{Q^2}{2C} = \tfrac{1}{2}Q\,\Delta V = \tfrac{1}{2}C(\Delta V)^2 \tag{26.11}$$

TABLE 26.2 Capacitance and Geometry

Geometry	Capacitance	Equation
Isolated charged sphere of radius R (second charged conductor assumed at infinity)	$C = 4\pi\epsilon_0 R$	26.2
Parallel-plate capacitor of plate area A and plate separation d	$C = \epsilon_0 \frac{A}{d}$	26.3
Cylindrical capacitor of length ℓ and inner and outer radii a and b, respectively	$C = \frac{\ell}{2k_e \ln\left(\frac{b}{a}\right)}$	26.4
Spherical capacitor with inner and outer radii a and b, respectively	$C = \frac{ab}{k_e(b-a)}$	26.6

When a dielectric material is inserted between the plates of a capacitor, the capacitance increases by a dimensionless factor κ, called the **dielectric constant:**

$$C = \kappa C_0 \tag{26.14}$$

where C_0 is the capacitance in the absence of the dielectric. The increase in capacitance is due to a decrease in the magnitude of the electric field in the presence of the dielectric and to a corresponding decrease in the potential difference between the plates—if we assume that the charging battery is removed from the circuit before the dielectric is inserted. The decrease in the magnitude of **E** arises from an internal electric field produced by aligned dipoles in the dielectric. This internal field produced by the dipoles opposes the applied field due to the capacitor plates, and the result is a reduction in the net electric field.

The **electric dipole moment p** of an electric dipole has a magnitude

$$p \equiv 2aq \tag{26.16}$$

The direction of the electric dipole moment vector is from the negative charge toward the positive charge.

The torque acting on an electric dipole in a uniform electric field **E** is

$$\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E} \tag{26.18}$$

The potential energy of an electric dipole in a uniform external electric field **E** is

$$U = -\mathbf{p} \cdot \mathbf{E} \tag{26.20}$$

Problem-Solving Hints

Capacitors

- Be careful with units. When you calculate capacitance in farads, make sure that distances are expressed in meters and that you use the SI value of ϵ_0. When checking consistency of units, remember that the unit for electric fields can be either N/C or V/m.
- When two or more capacitors are connected in parallel, the potential difference across each is the same. The charge on each capacitor is proportional to its capacitance; hence, the capacitances can be added directly to give the equivalent capacitance of the parallel combination. The equivalent capacitance is always larger than the individual capacitances.
- When two or more capacitors are connected in series, they carry the same charge, and the sum of the potential differences equals the total potential difference applied to the combination. The sum of the reciprocals of the capacitances equals the reciprocal of the equivalent capacitance, which is always less than the capacitance of the smallest individual capacitor.
- A dielectric increases the capacitance of a capacitor by a factor κ (the dielectric constant) over its capacitance when air is between the plates.
- For problems in which a battery is being connected or disconnected, note whether modifications to the capacitor are made while it is connected to the battery or after it has been disconnected. If the capacitor remains connected to the battery, the voltage across the capacitor remains unchanged (equal to the battery voltage), and the charge is proportional to the capaci-

tance, although it may be modified (for instance, by the insertion of a dielectric). If you disconnect the capacitor from the battery before making any modifications to the capacitor, then its charge remains fixed. In this case, as you vary the capacitance, the voltage across the plates changes according to the expression $\Delta V = Q/C$.

QUESTIONS

1. If you were asked to design a capacitor in a situation for which small size and large capacitance were required, what factors would be important in your design?
2. The plates of a capacitor are connected to a battery. What happens to the charge on the plates if the connecting wires are removed from the battery? What happens to the charge if the wires are removed from the battery and connected to each other?
3. A farad is a very large unit of capacitance. Calculate the length of one side of a square, air-filled capacitor that has a plate separation of 1 m. Assume that it has a capacitance of 1 F.
4. A pair of capacitors are connected in parallel, while an identical pair are connected in series. Which pair would be more dangerous to handle after being connected to the same voltage source? Explain.
5. If you are given three different capacitors C_1, C_2, C_3, how many different combinations of capacitance can you produce?
6. What advantage might there be in using two identical capacitors in parallel connected in series with another identical parallel pair rather than a single capacitor?
7. Is it always possible to reduce a combination of capacitors to one equivalent capacitor with the rules we have developed? Explain.
8. Because the net charge in a capacitor is always zero, what does a capacitor store?
9. Because the charges on the plates of a parallel-plate capacitor are of opposite sign, they attract each other. Hence, it would take positive work to increase the plate separation. What happens to the external work done in this process?
10. Explain why the work needed to move a charge Q through a potential difference ΔV is $W = Q\Delta V$, whereas the energy stored in a charged capacitor is $U = \frac{1}{2}Q\Delta V$. Where does the $\frac{1}{2}$ factor come from?
11. If the potential difference across a capacitor is doubled, by what factor does the stored energy change?
12. Why is it dangerous to touch the terminals of a high-voltage capacitor even after the applied voltage has been turned off? What can be done to make the capacitor safe to handle after the voltage source has been removed?
13. Describe how you can increase the maximum operating voltage of a parallel-plate capacitor for a fixed plate separation.
14. An air-filled capacitor is charged, disconnected from the power supply, and, finally, connected to a voltmeter. Explain how and why the voltage reading changes when a dielectric is inserted between the plates of the capacitor.
15. Using the polar molecule description of a dielectric, explain how a dielectric affects the electric field inside a capacitor.
16. Explain why a dielectric increases the maximum operating voltage of a capacitor even though the physical size of the capacitor does not change.
17. What is the difference between dielectric strength and the dielectric constant?
18. Explain why a water molecule is permanently polarized. What type of molecule has no permanent polarization?
19. If a dielectric-filled capacitor is heated, how does its capacitance change? (Neglect thermal expansion and assume that the dipole orientations are temperature dependent.)

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging   □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/**   = Computer useful in solving problem   = Interactive Physics
□ = paired numerical/symbolic problems

Section 26.1 Definition of Capacitance

1. (a) How much charge is on each plate of a 4.00-μF capacitor when it is connected to a 12.0-V battery? (b) If this same capacitor is connected to a 1.50-V battery, what charge is stored?
2. Two conductors having net charges of $+10.0\ \mu$C and $-10.0\ \mu$C have a potential difference of 10.0 V. Determine (a) the capacitance of the system and (b) the potential difference between the two conductors if the charges on each are increased to $+100\ \mu$C and $-100\ \mu$C.

Section 26.2 Calculating Capacitance

3. An isolated charged conducting sphere of radius 12.0 cm creates an electric field of 4.90×10^4 N/C at a distance 21.0 cm from its center. (a) What is its surface charge density? (b) What is its capacitance?

4. (a) If a drop of liquid has capacitance 1.00 pF, what is its radius? (b) If another drop has radius 2.00 mm, what is its capacitance? (c) What is the charge on the smaller drop if its potential is 100 V?

5. Two conducting spheres with diameters of 0.400 m and 1.00 m are separated by a distance that is large compared with the diameters. The spheres are connected by a thin wire and are charged to 7.00 μC. (a) How is this total charge shared between the spheres? (Neglect any charge on the wire.) (b) What is the potential of the system of spheres when the reference potential is taken to be $V = 0$ at $r = \infty$?

6. Regarding the Earth and a cloud layer 800 m above the Earth as the "plates" of a capacitor, calculate the capacitance if the cloud layer has an area of 1.00 km^2. Assume that the air between the cloud and the ground is pure and dry. Assume that charge builds up on the cloud and on the ground until a uniform electric field with a magnitude of 3.00×10^6 N/C throughout the space between them makes the air break down and conduct electricity as a lightning bolt. What is the maximum charge the cloud can hold?

WEB 7. An air-filled capacitor consists of two parallel plates, each with an area of 7.60 cm^2, separated by a distance of 1.80 mm. If a 20.0-V potential difference is applied to these plates, calculate (a) the electric field between the plates, (b) the surface charge density, (c) the capacitance, and (d) the charge on each plate.

8. A 1-megabit computer memory chip contains many 60.0-fF capacitors. Each capacitor has a plate area of 21.0×10^{-12} m^2. Determine the plate separation of such a capacitor (assume a parallel-plate configuration). The characteristic atomic diameter is 10^{-10} m $=$ 0.100 nm. Express the plate separation in nanometers.

9. When a potential difference of 150 V is applied to the plates of a parallel-plate capacitor, the plates carry a surface charge density of 30.0 nC/cm^2. What is the spacing between the plates?

10. A variable air capacitor used in tuning circuits is made of N semicircular plates each of radius R and positioned a distance d from each other. As shown in Figure P26.10, a second identical set of plates is enmeshed with its plates halfway between those of the first set. The second set can rotate as a unit. Determine the capacitance as a function of the angle of rotation θ, where $\theta = 0$ corresponds to the maximum capacitance.

WEB 11. A 50.0-m length of coaxial cable has an inner conductor that has a diameter of 2.58 mm and carries a charge of 8.10 μC. The surrounding conductor has an inner diameter of 7.27 mm and a charge of -8.10 μC. (a) What is the capacitance of this cable? (b) What is the potential difference between the two conductors? Assume the region between the conductors is air.

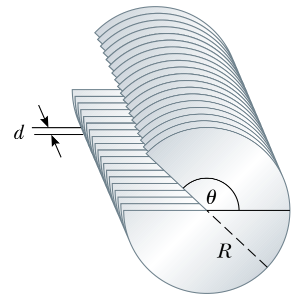


Figure P26.10

12. A 20.0-μF spherical capacitor is composed of two metallic spheres, one having a radius twice as large as the other. If the region between the spheres is a vacuum, determine the volume of this region.

13. A small object with a mass of 350 mg carries a charge of 30.0 nC and is suspended by a thread between the vertical plates of a parallel-plate capacitor. The plates are separated by 4.00 cm. If the thread makes an angle of 15.0° with the vertical, what is the potential difference between the plates?

14. A small object of mass m carries a charge q and is suspended by a thread between the vertical plates of a parallel-plate capacitor. The plate separation is d. If the thread makes an angle θ with the vertical, what is the potential difference between the plates?

15. An air-filled spherical capacitor is constructed with inner and outer shell radii of 7.00 and 14.0 cm, respectively. (a) Calculate the capacitance of the device. (b) What potential difference between the spheres results in a charge of 4.00 μC on the capacitor?

16. Find the capacitance of the Earth. (*Hint:* The outer conductor of the "spherical capacitor" may be considered as a conducting sphere at infinity where V approaches zero.)

Section 26.3 Combinations of Capacitors

17. Two capacitors $C_1 = 5.00$ μF and $C_2 = 12.0$ μF are connected in parallel, and the resulting combination is connected to a 9.00-V battery. (a) What is the value of the equivalent capacitance of the combination? What are (b) the potential difference across each capacitor and (c) the charge stored on each capacitor?

18. The two capacitors of Problem 17 are now connected in series and to a 9.00-V battery. Find (a) the value of the equivalent capacitance of the combination, (b) the voltage across each capacitor, and (c) the charge on each capacitor.

19. Two capacitors when connected in parallel give an equivalent capacitance of 9.00 pF and an equivalent ca-

pacitance of 2.00 pF when connected in series. What is the capacitance of each capacitor?

20. Two capacitors when connected in parallel give an equivalent capacitance of C_p and an equivalent capacitance of C_s when connected in series. What is the capacitance of each capacitor?

WEB **21.** Four capacitors are connected as shown in Figure P26.21. (a) Find the equivalent capacitance between points *a* and *b*. (b) Calculate the charge on each capacitor if $\Delta V_{ab} = 15.0$ V.

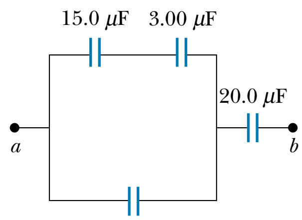


Figure P26.21

22. Evaluate the equivalent capacitance of the configuration shown in Figure P26.22. All the capacitors are identical, and each has capacitance *C*.

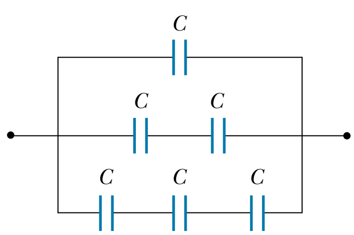


Figure P26.22

23. Consider the circuit shown in Figure P26.23, where $C_1 = 6.00\ \mu$F, $C_2 = 3.00\ \mu$F, and $\Delta V = 20.0$ V. Capacitor C_1 is first charged by the closing of switch S_1. Switch S_1 is then opened, and the charged capacitor is connected to the uncharged capacitor by the closing of S_2. Calculate the initial charge acquired by C_1 and the final charge on each.

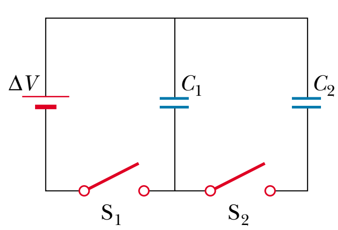


Figure P26.23

24. According to its design specification, the timer circuit delaying the closing of an elevator door is to have a capacitance of 32.0 μF between two points *A* and *B*. (a) When one circuit is being constructed, the inexpensive capacitor installed between these two points is found to have capacitance 34.8 μF. To meet the specification, one additional capacitor can be placed between the two points. Should it be in series or in parallel with the 34.8-μF capacitor? What should be its capacitance? (b) The next circuit comes down the assembly line with capacitance 29.8 μF between *A* and *B*. What additional capacitor should be installed in series or in parallel in that circuit, to meet the specification?

25. The circuit in Figure P26.25 consists of two identical parallel metallic plates connected by identical metallic springs to a 100-V battery. With the switch open, the plates are uncharged, are separated by a distance $d = 8.00$ mm, and have a capacitance $C = 2.00\ \mu$F. When the switch is closed, the distance between the plates decreases by a factor of 0.500. (a) How much charge collects on each plate and (b) what is the spring constant for each spring? (*Hint:* Use the result of Problem 35.)

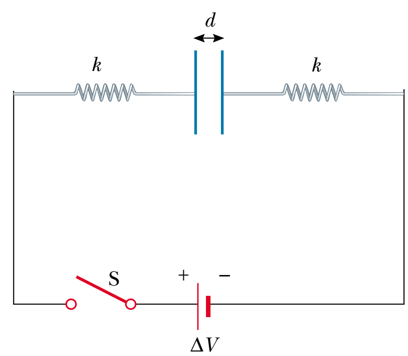


Figure P26.25

26. Figure P26.26 shows six concentric conducting spheres, A, B, C, D, E, and F having radii R, $2R$, $3R$, $4R$, $5R$, and $6R$, respectively. Spheres B and C are connected by a conducting wire, as are spheres D and E. Determine the equivalent capacitance of this system.

27. A group of identical capacitors is connected first in series and then in parallel. The combined capacitance in parallel is 100 times larger than for the series connection. How many capacitors are in the group?

28. Find the equivalent capacitance between points *a* and *b* for the group of capacitors connected as shown in Figure P26.28 if $C_1 = 5.00\ \mu$F, $C_2 = 10.0\ \mu$F, and $C_3 = 2.00\ \mu$F.

29. For the network described in the previous problem if the potential difference between points *a* and *b* is 60.0 V, what charge is stored on C_3?

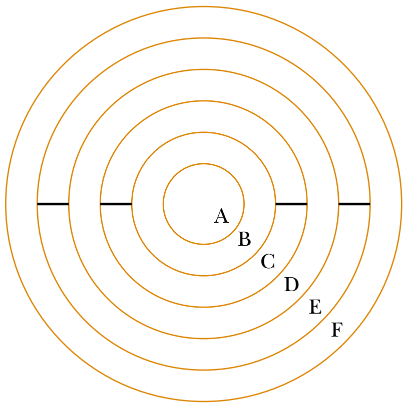


Figure P26.26

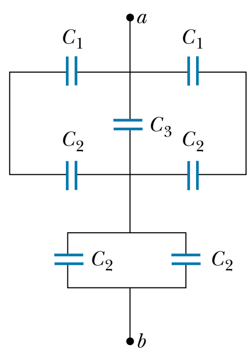


Figure P26.28 Problems 28 and 29.

30. Find the equivalent capacitance between points a and b in the combination of capacitors shown in Figure P26.30.

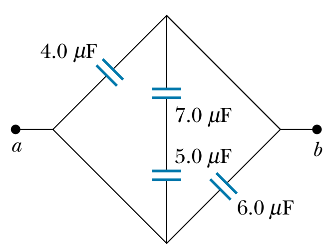


Figure P26.30

Section 26.4 Energy Stored in a Charged Capacitor

31. (a) A 3.00-μF capacitor is connected to a 12.0-V battery. How much energy is stored in the capacitor? (b) If the capacitor had been connected to a 6.00-V battery, how much energy would have been stored?

32. Two capacitors $C_1 = 25.0\ \mu$F and $C_2 = 5.00\ \mu$F are connected in parallel and charged with a 100-V power supply. (a) Draw a circuit diagram and calculate the total energy stored in the two capacitors. (b) What potential difference would be required across the same two capacitors connected in series so that the combination stores the same energy as in part (a)? Draw a circuit diagram of this circuit.

33. A parallel-plate capacitor is charged and then disconnected from a battery. By what fraction does the stored energy change (increase or decrease) when the plate separation is doubled?

34. A uniform electric field $E = 3\,000$ V/m exists within a certain region. What volume of space contains an energy equal to 1.00×10^{-7} J? Express your answer in cubic meters and in liters.

WEB **35.** A parallel-plate capacitor has a charge Q and plates of area A. Show that the force exerted on each plate by the other is $F = Q^2/2\epsilon_0 A$. (*Hint:* Let $C = \epsilon_0 A/x$ for an arbitrary plate separation x; then require that the work done in separating the two charged plates be $W = \int F\,dx$.)

36. Plate a of a parallel-plate, air-filled capacitor is connected to a spring having force constant k, and plate b is fixed. They rest on a table top as shown (top view) in Figure P26.36. If a charge $+Q$ is placed on plate a and a charge $-Q$ is placed on plate b, by how much does the spring expand?

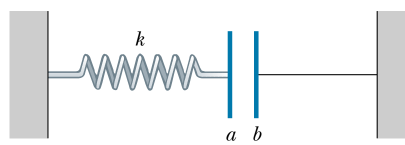


Figure P26.36

37. Review Problem. A certain storm cloud has a potential difference of 1.00×10^8 V relative to a tree. If, during a lightning storm, 50.0 C of charge is transferred through this potential difference and 1.00% of the energy is absorbed by the tree, how much water (sap in the tree) initially at 30.0°C can be boiled away? Water has a specific heat of 4 186 J/kg · °C, a boiling point of 100°C, and a heat of vaporization of 2.26×10^6 J/kg.

38. Show that the energy associated with a conducting sphere of radius R and charge Q surrounded by a vacuum is $U = k_e Q^2/2R$.

39. Einstein said that energy is associated with mass according to the famous relationship $E = mc^2$. Estimate the radius of an electron, assuming that its charge is distributed uniformly over the surface of a sphere of radius R and that the mass–energy of the electron is equal to the total energy stored in the resulting nonzero electric field between R and infinity. (See Problem 38. Experimentally, an electron nevertheless appears to be a point particle. The electric field close to the electron must be described by quantum electrodynamics, rather than the classical electrodynamics that we study.)

Section 26.5 Capacitors with Dielectrics

40. Find the capacitance of a parallel-plate capacitor that uses Bakelite as a dielectric, if each of the plates has an area of 5.00 cm^2 and the plate separation is 2.00 mm.

41. Determine (a) the capacitance and (b) the maximum voltage that can be applied to a Teflon-filled parallel-plate capacitor having a plate area of 1.75 cm^2 and plate separation of 0.040 0 mm.

42. (a) How much charge can be placed on a capacitor with air between the plates before it breaks down, if the area of each of the plates is 5.00 cm^2? (b) Find the maximum charge if polystyrene is used between the plates instead of air.

43. A commercial capacitor is constructed as shown in Figure 26.15a. This particular capacitor is rolled from two strips of aluminum separated by two strips of paraffin-coated paper. Each strip of foil and paper is 7.00 cm wide. The foil is 0.004 00 mm thick, and the paper is 0.025 0 mm thick and has a dielectric constant of 3.70. What length should the strips be if a capacitance of 9.50×10^{-8} F is desired? (Use the parallel-plate formula.)

44. The supermarket sells rolls of aluminum foil, plastic wrap, and waxed paper. Describe a capacitor made from supermarket materials. Compute order-of-magnitude estimates for its capacitance and its breakdown voltage.

45. A capacitor that has air between its plates is connected across a potential difference of 12.0 V and stores 48.0 μC of charge. It is then disconnected from the source while still charged. (a) Find the capacitance of the capacitor. (b) A piece of Teflon is inserted between the plates. Find its new capacitance. (c) Find the voltage and charge now on the capacitor.

46. A parallel-plate capacitor in air has a plate separation of 1.50 cm and a plate area of 25.0 cm^2. The plates are charged to a potential difference of 250 V and disconnected from the source. The capacitor is then immersed in distilled water. Determine (a) the charge on the plates before and after immersion, (b) the capacitance and voltage after immersion, and (c) the change in energy of the capacitor. Neglect the conductance of the liquid.

47. A conducting spherical shell has inner radius a and outer radius c. The space between these two surfaces is filled with a dielectric for which the dielectric constant is κ_1 between a and b, and κ_2 between b and c (Fig. P26.47). Determine the capacitance of this system.

48. A wafer of titanium dioxide ($\kappa = 173$) has an area of 1.00 cm^2 and a thickness of 0.100 mm. Aluminum is evaporated on the parallel faces to form a parallel-plate capacitor. (a) Calculate the capacitance. (b) When the capacitor is charged with a 12.0-V battery, what is the magnitude of charge delivered to each plate? (c) For the situation in part (b), what are the free and induced surface charge densities? (d) What is the magnitude E of the electric field?

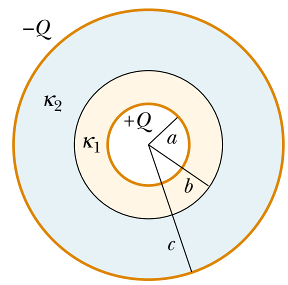


Figure P26.47

49. Each capacitor in the combination shown in Figure P26.49 has a breakdown voltage of 15.0 V. What is the breakdown voltage of the combination?

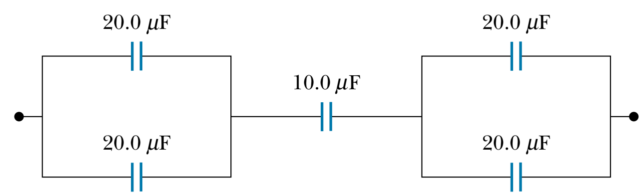


Figure P26.49

(Optional)
Section 26.6 Electric Dipole in an Electric Field

50. A small rigid object carries positive and negative 3.50-nC charges. It is oriented so that the positive charge is at the point (− 1.20 mm, 1.10 mm) and the negative charge is at the point (1.40 mm, − 1.30 mm). (a) Find the electric dipole moment of the object. The object is placed in an electric field $\mathbf{E} = (7\,800\mathbf{i} - 4\,900\mathbf{j})$ N/C. (b) Find the torque acting on the object. (c) Find the potential energy of the object in this orientation. (d) If the orientation of the object can change, find the difference between its maximum and its minimum potential energies.

51. A small object with electric dipole moment $\mathbf{p}$ is placed in a nonuniform electric field $\mathbf{E} = E(x)\,\mathbf{i}$. That is, the field is in the x direction, and its magnitude depends on the coordinate x. Let θ represent the angle between the dipole moment and the x direction. (a) Prove that the dipole experiences a net force $F = p(dE/dx)\cos\theta$ in the direction toward which the field increases. (b) Consider the field created by a spherical balloon centered at the origin. The balloon has a radius of 15.0 cm and carries a charge of 2.00 μC. Evaluate dE/dx at the point (16 cm, 0, 0). Assume that a water droplet at this point has an induced dipole moment of $(6.30\mathbf{i})$ nC·m. Find the force on it.

(Optional)
Section 26.7 An Atomic Description of Dielectrics

52. A detector of radiation called a Geiger–Muller counter consists of a closed, hollow, conducting cylinder with a

fine wire along its axis. Suppose that the internal diameter of the cylinder is 2.50 cm and that the wire along the axis has a diameter of 0.200 mm. If the dielectric strength of the gas between the central wire and the cylinder is 1.20×10^6 V/m, calculate the maximum voltage that can be applied between the wire and the cylinder before breakdown occurs in the gas.

53. The general form of Gauss's law describes how a charge creates an electric field in a material, as well as in a vacuum. It is

$$\oint \mathbf{E} \cdot d\mathbf{A} = \frac{q}{\epsilon}$$

where $\epsilon = \kappa\epsilon_0$ is the permittivity of the material. (a) A sheet with charge Q uniformly distributed over its area A is surrounded by a dielectric. Show that the sheet creates a uniform electric field with magnitude $E = Q/2A\epsilon$ at nearby points. (b) Two large sheets of area A carrying opposite charges of equal magnitude Q are a small distance d apart. Show that they create a uniform electric field of magnitude $E = Q/A\epsilon$ between them. (c) Assume that the negative plate is at zero potential. Show that the positive plate is at a potential $Qd/A\epsilon$. (d) Show that the capacitance of the pair of plates is $A\epsilon/d = \kappa A\epsilon_0/d$.

ADDITIONAL PROBLEMS

54. For the system of capacitors shown in Figure P26.54, find (a) the equivalent capacitance of the system, (b) the potential difference across each capacitor, (c) the charge on each capacitor, and (d) the total energy stored by the group.

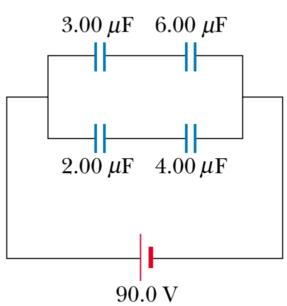


Figure P26.54

55. Consider two *long*, parallel, and oppositely charged wires of radius d with their centers separated by a distance D. Assuming the charge is distributed uniformly on the surface of each wire, show that the capacitance per unit length of this pair of wires is

$$\frac{C}{\ell} = \frac{\pi\epsilon_0}{\ln\left(\dfrac{D-d}{d}\right)}$$

56. A 2.00-nF parallel-plate capacitor is charged to an initial potential difference $\Delta V_i = 100$ V and then isolated. The dielectric material between the plates is mica ($\kappa = 5.00$). (a) How much work is required to withdraw the mica sheet? (b) What is the potential difference of the capacitor after the mica is withdrawn?

WEB **57.** A parallel-plate capacitor is constructed using a dielectric material whose dielectric constant is 3.00 and whose dielectric strength is 2.00×10^8 V/m. The desired capacitance is 0.250 μF, and the capacitor must withstand a maximum potential difference of 4 000 V. Find the minimum area of the capacitor plates.

58. A parallel-plate capacitor is constructed using three dielectric materials, as shown in Figure P26.58. You may assume that $\ell \gg d$. (a) Find an expression for the capacitance of the device in terms of the plate area A and d, κ_1, κ_2, and κ_3. (b) Calculate the capacitance using the values $A = 1.00\ \text{cm}^2$, $d = 2.00$ mm, $\kappa_1 = 4.90$, $\kappa_2 = 5.60$, and $\kappa_3 = 2.10$.

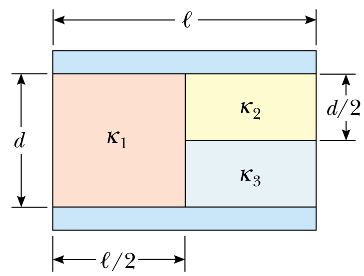


Figure P26.58

59. A conducting slab of thickness d and area A is inserted into the space between the plates of a parallel-plate capacitor with spacing s and surface area A, as shown in Figure P26.59. The slab is not necessarily halfway between the capacitor plates. What is the capacitance of the system?

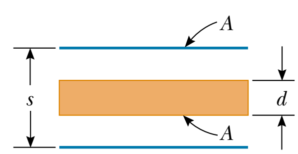


Figure P26.59

60. (a) Two spheres have radii a and b and their centers are a distance d apart. Show that the capacitance of this system is

$$C \approx \frac{4\pi\epsilon_0}{\dfrac{1}{a} + \dfrac{1}{b} - \dfrac{2}{d}}$$

provided that d is large compared with a and b. (*Hint:* Because the spheres are far apart, assume that the

charge on one sphere does not perturb the charge distribution on the other sphere. Thus, the potential of each sphere is expressed as that of a symmetric charge distribution, $V = k_e Q/r$, and the total potential at each sphere is the sum of the potentials due to each sphere. (b) Show that as d approaches infinity the above result reduces to that of two isolated spheres in series.

61. When a certain air-filled parallel-plate capacitor is connected across a battery, it acquires a charge (on each plate) of q_0. While the battery connection is maintained, a dielectric slab is inserted into and fills the region between the plates. This results in the accumulation of an *additional* charge q on each plate. What is the dielectric constant of the slab?

62. A capacitor is constructed from two square plates of sides ℓ and separation d. A material of dielectric constant κ is inserted a distance x into the capacitor, as shown in Figure P26.62. (a) Find the equivalent capacitance of the device. (b) Calculate the energy stored in the capacitor if the potential difference is ΔV. (c) Find the direction and magnitude of the force exerted on the dielectric, assuming a constant potential difference ΔV. Neglect friction. (d) Obtain a numerical value for the force assuming that $\ell = 5.00$ cm, $\Delta V = 2\ 000$ V, $d = 2.00$ mm, and the dielectric is glass ($\kappa = 4.50$). (*Hint:* The system can be considered as two capacitors connected in *parallel*.)

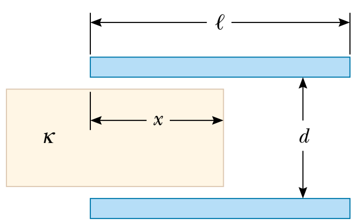


Figure P26.62 Problems 62 and 63.

63. A capacitor is constructed from two square plates of sides ℓ and separation d, as suggested in Figure P26.62. You may assume that d is much less than ℓ. The plates carry charges $+Q_0$ and $-Q_0$. A block of metal has a width ℓ, a length ℓ, and a thickness slightly less than d. It is inserted a distance x into the capacitor. The charges on the plates are not disturbed as the block slides in. In a static situation, a metal prevents an electric field from penetrating it. The metal can be thought of as a perfect dielectric, with $\kappa \to \infty$. (a) Calculate the stored energy as a function of x. (b) Find the direction and magnitude of the force that acts on the metallic block. (c) The area of the advancing front face of the block is essentially equal to ℓd. Considering the force on the block as acting on this face, find the stress (force per area) on it. (d) For comparison, express the energy density in the electric field between the capacitor plates in terms of Q_0, ℓ, d, and ϵ_0.

64. When considering the energy supply for an automobile, the energy per unit mass of the energy source is an important parameter. Using the following data, compare the energy per unit mass (J/kg) for gasoline, lead–acid batteries, and capacitors. (The ampere A will be introduced in Chapter 27 and is the SI unit of electric current. $1\ \text{A} = 1\ \text{C/s}$.)

Gasoline: 126 000 Btu/gal; density $= 670\ \text{kg/m}^3$
Lead–acid battery: 12.0 V; 100 A·h; mass $= 16.0$ kg
Capacitor: potential difference at full charge = 12.0 V; capacitance $= 0.100$ F; mass $= 0.100$ kg

65. An isolated capacitor of unknown capacitance has been charged to a potential difference of 100 V. When the charged capacitor is then connected in parallel to an uncharged 10.0-μF capacitor, the voltage across the combination is 30.0 V. Calculate the unknown capacitance.

66. A certain electronic circuit calls for a capacitor having a capacitance of 1.20 pF and a breakdown potential of 1 000 V. If you have a supply of 6.00-pF capacitors, each having a breakdown potential of 200 V, how could you meet this circuit requirement?

67. In the arrangement shown in Figure P26.67, a potential difference ΔV is applied, and C_1 is adjusted so that the voltmeter between points b and d reads zero. This "balance" occurs when $C_1 = 4.00\ \mu$F. If $C_3 = 9.00\ \mu$F and $C_4 = 12.0\ \mu$F, calculate the value of C_2.

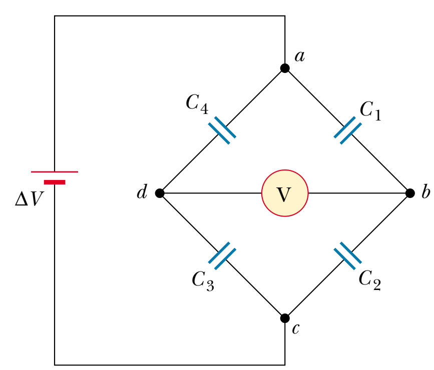


Figure P26.67

68. It is possible to obtain large potential differences by first charging a group of capacitors connected in parallel and then activating a switch arrangement that in effect disconnects the capacitors from the charging source and from each other and reconnects them in a series arrangement. The group of charged capacitors is then discharged in series. What is the maximum potential difference that can be obtained in this manner by using ten capacitors each of 500 μF and a charging source of 800 V?

69. A parallel-plate capacitor of plate separation d is charged to a potential difference ΔV_0. A dielectric slab

of thickness d and dielectric constant κ is introduced between the plates *while the battery remains connected to the plates.* (a) Show that the ratio of energy stored after the dielectric is introduced to the energy stored in the empty capacitor is $U/U_0 = \kappa$. Give a physical explanation for this increase in stored energy. (b) What happens to the charge on the capacitor? (Note that this situation is not the same as Example 26.7, in which the battery was removed from the circuit before the dielectric was introduced.)

70. A parallel-plate capacitor with plates of area A and plate separation d has the region between the plates filled with two dielectric materials as in Figure P26.70. Assume that $d \ll L$ and that $d \ll W$. (a) Determine the capacitance and (b) show that when $\kappa_1 = \kappa_2 = \kappa$ your result becomes the same as that for a capacitor containing a single dielectric, $C = \kappa\epsilon_0 A/d$.

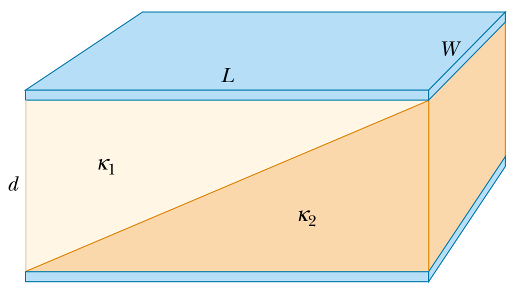


Figure P26.70

71. A vertical parallel-plate capacitor is half filled with a dielectric for which the dielectric constant is 2.00 (Fig. P26.71a). When this capacitor is positioned horizontally, what fraction of it should be filled with the same dielectric (Fig. P26.71b) so that the two capacitors have equal capacitance?

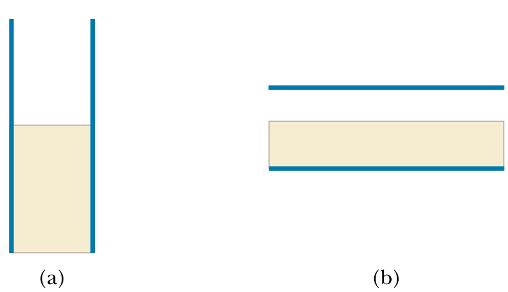


Figure P26.71

72. Capacitors $C_1 = 6.00\ \mu F$ and $C_2 = 2.00\ \mu F$ are charged as a parallel combination across a 250-V battery. The capacitors are disconnected from the battery and from each other. They are then connected positive plate to negative plate and negative plate to positive plate. Calculate the resulting charge on each capacitor.

73. The inner conductor of a coaxial cable has a radius of 0.800 mm, and the outer conductor's inside radius is 3.00 mm. The space between the conductors is filled with polyethylene, which has a dielectric constant of 2.30 and a dielectric strength of 18.0×10^6 V/m. What is the maximum potential difference that this cable can withstand?

74. You are optimizing coaxial cable design for a major manufacturer. Show that for a given outer conductor radius b, maximum potential difference capability is attained when the radius of the inner conductor is $a = b/e$ where e is the base of natural logarithms.

75. Calculate the equivalent capacitance between the points a and b in Figure P26.75. Note that this is not a simple series or parallel combination. (*Hint:* Assume a potential difference ΔV between points a and b. Write expressions for ΔV_{ab} in terms of the charges and capacitances for the various possible pathways from a to b, and require conservation of charge for those capacitor plates that are connected to each other.)

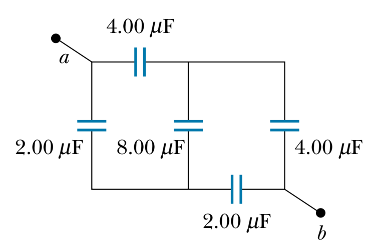


Figure P26.75

76. Determine the effective capacitance of the combination shown in Figure P26.76. (*Hint:* Consider the symmetry involved!)

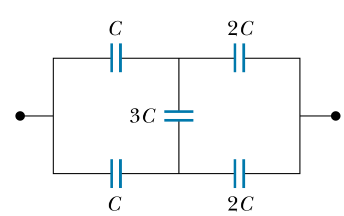


Figure P26.76

Answers to Quick Quizzes

26.1 (a) because the plate separation is decreased. Capacitance depends only on how a capacitor is constructed and not on the external circuit.

26.2 Zero. If you construct a spherical gaussian surface outside and concentric with the capacitor, the net charge inside the surface is zero. Applying Gauss's law to this configuration, we find that $E = 0$ at points outside the capacitor.

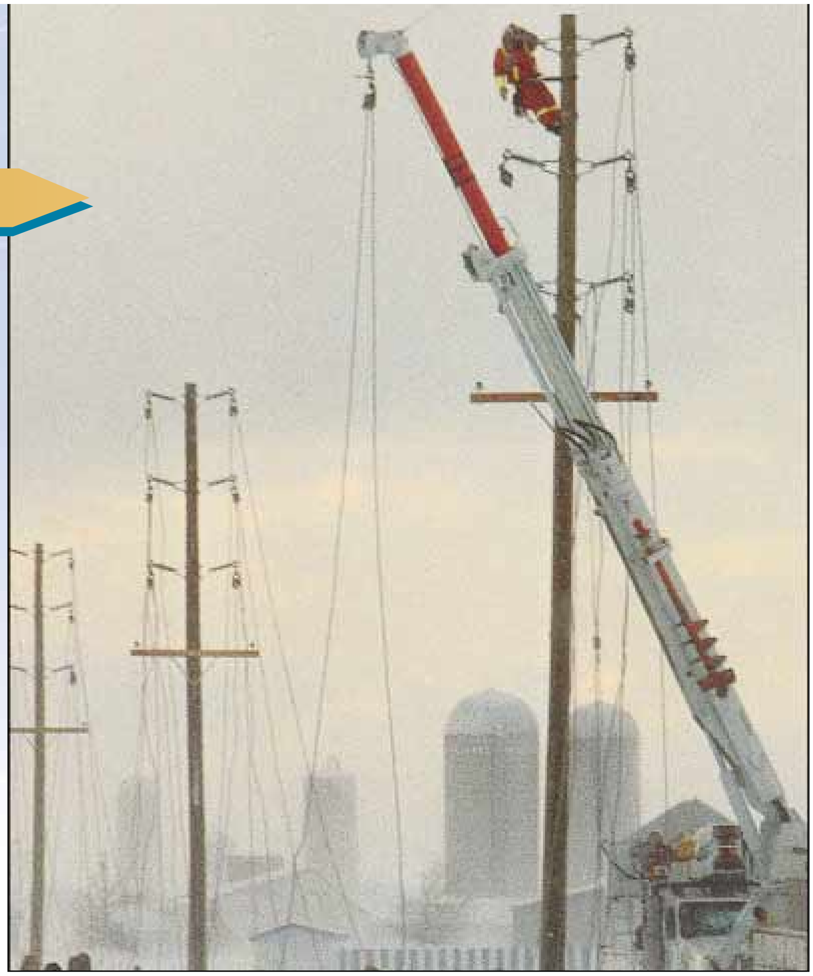
26.3 For a given voltage, the energy stored in a capacitor is proportional to C: $U = C(\Delta V)^2/2$. Thus, you want to maximize the equivalent capacitance. You do this by connecting the three capacitors in parallel, so that the capacitances add.

26.4 (a) C decreases (Eq. 26.3). (b) Q stays the same because there is no place for the charge to flow. (c) E remains constant (see Eq. 24.8 and the paragraph following it). (d) ΔV increases because $\Delta V = Q/C$, Q is constant (part b), and C decreases (part a). (e) The energy stored in the capacitor is proportional to both Q and ΔV (Eq. 26.11) and thus increases. The additional energy comes from the work you do in pulling the two plates apart.

26.5 (a) C decreases (Eq. 26.3). (b) Q decreases. The battery supplies a constant potential difference ΔV; thus, charge must flow out of the capacitor if $C = Q/\Delta V$ is to decrease. (c) E decreases because the charge density on the plates decreases. (d) ΔV remains constant because of the presence of the battery. (e) The energy stored in the capacitor decreases (Eq. 26.11).

26.6 It increases. The dielectric constant of wood (and of all other insulating materials, for that matter) is greater than 1; therefore, the capacitance increases (Eq. 26.14). This increase is sensed by the stud-finder's special circuitry, which causes an indicator on the device to light up.

26.7 (a) C increases (Eq. 26.14). (b) Q increases. Because the battery maintains a constant ΔV, Q must increase if $C\ (= Q/\Delta V)$ increases. (c) E between the plates remains constant because $\Delta V = Ed$ and neither ΔV nor d changes. The electric field due to the charges on the plates increases because more charge has flowed onto the plates. The induced surface charges on the dielectric create a field that opposes the increase in the field caused by the greater number of charges on the plates. (d) The battery maintains a constant ΔV. (e) The energy stored in the capacitor increases (Eq. 26.11). You would have to push the dielectric into the capacitor, just as you would have to do positive work to raise a mass and increase its gravitational potential energy.



PUZZLER

Electrical workers restoring power to the eastern Ontario town of St. Isadore, which was without power for several days in January 1998 because of a severe ice storm. It is very dangerous to touch fallen power transmission lines because of their high electric potential, which might be hundreds of thousands of volts relative to the ground. Why is such a high potential difference used in power transmission if it is so dangerous, and why aren't birds that perch on the wires electrocuted? *(AP/Wide World Photos/Fred Chartrand)*

chapter

27 Current and Resistance

Chapter Outline

Thus far our treatment of electrical phenomena has been confined to the study of charges at rest, or *electrostatics.* We now consider situations involving electric charges in motion. We use the term *electric current,* or simply *current,* to describe the rate of flow of charge through some region of space. Most practical applications of electricity deal with electric currents. For example, the battery in a flashlight supplies current to the filament of the bulb when the switch is turned on. A variety of home appliances operate on alternating current. In these common situations, the charges flow through a conductor, such as a copper wire. It also is possible for currents to exist outside a conductor. For instance, a beam of electrons in a television picture tube constitutes a current.

This chapter begins with the definitions of current and current density. A microscopic description of current is given, and some of the factors that contribute to the resistance to the flow of charge in conductors are discussed. A classical model is used to describe electrical conduction in metals, and some of the limitations of this model are cited.

27.1 ELECTRIC CURRENT

13.2

It is instructive to draw an analogy between water flow and current. In many localities it is common practice to install low-flow showerheads in homes as a water-conservation measure. We quantify the flow of water from these and similar devices by specifying the amount of water that emerges during a given time interval, which is often measured in liters per minute. On a grander scale, we can characterize a river current by describing the rate at which the water flows past a particular location. For example, the flow over the brink at Niagara Falls is maintained at rates between 1 400 m^3/s and 2 800 m^3/s.

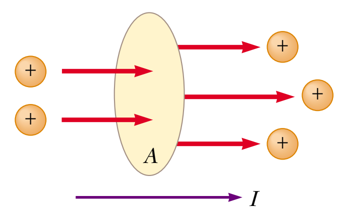


Figure 27.1 Charges in motion through an area A. The time rate at which charge flows through the area is defined as the current I. The direction of the current is the direction in which positive charges flow when free to do so.

Now consider a system of electric charges in motion. Whenever there is a net flow of charge through some region, a **current** is said to exist. To define current more precisely, suppose that the charges are moving perpendicular to a surface of area A, as shown in Figure 27.1. (This area could be the cross-sectional area of a wire, for example.) **The current is the rate at which charge flows through this surface.** If ΔQ is the amount of charge that passes through this area in a time interval Δt, the **average current** I_{av} is equal to the charge that passes through A per unit time:

$$I_{av} = \frac{\Delta Q}{\Delta t} \tag{27.1}$$

If the rate at which charge flows varies in time, then the current varies in time; we define the **instantaneous current** I as the differential limit of average current:

Electric current

$$I \equiv \frac{dQ}{dt} \tag{27.2}$$

The SI unit of current is the **ampere** (A):

$$1\ \text{A} = \frac{1\ \text{C}}{1\ \text{s}} \tag{27.3}$$

That is, 1 A of current is equivalent to 1 C of charge passing through the surface area in 1 s.

The direction of the current

The charges passing through the surface in Figure 27.1 can be positive or negative, or both. **It is conventional to assign to the current the same direction as the flow of positive charge.** In electrical conductors, such as copper or alu-

minum, the current is due to the motion of negatively charged electrons. Therefore, when we speak of current in an ordinary conductor, **the direction of the current is opposite the direction of flow of electrons.** However, if we are considering a beam of positively charged protons in an accelerator, the current is in the direction of motion of the protons. In some cases—such as those involving gases and electrolytes, for instance—the current is the result of the flow of both positive and negative charges.

If the ends of a conducting wire are connected to form a loop, all points on the loop are at the same electric potential, and hence the electric field is zero within and at the surface of the conductor. Because the electric field is zero, there is no net transport of charge through the wire, and therefore there is no current. The current in the conductor is zero even if the conductor has an excess of charge on it. However, if the ends of the conducting wire are connected to a battery, all points on the loop are not at the same potential. The battery sets up a potential difference between the ends of the loop, creating an electric field within the wire. The electric field exerts forces on the conduction electrons in the wire, causing them to move around the loop and thus creating a current.

It is common to refer to a moving charge (positive or negative) as a mobile **charge carrier.** For example, the mobile charge carriers in a metal are electrons.

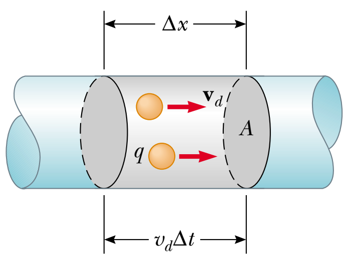


Figure 27.2 A section of a uniform conductor of cross-sectional area A. The mobile charge carriers move with a speed v_d, and the distance they travel in a time Δt is $\Delta x = v_d \Delta t$. The number of carriers in the section of length Δx is $nAv_d \Delta t$, where n is the number of carriers per unit volume.

Microscopic Model of Current

We can relate current to the motion of the charge carriers by describing a microscopic model of conduction in a metal. Consider the current in a conductor of cross-sectional area A (Fig. 27.2). The volume of a section of the conductor of length Δx (the gray region shown in Fig. 27.2) is $A\,\Delta x$. If n represents the number of mobile charge carriers per unit volume (in other words, the charge carrier density), the number of carriers in the gray section is $nA\,\Delta x$. Therefore, the charge ΔQ in this section is

$$\Delta Q = \text{number of carriers in section} \times \text{charge per carrier} = (nA\,\Delta x)q$$

where q is the charge on each carrier. If the carriers move with a speed v_d, the distance they move in a time Δt is $\Delta x = v_d\,\Delta t$. Therefore, we can write ΔQ in the form

$$\Delta Q = (nAv_d\,\Delta t)q$$

If we divide both sides of this equation by Δt, we see that the average current in the conductor is

Average current in a conductor

$$I_{\text{av}} = \frac{\Delta Q}{\Delta t} = nqv_dA \tag{27.4}$$

The speed of the charge carriers v_d is an average speed called the **drift speed.** To understand the meaning of drift speed, consider a conductor in which the charge carriers are free electrons. If the conductor is isolated—that is, the potential difference across it is zero—then these electrons undergo random motion that is analogous to the motion of gas molecules. As we discussed earlier, when a potential difference is applied across the conductor (for example, by means of a battery), an electric field is set up in the conductor; this field exerts an electric force on the electrons, producing a current. However, the electrons do not move in straight lines along the conductor. Instead, they collide repeatedly with the metal atoms, and their resultant motion is complicated and zigzag (Fig. 27.3). Despite the collisions, the electrons move slowly along the conductor (in a direction opposite that of $\mathbf{E}$) at the drift velocity $\mathbf{v}_d$.

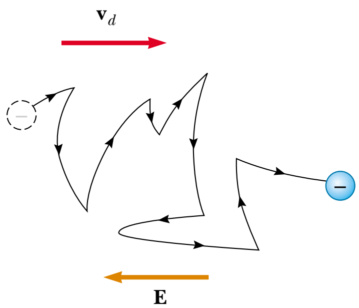


Figure 27.3 A schematic representation of the zigzag motion of an electron in a conductor. The changes in direction are the result of collisions between the electron and atoms in the conductor. Note that the net motion of the electron is opposite the direction of the electric field. Each section of the zigzag path is a parabolic segment.

We can think of the atom–electron collisions in a conductor as an effective internal friction (or drag force) similar to that experienced by the molecules of a liquid flowing through a pipe stuffed with steel wool. The energy transferred from the electrons to the metal atoms during collision causes an increase in the vibrational energy of the atoms and a corresponding increase in the temperature of the conductor.

Quick Quiz 27.1

Consider positive and negative charges moving horizontally through the four regions shown in Figure 27.4. Rank the current in these four regions, from lowest to highest.

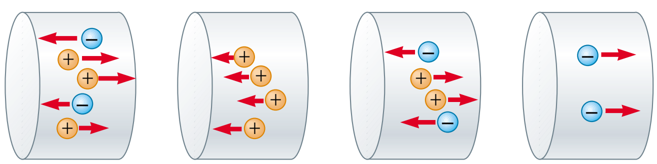


Figure 27.4

EXAMPLE 27.1 Drift Speed in a Copper Wire

The 12-gauge copper wire in a typical residential building has a cross-sectional area of 3.31×10^{-6} m^2. If it carries a current of 10.0 A, what is the drift speed of the electrons? Assume that each copper atom contributes one free electron to the current. The density of copper is 8.95 g/cm^3.

Solution From the periodic table of the elements in Appendix C, we find that the molar mass of copper is 63.5 g/mol. Recall that 1 mol of any substance contains Avogadro's number of atoms (6.02×10^{23}). Knowing the density of copper, we can calculate the volume occupied by 63.5 g (= 1 mol) of copper:

$$V = \frac{m}{\rho} = \frac{63.5 \text{ g}}{8.95 \text{ g/cm}^3} = 7.09 \text{ cm}^3$$

Because each copper atom contributes one free electron to the current, we have

$$n = \frac{6.02 \times 10^{23} \text{ electrons}}{7.09 \text{ cm}^3}(1.00 \times 10^6 \text{ cm}^3/\text{m}^3)$$
$$= 8.49 \times 10^{28} \text{ electrons/m}^3$$

From Equation 27.4, we find that the drift speed is

$$v_d = \frac{I}{nqA}$$

where q is the absolute value of the charge on each electron. Thus,

$$v_d = \frac{I}{nqA}$$
$$= \frac{10.0 \text{ C/s}}{(8.49 \times 10^{28} \text{ m}^{-3})(1.60 \times 10^{-19} \text{ C})(3.31 \times 10^{-6} \text{ m}^2)}$$
$$= \boxed{2.22 \times 10^{-4} \text{ m/s}}$$

Exercise If a copper wire carries a current of 80.0 mA, how many electrons flow past a given cross-section of the wire in 10.0 min?

Answer 3.0×10^{20} electrons.

Example 27.1 shows that typical drift speeds are very low. For instance, electrons traveling with a speed of 2.46×10^{-4} m/s would take about 68 min to travel 1 m! In view of this, you might wonder why a light turns on almost instantaneously when a switch is thrown. In a conductor, the electric field that drives the free electrons travels through the conductor with a speed close to that of light. Thus, when you flip on a light switch, the message for the electrons to start moving through the wire (the electric field) reaches them at a speed on the order of 10^8 m/s.

27.2 RESISTANCE AND OHM'S LAW

13.3 In Chapter 24 we found that no electric field can exist inside a conductor. However, this statement is true *only* if the conductor is in static equilibrium. The purpose of this section is to describe what happens when the charges in the conductor are allowed to move.

Charges moving in a conductor produce a current under the action of an electric field, which is maintained by the connection of a battery across the conductor. An electric field can exist in the conductor because the charges in this situation are in motion—that is, this is a *nonelectrostatic* situation.

Consider a conductor of cross-sectional area A carrying a current I. The **current density** J in the conductor is defined as the current per unit area. Because the current $I = nqv_dA$, the current density is

$$J \equiv \frac{I}{A} = nqv_d \qquad \textbf{(27.5)}$$

where J has SI units of A/m^2. This expression is valid only if the current density is uniform and only if the surface of cross-sectional area A is perpendicular to the direction of the current. In general, the current density is a vector quantity:

Current density

$$\mathbf{J} = nq\mathbf{v}_d \qquad \textbf{(27.6)}$$

From this equation, we see that current density, like current, is in the direction of charge motion for positive charge carriers and opposite the direction of motion for negative charge carriers.

A current density J and an electric field E are established in a conductor whenever a potential difference is maintained across the conductor. If the potential difference is constant, then the current also is constant. In some materials, the current density is proportional to the electric field:

Ohm's law

$$\mathbf{J} = \sigma\mathbf{E} \qquad \textbf{(27.7)}$$

where the constant of proportionality σ is called the **conductivity** of the conductor.[1] Materials that obey Equation 27.7 are said to follow **Ohm's law,** named after Georg Simon Ohm (1787–1854). More specifically, Ohm's law states that

> for many materials (including most metals), the ratio of the current density to the electric field is a constant σ that is independent of the electric field producing the current.

Materials that obey Ohm's law and hence demonstrate this simple relationship between **E** and **J** are said to be *ohmic*. Experimentally, it is found that not all materials have this property, however, and materials that do not obey Ohm's law are said to

[1] Do not confuse conductivity σ with surface charge density, for which the same symbol is used.

be *nonohmic.* Ohm's law is not a fundamental law of nature but rather an empirical relationship valid only for certain materials.

Quick Quiz 27.2

Suppose that a current-carrying ohmic metal wire has a cross-sectional area that gradually becomes smaller from one end of the wire to the other. How do drift velocity, current density, and electric field vary along the wire? Note that the current must have the same value everywhere in the wire so that charge does not accumulate at any one point.

We can obtain a form of Ohm's law useful in practical applications by considering a segment of straight wire of uniform cross-sectional area A and length ℓ, as shown in Figure 27.5. A potential difference $\Delta V = V_b - V_a$ is maintained across the wire, creating in the wire an electric field and a current. If the field is assumed to be uniform, the potential difference is related to the field through the relationship[2]

$$\Delta V = E\ell$$

Therefore, we can express the magnitude of the current density in the wire as

$$J = \sigma E = \sigma \frac{\Delta V}{\ell}$$

Because $J = I/A$, we can write the potential difference as

$$\Delta V = \frac{\ell}{\sigma} J = \left(\frac{\ell}{\sigma A}\right) I$$

The quantity $\ell/\sigma A$ is called the **resistance** R of the conductor. We can define the resistance as the ratio of the potential difference across a conductor to the current through the conductor:

$$R \equiv \frac{\ell}{\sigma A} \equiv \frac{\Delta V}{I} \tag{27.8}$$

Resistance of a conductor

From this result we see that resistance has SI units of volts per ampere. One volt per ampere is defined to be 1 **ohm** (Ω):

$$1\ \Omega \equiv \frac{1\ \text{V}}{1\ \text{A}} \tag{27.9}$$

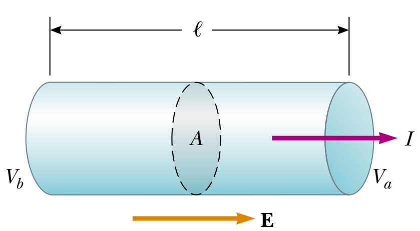


Figure 27.5 A uniform conductor of length ℓ and cross-sectional area A. A potential difference $\Delta V = V_b - V_a$ maintained across the conductor sets up an electric field $\mathbf{E}$, and this field produces a current I that is proportional to the potential difference.

[2] This result follows from the definition of potential difference:

$$V_b - V_a = -\int_a^b \mathbf{E} \cdot d\mathbf{s} = E\int_0^{\ell} dx = E\ell$$

An assortment of resistors used in electric circuits.

This expression shows that if a potential difference of 1 V across a conductor causes a current of 1 A, the resistance of the conductor is 1 Ω. For example, if an electrical appliance connected to a 120-V source of potential difference carries a current of 6 A, its resistance is 20 Ω.

Equation 27.8 solved for potential difference ($\Delta V = I\ell/\sigma A$) explains part of the chapter-opening puzzler: How can a bird perch on a high-voltage power line without being electrocuted? Even though the potential difference between the ground and the wire might be hundreds of thousands of volts, that between the bird's feet (which is what determines how much current flows through the bird) is very small.

The inverse of conductivity is **resistivity**[3] ρ:

Resistivity

$$\rho \equiv \frac{1}{\sigma} \tag{27.10}$$

where ρ has the units ohm-meters ($\Omega \cdot \text{m}$). We can use this definition and Equation 27.8 to express the resistance of a uniform block of material as

Resistance of a uniform conductor

$$R = \rho\frac{\ell}{A} \tag{27.11}$$

Every ohmic material has a characteristic resistivity that depends on the properties of the material and on temperature. Additionally, as you can see from Equation 27.11, the resistance of a sample depends on geometry as well as on resistivity. Table 27.1 gives the resistivities of a variety of materials at 20°C. Note the enormous range, from very low values for good conductors such as copper and silver, to very high values for good insulators such as glass and rubber. An ideal conductor would have zero resistivity, and an ideal insulator would have infinite resistivity.

Equation 27.11 shows that the resistance of a given cylindrical conductor is proportional to its length and inversely proportional to its cross-sectional area. If the length of a wire is doubled, then its resistance doubles. If its cross-sectional area is doubled, then its resistance decreases by one half. The situation is analogous to the flow of a liquid through a pipe. As the pipe's length is increased, the

[3] Do not confuse resistivity with mass density or charge density, for which the same symbol is used.

TABLE 27.1 Resistivities and Temperature Coefficients of Resistivity for Various Materials

Material	Resistivity[a] ($\Omega \cdot m$)	Temperature Coefficient $\alpha[(°C)^{-1}]$
Silver	1.59×10^{-8}	3.8×10^{-3}
Copper	1.7×10^{-8}	3.9×10^{-3}
Gold	2.44×10^{-8}	3.4×10^{-3}
Aluminum	2.82×10^{-8}	3.9×10^{-3}
Tungsten	5.6×10^{-8}	4.5×10^{-3}
Iron	10×10^{-8}	5.0×10^{-3}
Platinum	11×10^{-8}	3.92×10^{-3}
Lead	22×10^{-8}	3.9×10^{-3}
Nichrome[b]	1.50×10^{-6}	0.4×10^{-3}
Carbon	3.5×10^{-5}	-0.5×10^{-3}
Germanium	0.46	-48×10^{-3}
Silicon	640	-75×10^{-3}
Glass	10^{10} to 10^{14}	
Hard rubber	$\approx 10^{13}$	
Sulfur	10^{15}	
Quartz (fused)	75×10^{16}	

[a] All values at 20°C.

[b] A nickel–chromium alloy commonly used in heating elements.

resistance to flow increases. As the pipe's cross-sectional area is increased, more liquid crosses a given cross-section of the pipe per unit time. Thus, more liquid flows for the same pressure differential applied to the pipe, and the resistance to flow decreases.

Most electric circuits use devices called **resistors** to control the current level in the various parts of the circuit. Two common types of resistors are the *composition resistor,* which contains carbon, and the *wire-wound resistor,* which consists of a coil of wire. Resistors' values in ohms are normally indicated by color-coding, as shown in Figure 27.6 and Table 27.2.

Ohmic materials have a linear current–potential difference relationship over a broad range of applied potential differences (Fig. 27.7a). The slope of the I-versus-ΔV curve in the linear region yields a value for $1/R$. Nonohmic materials

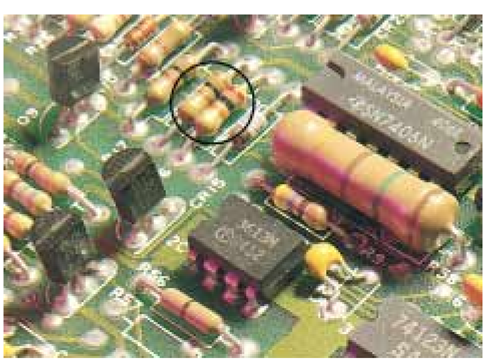


Figure 27.6 The colored bands on a resistor represent a code for determining resistance. The first two colors give the first two digits in the resistance value. The third color represents the power of ten for the multiplier of the resistance value. The last color is the tolerance of the resistance value. As an example, the four colors on the circled resistors are red ($= 2$), black ($= 0$), orange ($= 10^3$), and gold ($= 5\%$), and so the resistance value is $20 \times 10^3\ \Omega = 20\ k\Omega$ with a tolerance value of $5\% = 1\ k\Omega$. (The values for the colors are from Table 27.2.)

TABLE 27.2 Color Coding for Resistors

Color	Number	Multiplier	Tolerance
Black	0	1	
Brown	1	10^1	
Red	2	10^2	
Orange	3	10^3	
Yellow	4	10^4	
Green	5	10^5	
Blue	6	10^6	
Violet	7	10^7	
Gray	8	10^8	
White	9	10^9	
Gold		10^{-1}	5%
Silver		10^{-2}	10%
Colorless			20%

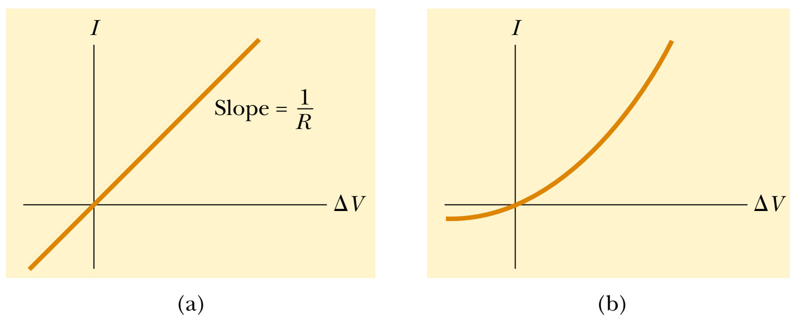


Figure 27.7 (a) The current–potential difference curve for an ohmic material. The curve is linear, and the slope is equal to the inverse of the resistance of the conductor. (b) A nonlinear current–potential difference curve for a semiconducting diode. This device does not obey Ohm's law.

have a nonlinear current–potential difference relationship. One common semiconducting device that has nonlinear I-versus-ΔV characteristics is the *junction diode* (Fig. 27.7b). The resistance of this device is low for currents in one direction (positive ΔV) and high for currents in the reverse direction (negative ΔV). In fact, most modern electronic devices, such as transistors, have nonlinear current–potential difference relationships; their proper operation depends on the particular way in which they violate Ohm's law.

Quick Quiz 27.3

What does the slope of the curved line in Figure 27.7b represent?

Quick Quiz 27.4

Your boss asks you to design an automobile battery jumper cable that has a low resistance. In view of Equation 27.11, what factors would you consider in your design?

EXAMPLE 27.2 The Resistance of a Conductor

Calculate the resistance of an aluminum cylinder that is 10.0 cm long and has a cross-sectional area of $2.00 \times 10^{-4}\ \text{m}^2$. Repeat the calculation for a cylinder of the same dimensions and made of glass having a resistivity of $3.0 \times 10^{10}\ \Omega\cdot\text{m}$.

Solution From Equation 27.11 and Table 27.1, we can calculate the resistance of the aluminum cylinder as follows:

$$R = \rho \frac{\ell}{A} = (2.82 \times 10^{-8}\ \Omega\cdot\text{m})\left(\frac{0.100\ \text{m}}{2.00 \times 10^{-4}\ \text{m}^2}\right)$$

$$= \boxed{1.41 \times 10^{-5}\ \Omega}$$

Similarly, for glass we find that

$$R = \rho \frac{\ell}{A} = (3.0 \times 10^{10}\ \Omega\cdot\text{m})\left(\frac{0.100\ \text{m}}{2.00 \times 10^{-4}\ \text{m}^2}\right)$$

$$= \boxed{1.5 \times 10^{13}\ \Omega}$$

As you might guess from the large difference in resistivities, the resistance of identically shaped cylinders of aluminum and glass differ widely. The resistance of the glass cylinder is 18 orders of magnitude greater than that of the aluminum cylinder.

Electrical insulators on telephone poles are often made of glass because of its low electrical conductivity.

EXAMPLE 27.3 The Resistance of Nichrome Wire

(a) Calculate the resistance per unit length of a 22-gauge Nichrome wire, which has a radius of 0.321 mm.

Solution The cross-sectional area of this wire is

$$A = \pi r^2 = \pi(0.321 \times 10^{-3}\ \text{m})^2 = 3.24 \times 10^{-7}\ \text{m}^2$$

The resistivity of Nichrome is $1.5 \times 10^{-6}\ \Omega\cdot\text{m}$ (see Table 27.1). Thus, we can use Equation 27.11 to find the resistance per unit length:

$$\frac{R}{\ell} = \frac{\rho}{A} = \frac{1.5 \times 10^{-6}\ \Omega\cdot\text{m}}{3.24 \times 10^{-7}\ \text{m}^2} = \boxed{4.6\ \Omega/\text{m}}$$

(b) If a potential difference of 10 V is maintained across a 1.0-m length of the Nichrome wire, what is the current in the wire?

Solution Because a 1.0-m length of this wire has a resistance of 4.6 Ω, Equation 27.8 gives

$$I = \frac{\Delta V}{R} = \frac{10\ \text{V}}{4.6\ \Omega} = \boxed{2.2\ \text{A}}$$

Note from Table 27.1 that the resistivity of Nichrome wire is about 100 times that of copper. A copper wire of the same radius would have a resistance per unit length of only 0.052 Ω/m. A 1.0-m length of copper wire of the same radius would carry the same current (2.2 A) with an applied potential difference of only 0.11 V.

Because of its high resistivity and its resistance to oxidation, Nichrome is often used for heating elements in toasters, irons, and electric heaters.

Exercise What is the resistance of a 6.0-m length of 22-gauge Nichrome wire? How much current does the wire carry when connected to a 120-V source of potential difference?

Answer 28 Ω; 4.3 A.

Exercise Calculate the current density and electric field in the wire when it carries a current of 2.2 A.

Answer $6.8 \times 10^6\ \text{A/m}^2$; 10 N/C.

EXAMPLE 27.4 The Radial Resistance of a Coaxial Cable

Coaxial cables are used extensively for cable television and other electronic applications. A coaxial cable consists of two cylindrical conductors. The gap between the conductors is completely filled with silicon, as shown in Figure 27.8a, and current leakage through the silicon is unwanted. (The cable is designed to conduct current along its length.) The radius

of the inner conductor is $a = 0.500$ cm, the radius of the outer one is $b = 1.75$ cm, and the length of the cable is $L = 15.0$ cm. Calculate the resistance of the silicon between the two conductors.

Solution In this type of problem, we must divide the object whose resistance we are calculating into concentric elements of infinitesimal thickness dr (Fig. 27.8b). We start by using the differential form of Equation 27.11, replacing ℓ with r for the distance variable: $dR = \rho\, dr/A$, where dR is the resistance of an element of silicon of thickness dr and surface area A. In this example, we take as our representative concentric element a hollow silicon cylinder of radius r, thickness dr, and length L, as shown in Figure 27.8. Any current that passes from the inner conductor to the outer one must pass radially through this concentric element, and the area through which this current passes is $A = 2\pi rL$. (This is the curved surface area—circumference multiplied by length—of our hollow silicon cylinder of thickness dr.) Hence, we can write the resistance of our hollow cylinder of silicon as

$$dR = \frac{\rho}{2\pi rL}\, dr$$

Because we wish to know the total resistance across the entire thickness of the silicon, we must integrate this expression from $r = a$ to $r = b$:

$$R = \int_a^b dR = \frac{\rho}{2\pi L}\int_a^b \frac{dr}{r} = \frac{\rho}{2\pi L}\ln\left(\frac{b}{a}\right)$$

Substituting in the values given, and using $\rho = 640\ \Omega\cdot\text{m}$ for silicon, we obtain

$$R = \frac{640\ \Omega\cdot\text{m}}{2\pi(0.150\text{ m})}\ln\left(\frac{1.75\text{ cm}}{0.500\text{ cm}}\right) = \boxed{851\ \Omega}$$

Exercise If a potential difference of 12.0 V is applied between the inner and outer conductors, what is the value of the total current that passes between them?

Answer 14.1 mA.

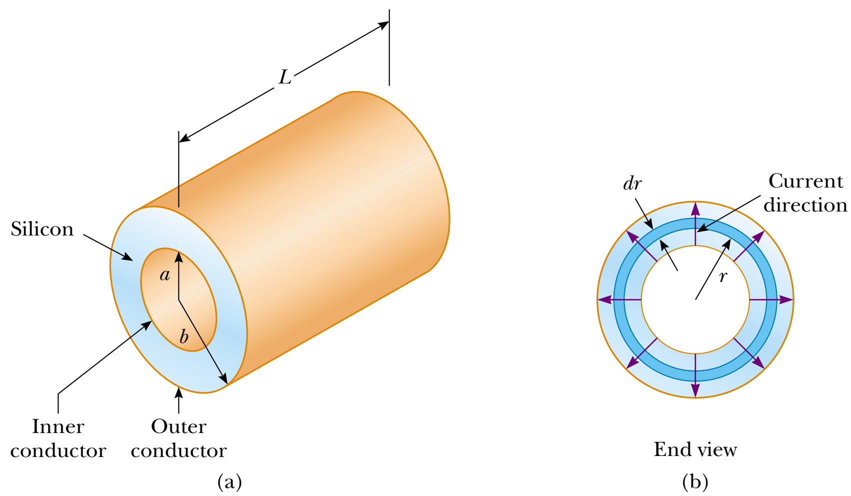


Figure 27.8 A coaxial cable. (a) Silicon fills the gap between the two conductors. (b) End view, showing current leakage.

27.3 A MODEL FOR ELECTRICAL CONDUCTION

In this section we describe a classical model of electrical conduction in metals that was first proposed by Paul Drude in 1900. This model leads to Ohm's law and shows that resistivity can be related to the motion of electrons in metals. Although the Drude model described here does have limitations, it nevertheless introduces concepts that are still applied in more elaborate treatments.

Consider a conductor as a regular array of atoms plus a collection of free electrons, which are sometimes called *conduction* electrons. The conduction electrons, although bound to their respective atoms when the atoms are not part of a solid, gain mobility when the free atoms condense into a solid. In the absence of an electric field, the conduction electrons move in random directions through the con-

ductor with average speeds of the order of 10^6 m/s. The situation is similar to the motion of gas molecules confined in a vessel. In fact, some scientists refer to conduction electrons in a metal as an *electron gas*. There is no current through the conductor in the absence of an electric field because the drift velocity of the free electrons is zero. That is, on the average, just as many electrons move in one direction as in the opposite direction, and so there is no net flow of charge.

This situation changes when an electric field is applied. Now, in addition to undergoing the random motion just described, the free electrons drift slowly in a direction opposite that of the electric field, with an average drift speed v_d that is much smaller (typically 10^{-4} m/s) than their average speed between collisions (typically 10^6 m/s).

Figure 27.9 provides a crude description of the motion of free electrons in a conductor. In the absence of an electric field, there is no net displacement after many collisions (Fig. 27.9a). An electric field $\mathbf{E}$ modifies the random motion and causes the electrons to drift in a direction opposite that of $\mathbf{E}$ (Fig. 27.9b). The slight curvature in the paths shown in Figure 27.9b results from the acceleration of the electrons between collisions, which is caused by the applied field.

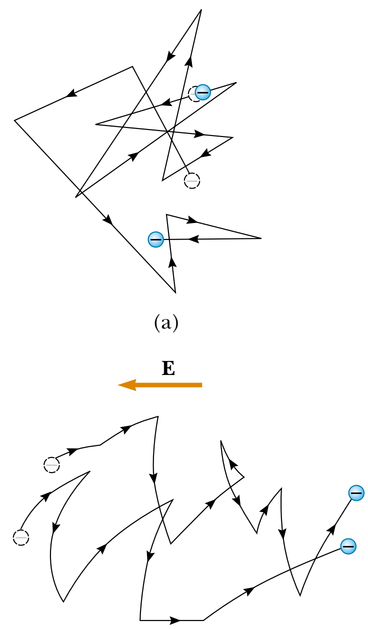


Figure 27.9 (a) A schematic diagram of the random motion of two charge carriers in a conductor in the absence of an electric field. The drift velocity is zero. (b) The motion of the charge carriers in a conductor in the presence of an electric field. Note that the random motion is modified by the field, and the charge carriers have a drift velocity.

In our model, we assume that the motion of an electron after a collision is independent of its motion before the collision. We also assume that the excess energy acquired by the electrons in the electric field is lost to the atoms of the conductor when the electrons and atoms collide. The energy given up to the atoms increases their vibrational energy, and this causes the temperature of the conductor to increase. The temperature increase of a conductor due to resistance is utilized in electric toasters and other familiar appliances.

We are now in a position to derive an expression for the drift velocity. When a free electron of mass m_e and charge $q\ (= -e)$ is subjected to an electric field $\mathbf{E}$, it experiences a force $\mathbf{F} = q\mathbf{E}$. Because $\Sigma\mathbf{F} = m_e\mathbf{a}$, we conclude that the acceleration of the electron is

$$\mathbf{a} = \frac{q\mathbf{E}}{m_e} \tag{27.12}$$

This acceleration, which occurs for only a short time between collisions, enables the electron to acquire a small drift velocity. If t is the time since the last collision and $\mathbf{v}_i$ is the electron's initial velocity the instant after that collision, then the velocity of the electron after a time t is

$$\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t = \mathbf{v}_i + \frac{q\mathbf{E}}{m_e}\,t \tag{27.13}$$

We now take the average value of $\mathbf{v}_f$ over all possible times t and all possible values of $\mathbf{v}_i$. If we assume that the initial velocities are randomly distributed over all possible values, we see that the average value of $\mathbf{v}_i$ is zero. The term $(q\mathbf{E}/m_e)t$ is the velocity added by the field during one trip between atoms. If the electron starts with zero velocity, then the average value of the second term of Equation 27.13 is $(q\mathbf{E}/m_e)\tau$, where τ is the *average time interval between successive collisions.* Because the average value of $\mathbf{v}_f$ is equal to the drift velocity,[4] we have

Drift velocity

$$\overline{\mathbf{v}}_f = \mathbf{v}_d = \frac{q\mathbf{E}}{m_e}\,\tau \tag{27.14}$$

[4] Because the collision process is random, each collision event is *independent* of what happened earlier. This is analogous to the random process of throwing a die. The probability of rolling a particular number on one throw is independent of the result of the previous throw. On average, the particular number comes up every sixth throw, starting at any arbitrary time.

We can relate this expression for drift velocity to the current in the conductor. Substituting Equation 27.14 into Equation 27.6, we find that the magnitude of the current density is

Current density

$$J = nqv_d = \frac{nq^2E}{m_e}\tau \tag{27.15}$$

where n is the number of charge carriers per unit volume. Comparing this expression with Ohm's law, $J = \sigma E$, we obtain the following relationships for conductivity and resistivity:

Conductivity

$$\sigma = \frac{nq^2\tau}{m_e} \tag{27.16}$$

Resistivity

$$\rho = \frac{1}{\sigma} = \frac{m_e}{nq^2\tau} \tag{27.17}$$

According to this classical model, conductivity and resistivity do not depend on the strength of the electric field. This feature is characteristic of a conductor obeying Ohm's law.

The average time between collisions τ is related to the average distance between collisions ℓ (that is, the *mean free path;* see Section 21.7) and the average speed $\bar{v}$ through the expression

$$\tau = \frac{\ell}{\bar{v}} \tag{27.18}$$

EXAMPLE 27.5 Electron Collisions in a Wire

(a) Using the data and results from Example 27.1 and the classical model of electron conduction, estimate the average time between collisions for electrons in household copper wiring.

Solution From Equation 27.17, we see that

$$\tau = \frac{m_e}{nq^2\rho}$$

where $\rho = 1.7 \times 10^{-8}\ \Omega\cdot\text{m}$ for copper and the carrier density is $n = 8.49 \times 10^{28}$ electrons/m^3 for the wire described in Example 27.1. Substitution of these values into the expression above gives

$$\tau = \frac{(9.11 \times 10^{-31}\ \text{kg})}{(8.49 \times 10^{28}\ \text{m}^{-3})(1.6 \times 10^{-19}\ \text{C})^2(1.7 \times 10^{-8}\ \Omega\cdot\text{m})}$$

$$= \boxed{2.5 \times 10^{-14}\ \text{s}}$$

(b) Assuming that the average speed for free electrons in copper is 1.6×10^6 m/s and using the result from part (a), calculate the mean free path for electrons in copper.

Solution

$$\ell = \bar{v}\tau = (1.6 \times 10^6\ \text{m/s})(2.5 \times 10^{-14}\ \text{s})$$

$$= \boxed{4.0 \times 10^{-8}\ \text{m}}$$

which is equivalent to 40 nm (compared with atomic spacings of about 0.2 nm). Thus, although the time between collisions is very short, an electron in the wire travels about 200 atomic spacings between collisions.

Although this classical model of conduction is consistent with Ohm's law, it is not satisfactory for explaining some important phenomena. For example, classical values for $\bar{v}$ calculated on the basis of an ideal-gas model (see Section 21.6) are smaller than the true values by about a factor of ten. Furthermore, if we substitute $\ell/\bar{v}$ for τ in Equation 27.17 and rearrange terms so that $\bar{v}$ appears in the numerator, we find that the resistivity ρ is proportional to $\bar{v}$. According to the ideal-gas model, $\bar{v}$ is proportional to $\sqrt{T}$; hence, it should also be true that $\rho \propto \sqrt{T}$. This is in disagreement with the fact that, for pure metals, resistivity depends linearly on temperature. We are able to account for the linear dependence only by using a quantum mechanical model, which we now describe briefly.

According to quantum mechanics, electrons have wave-like properties. If the array of atoms in a conductor is regularly spaced (that is, it is periodic), then the wave-like character of the electrons enables them to move freely through the conductor, and a collision with an atom is unlikely. For an idealized conductor, no collisions would occur, the mean free path would be infinite, and the resistivity would be zero. Electron waves are scattered only if the atomic arrangement is irregular (not periodic) as a result of, for example, structural defects or impurities. At low temperatures, the resistivity of metals is dominated by scattering caused by collisions between electrons and defects or impurities. At high temperatures, the resistivity is dominated by scattering caused by collisions between electrons and atoms of the conductor, which are continuously displaced from the regularly spaced array as a result of thermal agitation. The thermal motion of the atoms causes the structure to be irregular (compared with an atomic array at rest), thereby reducing the electron's mean free path.

27.4 RESISTANCE AND TEMPERATURE

Over a limited temperature range, the resistivity of a metal varies approximately linearly with temperature according to the expression

$$\rho = \rho_0[1 + \alpha(T - T_0)] \qquad (27.19)$$

Variation of ρ with temperature

where ρ is the resistivity at some temperature T (in degrees Celsius), ρ_0 is the resistivity at some reference temperature T_0 (usually taken to be 20°C), and α is the **temperature coefficient of resistivity.** From Equation 27.19, we see that the temperature coefficient of resistivity can be expressed as

$$\alpha = \frac{1}{\rho_0}\frac{\Delta\rho}{\Delta T} \qquad (27.20)$$

Temperature coefficient of resistivity

where $\Delta\rho = \rho - \rho_0$ is the change in resistivity in the temperature interval $\Delta T = T - T_0$.

The temperature coefficients of resistivity for various materials are given in Table 27.1. Note that the unit for α is degrees Celsius^{-1} $[(°\mathrm{C})^{-1}]$. Because resistance is proportional to resistivity (Eq. 27.11), we can write the variation of resistance as

$$R = R_0[1 + \alpha(T - T_0)] \qquad (27.21)$$

Use of this property enables us to make precise temperature measurements, as shown in the following example.

EXAMPLE 27.6 A Platinum Resistance Thermometer

A resistance thermometer, which measures temperature by measuring the change in resistance of a conductor, is made from platinum and has a resistance of 50.0 Ω at 20.0°C. When immersed in a vessel containing melting indium, its resistance increases to 76.8 Ω. Calculate the melting point of the indium.

Solution Solving Equation 27.21 for ΔT and using the α value for platinum given in Table 27.1, we obtain

$$\Delta T = \frac{R - R_0}{\alpha R_0} = \frac{76.8\ \Omega - 50.0\ \Omega}{[3.92 \times 10^{-3}\ (°\mathrm{C})^{-1}](50.0\ \Omega)} = 137°\mathrm{C}$$

Because $T_0 = 20.0°\mathrm{C}$, we find that T, the temperature of the melting indium sample, is 157°C.

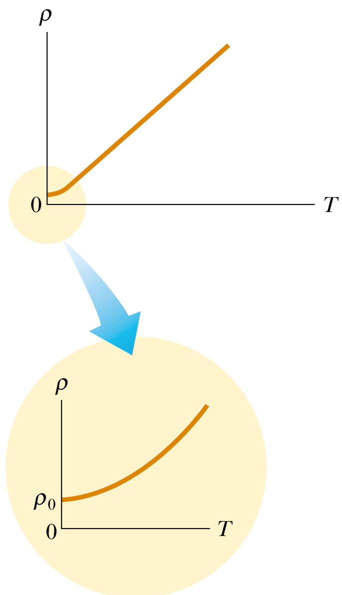


Figure 27.10 Resistivity versus temperature for a metal such as copper. The curve is linear over a wide range of temperatures, and ρ increases with increasing temperature. As T approaches absolute zero (inset), the resistivity approaches a finite value ρ_0.

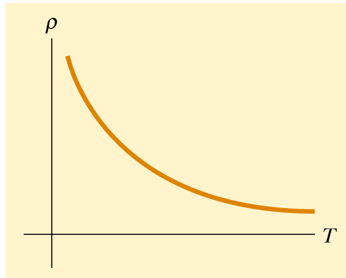


Figure 27.11 Resistivity versus temperature for a pure semiconductor, such as silicon or germanium.

For metals like copper, resistivity is nearly proportional to temperature, as shown in Figure 27.10. However, a nonlinear region always exists at very low temperatures, and the resistivity usually approaches some finite value as the temperature nears absolute zero. This residual resistivity near absolute zero is caused primarily by the collision of electrons with impurities and imperfections in the metal. In contrast, high-temperature resistivity (the linear region) is predominantly characterized by collisions between electrons and metal atoms.

Notice that three of the α values in Table 27.1 are negative; this indicates that the resistivity of these materials decreases with increasing temperature (Fig. 27.11). This behavior is due to an increase in the density of charge carriers at higher temperatures.

Because the charge carriers in a semiconductor are often associated with impurity atoms, the resistivity of these materials is very sensitive to the type and concentration of such impurities. We shall return to the study of semiconductors in Chapter 43 of the extended version of this text.

Quick Quiz 27.5

When does a lightbulb carry more current—just after it is turned on and the glow of the metal filament is increasing, or after it has been on for a few milliseconds and the glow is steady?

Optional Section

27.5 SUPERCONDUCTORS

There is a class of metals and compounds whose resistance decreases to zero when they are below a certain temperature T_c, known as the *critical temperature.* These materials are known as **superconductors.** The resistance–temperature graph for a superconductor follows that of a normal metal at temperatures above T_c (Fig. 27.12). When the temperature is at or below T_c, the resistivity drops suddenly to zero. This phenomenon was discovered in 1911 by the Dutch physicist Heike Kamerlingh-Onnes (1853–1926) as he worked with mercury, which is a superconductor below 4.2 K. Recent measurements have shown that the resistivities of superconductors below their T_c values are less than $4 \times 10^{-25}\ \Omega \cdot \text{m}$—around 10^{17} times smaller than the resistivity of copper and in practice considered to be zero.

Today thousands of superconductors are known, and as Figure 27.13 illustrates, the critical temperatures of recently discovered superconductors are substantially higher than initially thought possible. Two kinds of superconductors are recognized. The more recently identified ones, such as $YBa_2Cu_3O_7$, are essentially ceramics with high critical temperatures, whereas superconducting materials such

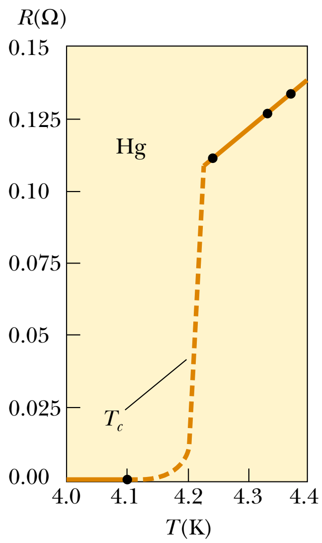


Figure 27.12 Resistance versus temperature for a sample of mercury (Hg). The graph follows that of a normal metal above the critical temperature T_c. The resistance drops to zero at T_c, which is 4.2 K for mercury.

A small permanent magnet levitated above a disk of the superconductor $YBa_2Cu_3O_7$, which is at 77 K.

as those observed by Kamerlingh-Onnes are metals. If a room-temperature superconductor is ever identified, its impact on technology could be tremendous.

The value of T_c is sensitive to chemical composition, pressure, and molecular structure. It is interesting to note that copper, silver, and gold, which are excellent conductors, do not exhibit superconductivity.

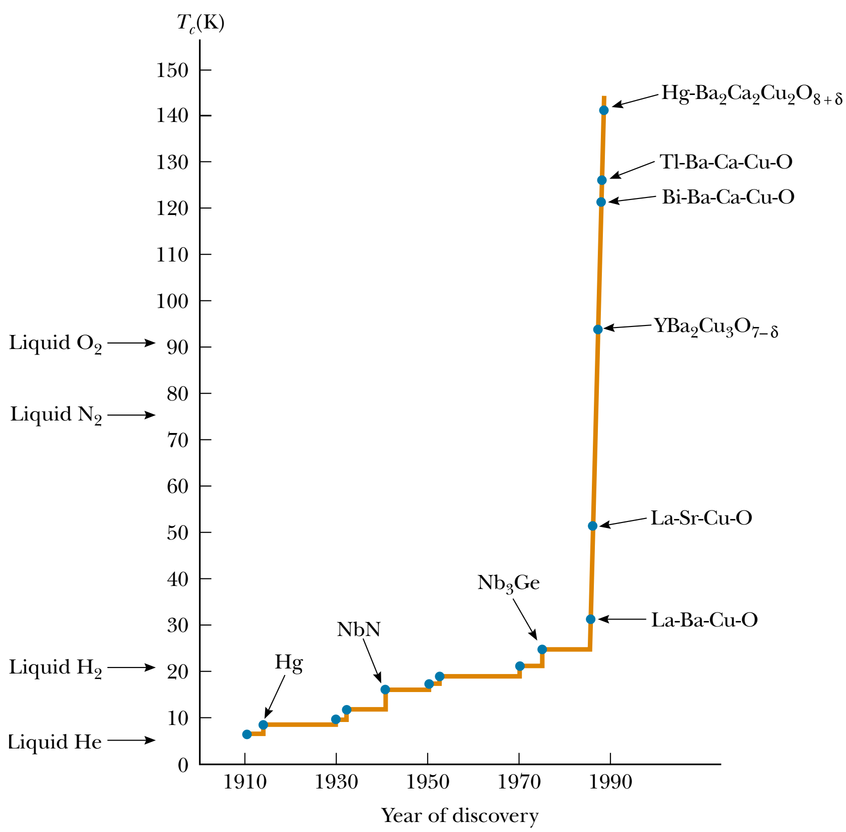


Figure 27.13 Evolution of the superconducting critical temperature since the discovery of the phenomenon.

One of the truly remarkable features of superconductors is that once a current is set up in them, it persists *without any applied potential difference* (because $R = 0$). Steady currents have been observed to persist in superconducting loops for several years with no apparent decay!

An important and useful application of superconductivity is in the development of superconducting magnets, in which the magnitudes of the magnetic field are about ten times greater than those produced by the best normal electromagnets. Such superconducting magnets are being considered as a means of storing energy. Superconducting magnets are currently used in medical magnetic resonance imaging (MRI) units, which produce high-quality images of internal organs without the need for excessive exposure of patients to x-rays or other harmful radiation.

For further information on superconductivity, see Section 43.8.

27.6 ELECTRICAL ENERGY AND POWER

13.3

If a battery is used to establish an electric current in a conductor, the chemical energy stored in the battery is continuously transformed into kinetic energy of the charge carriers. In the conductor, this kinetic energy is quickly lost as a result of collisions between the charge carriers and the atoms making up the conductor, and this leads to an increase in the temperature of the conductor. In other words, the chemical energy stored in the battery is continuously transformed to internal energy associated with the temperature of the conductor.

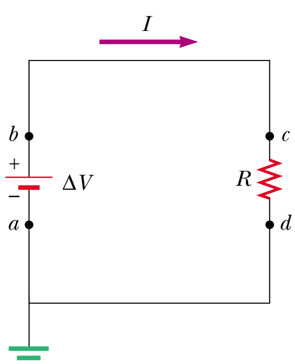


Figure 27.14 A circuit consisting of a resistor of resistance R and a battery having a potential difference ΔV across its terminals. Positive charge flows in the clockwise direction. Points a and d are grounded.

Consider a simple circuit consisting of a battery whose terminals are connected to a resistor, as shown in Figure 27.14. (Resistors are designated by the symbol .) Now imagine following a positive quantity of charge ΔQ that is moving clockwise around the circuit from point a through the battery and resistor back to point a. Points a and d are *grounded* (ground is designated by the symbol); that is, we take the electric potential at these two points to be zero. As the charge moves from a to b through the battery, its electric potential energy U *increases* by an amount $\Delta V \Delta Q$ (where ΔV is the potential difference between b and a), while the chemical potential energy in the battery *decreases* by the same amount. (Recall from Eq. 25.9 that $\Delta U = q\Delta V$.) However, as the charge moves from c to d through the resistor, it *loses* this electric potential energy as it collides with atoms in the resistor, thereby producing internal energy. If we neglect the resistance of the connecting wires, no loss in energy occurs for paths bc and da. When the charge arrives at point a, it must have the same electric potential energy (zero) that it had at the start.[5] Note that because charge cannot build up at any point, the current is the same everywhere in the circuit.

The rate at which the charge ΔQ loses potential energy in going through the resistor is

$$\frac{\Delta U}{\Delta t} = \frac{\Delta Q}{\Delta t}\Delta V = I\Delta V$$

where I is the current in the circuit. In contrast, the charge regains this energy when it passes through the battery. Because the rate at which the charge loses energy equals the power $\mathcal{P}$ delivered to the resistor (which appears as internal energy), we have

Power

$$\mathcal{P} = I\Delta V \qquad \textbf{(27.22)}$$

[5] Note that once the current reaches its steady-state value, there is *no* change in the kinetic energy of the charge carriers creating the current.

In this case, the power is supplied to a resistor by a battery. However, we can use Equation 27.22 to determine the power transferred to *any* device carrying a current I and having a potential difference ΔV between its terminals.

Using Equation 27.22 and the fact that $\Delta V = IR$ for a resistor, we can express the power delivered to the resistor in the alternative forms

$$\mathcal{P} = I^2R = \frac{(\Delta V)^2}{R} \tag{27.23}$$

Power delivered to a resistor

When I is expressed in amperes, ΔV in volts, and R in ohms, the SI unit of power is the watt, as it was in Chapter 7 in our discussion of mechanical power. The power lost as internal energy in a conductor of resistance R is called *joule heating*[6]; this transformation is also often referred to as an I^2R loss.

A battery, a device that supplies electrical energy, is called either a *source of electromotive force* or, more commonly, an *emf source*. The concept of emf is discussed in greater detail in Chapter 28. (The phrase *electromotive force* is an unfortunate choice because it describes not a force but rather a potential difference in volts.) **When the internal resistance of the battery is neglected, the potential difference between points a and b in Figure 27.14 is equal to the emf $\mathcal{E}$ of the battery**—that is, $\Delta V = V_b - V_a = \mathcal{E}$. This being true, we can state that the current in the circuit is $I = \Delta V/R = \mathcal{E}/R$. Because $\Delta V = \mathcal{E}$, the power supplied by the emf source can be expressed as $\mathcal{P} = I\mathcal{E}$, which equals the power delivered to the resistor, I^2R.

When transporting electrical energy through power lines, such as those shown in Figure 27.15, utility companies seek to minimize the power transformed to internal energy in the lines and maximize the energy delivered to the consumer. Because $\mathcal{P} = I\Delta V$, the same amount of power can be transported either at high currents and low potential differences or at low currents and high potential differences. Utility companies choose to transport electrical energy at low currents and high potential differences primarily for economic reasons. Copper wire is very expensive, and so it is cheaper to use high-resistance wire (that is, wire having a small cross-sectional area; see Eq. 27.11). Thus, in the expression for the power delivered to a resistor, $\mathcal{P} = I^2R$, the resistance of the wire is fixed at a relatively high value for economic considerations. The I^2R loss can be reduced by keeping the current I as low as possible. In some instances, power is transported at potential differences as great as 765 kV. Once the electricity reaches your city, the potential difference is usually reduced to 4 kV by a device called a *transformer.* Another transformer drops the potential difference to 240 V before the electricity finally reaches your home. Of course, each time the potential difference decreases, the current increases by the same factor, and the power remains the same. We shall discuss transformers in greater detail in Chapter 33.

Figure 27.15 Power companies transfer electrical energy at high potential differences.

Quick Quiz 27.6

The same potential difference is applied to the two lightbulbs shown in Figure 27.16. Which one of the following statements is true?

(a) The 30-W bulb carries the greater current and has the higher resistance.

(b) The 30-W bulb carries the greater current, but the 60-W bulb has the higher resistance.

QuickLab

If you have access to an ohmmeter, verify your answer to Quick Quiz 27.6 by testing the resistance of a few lightbulbs.

[6] It is called *joule heating* even though the process of heat does not occur. This is another example of incorrect usage of the word *heat* that has become entrenched in our language.

Figure 27.16 These lightbulbs operate at their rated power only when they are connected to a 120-V source.

(c) The 30-W bulb has the higher resistance, but the 60-W bulb carries the greater current.
(d) The 60-W bulb carries the greater current and has the higher resistance.

QuickLab

From the labels on household appliances such as hair dryers, televisions, and stereos, estimate the annual cost of operating them.

Quick Quiz 27.7

For the two lightbulbs shown in Figure 27.17, rank the current values at points a through f, from greatest to least.

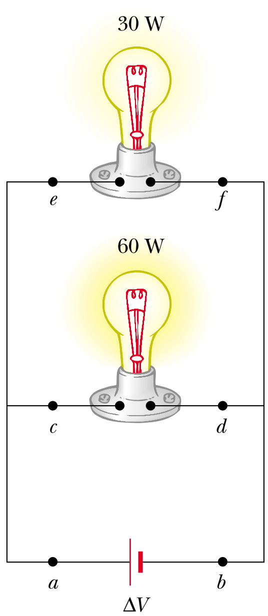


Figure 27.17 Two lightbulbs connected across the same potential difference. The bulbs operate at their rated power only if they are connected to a 120-V battery.

EXAMPLE 27.7 Power in an Electric Heater

An electric heater is constructed by applying a potential difference of 120 V to a Nichrome wire that has a total resistance of 8.00 Ω. Find the current carried by the wire and the power rating of the heater.

Solution Because $\Delta V = IR$, we have

$$I = \frac{\Delta V}{R} = \frac{120\text{ V}}{8.00\ \Omega} = \boxed{15.0\text{ A}}$$

We can find the power rating using the expression $\mathcal{P} = I^2R$:

$$\mathcal{P} = I^2R = (15.0\text{ A})^2(8.00\ \Omega) = \boxed{1.80\text{ kW}}$$

If we doubled the applied potential difference, the current would double but the power would quadruple because $\mathcal{P} = (\Delta V)^2/R$.

EXAMPLE 27.8 The Cost of Making Dinner

Estimate the cost of cooking a turkey for 4 h in an oven that operates continuously at 20.0 A and 240 V.

Solution The power used by the oven is

$$\mathcal{P} = I\Delta V = (20.0\text{ A})(240\text{ V}) = 4\,800\text{ W} = 4.80\text{ kW}$$

Because the energy consumed equals power × time, the amount of energy for which you must pay is

$$\text{Energy} = \mathcal{P}t = (4.80\text{ kW})(4\text{ h}) = 19.2\text{ kWh}$$

If the energy is purchased at an estimated price of 8.00¢ per kilowatt hour, the cost is

$$\text{Cost} = (19.2\text{ kWh})(\$0.080/\text{kWh}) = \boxed{\$1.54}$$

Demands on our dwindling energy supplies have made it necessary for us to be aware of the energy requirements of our electrical devices. Every electrical appliance carries a label that contains the information you need to calculate the appliance's power requirements. In many cases, the power consumption in watts is stated directly, as it is on a lightbulb. In other cases, the amount of current used by the device and the potential difference at which it operates are given. This information and Equation 27.22 are sufficient for calculating the operating cost of any electrical device.

Exercise What does it cost to operate a 100-W lightbulb for 24 h if the power company charges \$0.08/kWh?

Answer \$0.19.

EXAMPLE 27.9 Current in an Electron Beam

In a certain particle accelerator, electrons emerge with an energy of 40.0 MeV ($1\text{ MeV} = 1.60 \times 10^{-13}\text{ J}$). The electrons emerge not in a steady stream but rather in pulses at the rate of 250 pulses/s. This corresponds to a time between pulses of 4.00 ms (Fig. 27.18). Each pulse has a duration of 200 ns, and the electrons in the pulse constitute a current of 250 mA. The current is zero between pulses. (a) How many electrons are delivered by the accelerator per pulse?

Solution We use Equation 27.2 in the form $dQ = I\,dt$ and integrate to find the charge per pulse. While the pulse is on, the current is constant; thus,

$$Q_{\text{pulse}} = I\int dt = I\Delta t = (250 \times 10^{-3}\text{ A})(200 \times 10^{-9}\text{ s})$$
$$= 5.00 \times 10^{-8}\text{ C}$$

Dividing this quantity of charge per pulse by the electronic charge gives the number of electrons per pulse:

$$\text{Electrons per pulse} = \frac{5.00 \times 10^{-8}\text{ C/pulse}}{1.60 \times 10^{-19}\text{ C/electron}}$$
$$= \boxed{3.13 \times 10^{11}\text{ electrons/pulse}}$$

(b) What is the average current per pulse delivered by the accelerator?

Solution Average current is given by Equation 27.1, $I_{\text{av}} = \Delta Q/\Delta t$. Because the time interval between pulses is 4.00 ms, and because we know the charge per pulse from part (a), we obtain

$$I_{\text{av}} = \frac{Q_{\text{pulse}}}{\Delta t} = \frac{5.00 \times 10^{-8}\text{ C}}{4.00 \times 10^{-3}\text{ s}} = \boxed{12.5\ \mu\text{A}}$$

This represents only 0.005% of the peak current, which is 250 mA.

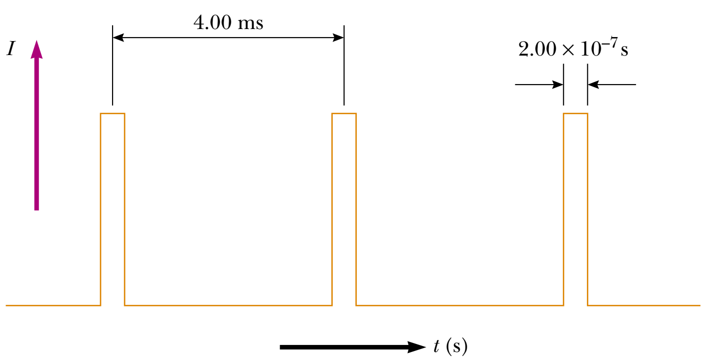


Figure 27.18 Current versus time for a pulsed beam of electrons.

(c) What is the maximum power delivered by the electron beam?

Solution By definition, power is energy delivered per unit time. Thus, the maximum power is equal to the energy delivered by a pulse divided by the pulse duration:

$$\mathcal{P} = \frac{E}{\Delta t}$$

$$= \frac{(3.13 \times 10^{11} \text{ electrons/pulse})(40.0 \text{ MeV/electron})}{2.00 \times 10^{-7} \text{ s/pulse}}$$

$$= (6.26 \times 10^{19} \text{ MeV/s})(1.60 \times 10^{-13} \text{ J/MeV})$$

$$= 1.00 \times 10^{7} \text{ W} = \boxed{10.0 \text{ MW}}$$

We could also compute this power directly. We assume that each electron had zero energy before being accelerated. Thus, by definition, each electron must have gone through a potential difference of 40.0 MV to acquire a final energy of 40.0 MeV. Hence, we have

$$\mathcal{P} = I\,\Delta V = (250 \times 10^{-3} \text{ A})(40.0 \times 10^{6} \text{ V}) = \boxed{10.0 \text{ MW}}$$

SUMMARY

The **electric current** I in a conductor is defined as

$$I \equiv \frac{dQ}{dt} \quad \textbf{(27.2)}$$

where dQ is the charge that passes through a cross-section of the conductor in a time dt. The SI unit of current is the **ampere** (A), where 1 A = 1 C/s.

The average current in a conductor is related to the motion of the charge carriers through the relationship

$$I_{\text{av}} = nqv_d A \quad \textbf{(27.4)}$$

where n is the density of charge carriers, q is the charge on each carrier, v_d is the drift speed, and A is the cross-sectional area of the conductor.

The magnitude of the **current density** J in a conductor is the current per unit area:

$$J \equiv \frac{I}{A} = nqv_d \quad \textbf{(27.5)}$$

The current density in a conductor is proportional to the electric field according to the expression

$$\mathbf{J} = \sigma \mathbf{E} \quad \textbf{(27.7)}$$

The proportionality constant σ is called the **conductivity** of the material of which the conductor is made. The inverse of σ is known as **resistivity** ρ ($\rho = 1/\sigma$). Equation 27.7 is known as **Ohm's law,** and a material is said to obey this law if the ratio of its current density $\mathbf{J}$ to its applied electric field $\mathbf{E}$ is a constant that is independent of the applied field.

The **resistance** R of a conductor is defined either in terms of the length of the conductor or in terms of the potential difference across it:

$$R \equiv \frac{\ell}{\sigma A} \equiv \frac{\Delta V}{I} \quad \textbf{(27.8)}$$

where ℓ is the length of the conductor, σ is the conductivity of the material of which it is made, A is its cross-sectional area, ΔV is the potential difference across it, and I is the current it carries.

The SI unit of resistance is volts per ampere, which is defined to be 1 **ohm** (Ω); that is, 1 Ω = 1 V/A. If the resistance is independent of the applied potential difference, the conductor obeys Ohm's law.

In a classical model of electrical conduction in metals, the electrons are treated as molecules of a gas. In the absence of an electric field, the average velocity of the electrons is zero. When an electric field is applied, the electrons move (on the average) with a **drift velocity** $\mathbf{v}_d$ that is opposite the electric field and given by the expression

$$\mathbf{v}_d = \frac{q\mathbf{E}}{m_e}\tau \tag{27.14}$$

where τ is the average time between electron–atom collisions, m_e is the mass of the electron, and q is its charge. According to this model, the resistivity of the metal is

$$\rho = \frac{m_e}{nq^2\tau} \tag{27.17}$$

where n is the number of free electrons per unit volume.

The resistivity of a conductor varies approximately linearly with temperature according to the expression

$$\rho = \rho_0[1 + \alpha(T - T_0)] \tag{27.19}$$

where α is the **temperature coefficient of resistivity** and ρ_0 is the resistivity at some reference temperature T_0.

If a potential difference ΔV is maintained across a resistor, the **power,** or rate at which energy is supplied to the resistor, is

$$\mathcal{P} = I\Delta V \tag{27.22}$$

Because the potential difference across a resistor is given by $\Delta V = IR$, we can express the power delivered to a resistor in the form

$$\mathcal{P} = I^2R = \frac{(\Delta V)^2}{R} \tag{27.23}$$

The electrical energy supplied to a resistor appears in the form of internal energy in the resistor.

Questions

1. Newspaper articles often contain statements such as "10 000 volts of electricity surged through the victim's body." What is wrong with this statement?
2. What is the difference between resistance and resistivity?
3. Two wires A and B of circular cross-section are made of the same metal and have equal lengths, but the resistance of wire A is three times greater than that of wire B. What is the ratio of their cross-sectional areas? How do their radii compare?
4. What is required in order to maintain a steady current in a conductor?
5. Do all conductors obey Ohm's law? Give examples to justify your answer.
6. When the voltage across a certain conductor is doubled, the current is observed to increase by a factor of three. What can you conclude about the conductor?
7. In the water analogy of an electric circuit, what corresponds to the power supply, resistor, charge, and potential difference?
8. Why might a "good" electrical conductor also be a "good" thermal conductor?
9. On the basis of the atomic theory of matter, explain why the resistance of a material should increase as its temperature increases.
10. How does the resistance for copper and silicon change with temperature? Why are the behaviors of these two materials different?
11. Explain how a current can persist in a superconductor in the absence of any applied voltage.
12. What single experimental requirement makes superconducting devices expensive to operate? In principle, can this limitation be overcome?

13. What would happen to the drift velocity of the electrons in a wire and to the current in the wire if the electrons could move freely without resistance through the wire?
14. If charges flow very slowly through a metal, why does it not require several hours for a light to turn on when you throw a switch?
15. In a conductor, the electric field that drives the electrons through the conductor propagates with a speed that is almost the same as the speed of light, even though the drift velocity of the electrons is very small. Explain how these can both be true. Does a given electron move from one end of the conductor to the other?
16. Two conductors of the same length and radius are connected across the same potential difference. One conductor has twice the resistance of the other. To which conductor is more power delivered?
17. Car batteries are often rated in ampere-hours. Does this designate the amount of current, power, energy, or charge that can be drawn from the battery?
18. If you were to design an electric heater using Nichrome wire as the heating element, what parameters of the wire could you vary to meet a specific power output, such as 1 000 W?
19. Consider the following typical monthly utility rate structure: \$2.00 for the first 16 kWh, 8.00¢/kWh for the next 34 kWh, 6.50¢/kWh for the next 50 kWh, 5.00¢/kWh for the next 100 kWh, 4.00¢/kWh for the next 200 kWh, and 3.50¢/kWh for all kilowatt-hours in excess of 400 kWh. On the basis of these rates, determine the amount charged for 327 kWh.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
 = paired numerical/symbolic problems

Section 27.1 Electric Current

1. In a particular cathode ray tube, the measured beam current is 30.0 μA. How many electrons strike the tube screen every 40.0 s?
2. A teapot with a surface area of 700 cm^2 is to be silver plated. It is attached to the negative electrode of an electrolytic cell containing silver nitrate ($Ag^+NO_3^-$). If the cell is powered by a 12.0-V battery and has a resistance of 1.80 Ω, how long does it take for a 0.133-mm layer of silver to build up on the teapot? (The density of silver is 10.5×10^3 kg/m^3.)

WEB 3. Suppose that the current through a conductor decreases exponentially with time according to the expression $I(t) = I_0 e^{-t/\tau}$, where I_0 is the initial current (at $t = 0$) and τ is a constant having dimensions of time. Consider a fixed observation point within the conductor. (a) How much charge passes this point between $t = 0$ and $t = \tau$? (b) How much charge passes this point between $t = 0$ and $t = 10\tau$? (c) How much charge passes this point between $t = 0$ and $t = \infty$?
4. In the Bohr model of the hydrogen atom, an electron in the lowest energy state follows a circular path at a distance of 5.29×10^{-11} m from the proton. (a) Show that the speed of the electron is 2.19×10^6 m/s. (b) What is the effective current associated with this orbiting electron?
5. A small sphere that carries a charge of 8.00 nC is whirled in a circle at the end of an insulating string. The angular frequency of rotation is 100π rad/s. What average current does this rotating charge represent?
6. A small sphere that carries a charge q is whirled in a circle at the end of an insulating string. The angular frequency of rotation is ω. What average current does this rotating charge represent?
7. The quantity of charge q (in coulombs) passing through a surface of area 2.00 cm^2 varies with time according to the equation $q = 4.00t^3 + 5.00t + 6.00$, where t is in seconds. (a) What is the instantaneous current through the surface at $t = 1.00$ s? (b) What is the value of the current density?
8. An electric current is given by the expression $I(t) = 100 \sin(120\pi t)$, where I is in amperes and t is in seconds. What is the total charge carried by the current from $t = 0$ to $t = 1/240$ s?
9. Figure P27.9 represents a section of a circular conductor of nonuniform diameter carrying a current of 5.00 A. The radius of cross-section A_1 is 0.400 cm. (a) What is the magnitude of the current density across A_1? (b) If the current density across A_2 is one-fourth the value across A_1, what is the radius of the conductor at A_2?

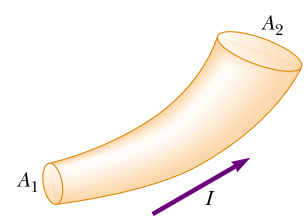


Figure P27.9

10. A Van de Graaff generator produces a beam of 2.00-MeV *deuterons,* which are heavy hydrogen nuclei containing a proton and a neutron. (a) If the beam current is 10.0 μA, how far apart are the deuterons? (b) Is their electrostatic repulsion a factor in beam stability? Explain.

11. The electron beam emerging from a certain high-energy electron accelerator has a circular cross-section of radius 1.00 mm. (a) If the beam current is 8.00 μA, what is the current density in the beam, assuming that it is uniform throughout? (b) The speed of the electrons is so close to the speed of light that their speed can be taken as $c = 3.00 \times 10^8$ m/s with negligible error. Find the electron density in the beam. (c) How long does it take for Avogadro's number of electrons to emerge from the accelerator?

12. An aluminum wire having a cross-sectional area of 4.00×10^{-6} m^2 carries a current of 5.00 A. Find the drift speed of the electrons in the wire. The density of aluminum is 2.70 g/cm^3. (Assume that one electron is supplied by each atom.)

Section 27.2 Resistance and Ohm's Law

13. A lightbulb has a resistance of 240 Ω when operating at a voltage of 120 V. What is the current through the lightbulb?

14. A resistor is constructed of a carbon rod that has a uniform cross-sectional area of 5.00 mm^2. When a potential difference of 15.0 V is applied across the ends of the rod, there is a current of 4.00×10^{-3} A in the rod. Find (a) the resistance of the rod and (b) the rod's length.

WEB 15. A 0.900-V potential difference is maintained across a 1.50-m length of tungsten wire that has a cross-sectional area of 0.600 mm^2. What is the current in the wire?

16. A conductor of uniform radius 1.20 cm carries a current of 3.00 A produced by an electric field of 120 V/m. What is the resistivity of the material?

17. Suppose that you wish to fabricate a uniform wire out of 1.00 g of copper. If the wire is to have a resistance of $R = 0.500$ Ω, and if all of the copper is to be used, what will be (a) the length and (b) the diameter of this wire?

18. (a) Make an order-of-magnitude estimate of the resistance between the ends of a rubber band. (b) Make an order-of-magnitude estimate of the resistance between the 'heads' and 'tails' sides of a penny. In each case, state what quantities you take as data and the values you measure or estimate for them. (c) What would be the order of magnitude of the current that each carries if it were connected across a 120-V power supply? (WARNING! Do not try this at home!)

19. A solid cube of silver (density = 10.5 g/cm^3) has a mass of 90.0 g. (a) What is the resistance between opposite faces of the cube? (b) If there is one conduction electron for each silver atom, what is the average drift speed of electrons when a potential difference of 1.00×10^{-5} V is applied to opposite faces? (The atomic number of silver is 47, and its molar mass is 107.87 g/mol.)

20. A metal wire of resistance R is cut into three equal pieces that are then connected side by side to form a new wire whose length is equal to one-third the original length. What is the resistance of this new wire?

21. A wire with a resistance R is lengthened to 1.25 times its original length by being pulled through a small hole. Find the resistance of the wire after it has been stretched.

22. Aluminum and copper wires of equal length are found to have the same resistance. What is the ratio of their radii?

23. A current density of 6.00×10^{-13} A/m^2 exists in the atmosphere where the electric field (due to charged thunderclouds in the vicinity) is 100 V/m. Calculate the electrical conductivity of the Earth's atmosphere in this region.

24. The rod in Figure P27.24 (not drawn to scale) is made of two materials. Both have a square cross section of 3.00 mm on a side. The first material has a resistivity of 4.00×10^{-3} Ω · m and is 25.0 cm long, while the second material has a resistivity of 6.00×10^{-3} Ω · m and is 40.0 cm long. What is the resistance between the ends of the rod?

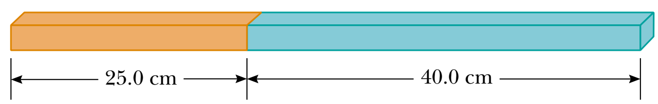


Figure P27.24

Section 27.3 A Model for Electrical Conduction

WEB 25. If the drift velocity of free electrons in a copper wire is 7.84×10^{-4} m/s, what is the electric field in the conductor?

26. If the current carried by a conductor is doubled, what happens to the (a) charge carrier density? (b) current density? (c) electron drift velocity? (d) average time between collisions?

27. Use data from Example 27.1 to calculate the collision mean free path of electrons in copper, assuming that the average thermal speed of conduction electrons is 8.60×10^5 m/s.

Section 27.4 Resistance and Temperature

28. While taking photographs in Death Valley on a day when the temperature is 58.0°C, Bill Hiker finds that a certain voltage applied to a copper wire produces a current of 1.000 A. Bill then travels to Antarctica and applies the same voltage to the same wire. What current does he register there if the temperature is −88.0°C? Assume that no change occurs in the wire's shape and size.

29. A certain lightbulb has a tungsten filament with a resistance of 19.0 Ω when cold and of 140 Ω when hot. Assuming that Equation 27.21 can be used over the large

temperature range involved here, find the temperature of the filament when hot. (Assume an initial temperature of 20.0°C.)

30. A carbon wire and a Nichrome wire are connected in series. If the combination has a resistance of 10.0 kΩ at 0°C, what is the resistance of each wire at 0°C such that the resistance of the combination does not change with temperature? (Note that the equivalent resistance of two resistors in series is the sum of their resistances.)

31. An aluminum wire with a diameter of 0.100 mm has a uniform electric field with a magnitude of 0.200 V/m imposed along its entire length. The temperature of the wire is 50.0°C. Assume one free electron per atom. (a) Using the information given in Table 27.1, determine the resistivity. (b) What is the current density in the wire? (c) What is the total current in the wire? (d) What is the drift speed of the conduction electrons? (e) What potential difference must exist between the ends of a 2.00-m length of the wire if the stated electric field is to be produced?

32. **Review Problem.** An aluminum rod has a resistance of 1.234 Ω at 20.0°C. Calculate the resistance of the rod at 120°C by accounting for the changes in both the resistivity and the dimensions of the rod.

33. What is the fractional change in the resistance of an iron filament when its temperature changes from 25.0°C to 50.0°C?

34. The resistance of a platinum wire is to be calibrated for low-temperature measurements. A platinum wire with a resistance of 1.00 Ω at 20.0°C is immersed in liquid nitrogen at 77 K (− 196°C). If the temperature response of the platinum wire is linear, what is the expected resistance of the platinum wire at − 196°C? ($\alpha_{\text{platinum}} = 3.92 \times 10^{-3}/°\text{C}$)

35. The temperature of a tungsten sample is raised while a copper sample is maintained at 20°C. At what temperature will the resistivity of the tungsten sample be four times that of the copper sample?

36. A segment of Nichrome wire is initially at 20.0°C. Using the data from Table 27.1, calculate the temperature to which the wire must be heated if its resistance is to be doubled.

Section 27.6 Electrical Energy and Power

37. A toaster is rated at 600 W when connected to a 120-V source. What current does the toaster carry, and what is its resistance?

38. In a hydroelectric installation, a turbine delivers 1 500 hp to a generator, which in turn converts 80.0% of the mechanical energy into electrical energy. Under these conditions, what current does the generator deliver at a terminal potential difference of 2 000 V?

WEB **39.** **Review Problem.** What is the required resistance of an immersion heater that increases the temperature of 1.50 kg of water from 10.0°C to 50.0°C in 10.0 min while operating at 110 V?

40. **Review Problem.** What is the required resistance of an immersion heater that increases the temperature of a mass m of liquid water from T_1 to T_2 in a time t while operating at a voltage ΔV?

41. Suppose that a voltage surge produces 140 V for a moment. By what percentage does the power output of a 120-V, 100-W lightbulb increase? (Assume that its resistance does not change.)

42. A 500-W heating coil designed to operate from 110 V is made of Nichrome wire 0.500 mm in diameter. (a) Assuming that the resistivity of the Nichrome remains constant at its 20.0°C value, find the length of wire used. (b) Now consider the variation of resistivity with temperature. What power does the coil of part (a) actually deliver when it is heated to 1 200°C?

43. A coil of Nichrome wire is 25.0 m long. The wire has a diameter of 0.400 mm and is at 20.0°C. If it carries a current of 0.500 A, what are (a) the magnitude of the electric field in the wire and (b) the power delivered to it? (c) If the temperature is increased to 340°C and the potential difference across the wire remains constant, what is the power delivered?

44. Batteries are rated in terms of ampere-hours (A · h): For example, a battery that can produce a current of 2.00 A for 3.00 h is rated at 6.00 A · h. (a) What is the total energy, in kilowatt-hours, stored in a 12.0-V battery rated at 55.0 A · h? (b) At a rate of $0.060 0 per kilowatt-hour, what is the value of the electricity produced by this battery?

45. A 10.0-V battery is connected to a 120-Ω resistor. Neglecting the internal resistance of the battery, calculate the power delivered to the resistor.

46. It is estimated that each person in the United States (population = 270 million) has one electric clock, and that each clock uses energy at a rate of 2.50 W. To supply this energy, about how many metric tons of coal are burned per hour in coal-fired electricity generating plants that are, on average, 25.0% efficient? (The heat of combustion for coal is 33.0 MJ/kg.)

47. Compute the cost per day of operating a lamp that draws 1.70 A from a 110-V line if the cost of electrical energy is $0.060 0/kWh.

48. **Review Problem.** The heating element of a coffee-maker operates at 120 V and carries a current of 2.00 A. Assuming that all of the energy transferred from the heating element is absorbed by the water, calculate how long it takes to heat 0.500 kg of water from room temperature (23.0°C) to the boiling point.

49. A certain toaster has a heating element made of Nichrome resistance wire. When the toaster is first connected to a 120-V source of potential difference (and the wire is at a temperature of 20.0°C), the initial current is 1.80 A. However, the current begins to decrease as the resistive element warms up. When the toaster has reached its final operating temperature, the current has dropped to 1.53 A. (a) Find the power the toaster con-

sumes when it is at its operating temperature. (b) What is the final temperature of the heating element?

50. To heat a room having ceilings 8.0 ft high, about 10.0 W of electric power are required per square foot. At a cost of \$0.080 0/kWh, how much does it cost per day to use electricity to heat a room measuring 10.0 ft × 15.0 ft?

51. Estimate the cost of one person's routine use of a hair dryer for 1 yr. If you do not use a blow dryer yourself, observe or interview someone who does. State the quantities you estimate and their values.

ADDITIONAL PROBLEMS

52. One lightbulb is marked "25 W 120 V," and another "100 W 120 V"; this means that each bulb converts its respective power when plugged into a constant 120-V potential difference. (a) Find the resistance of each bulb. (b) How long does it take for 1.00 C to pass through the dim bulb? How is this charge different at the time of its exit compared with the time of its entry? (c) How long does it take for 1.00 J to pass through the dim bulb? How is this energy different at the time of its exit compared with the time of its entry? (d) Find the cost of running the dim bulb continuously for 30.0 days if the electric company sells its product at \$0.070 0 per kWh. What product *does* the electric company sell? What is its price for one SI unit of this quantity?

53. A high-voltage transmission line with a diameter of 2.00 cm and a length of 200 km carries a steady current of 1 000 A. If the conductor is copper wire with a free charge density of 8.00×10^{28} electrons/m^3, how long does it take one electron to travel the full length of the cable?

54. A high-voltage transmission line carries 1 000 A starting at 700 kV for a distance of 100 mi. If the resistance in the wire is 0.500 Ω/mi, what is the power loss due to resistive losses?

55. A more general definition of the temperature coefficient of resistivity is

$$\alpha = \frac{1}{\rho}\frac{d\rho}{dT}$$

where ρ is the resistivity at temperature T. (a) Assuming that α is constant, show that

$$\rho = \rho_0 e^{\alpha(T-T_0)}$$

where ρ_0 is the resistivity at temperature T_0. (b) Using the series expansion ($e^x \cong 1 + x$ for $x \ll 1$), show that the resistivity is given approximately by the expression $\rho = \rho_0[1 + \alpha(T - T_0)]$ for $\alpha(T - T_0) \ll 1$.

56. A copper cable is to be designed to carry a current of 300 A with a power loss of only 2.00 W/m. What is the required radius of the copper cable?

WEB **57.** An experiment is conducted to measure the electrical resistivity of Nichrome in the form of wires with different lengths and cross-sectional areas. For one set of measurements, a student uses 30-gauge wire, which has a cross-sectional area of 7.30×10^{-8} m^2. The student measures the potential difference across the wire and the current in the wire with a voltmeter and ammeter, respectively. For each of the measurements given in the table taken on wires of three different lengths, calculate the resistance of the wires and the corresponding values of the resistivity. What is the average value of the resistivity, and how does this value compare with the value given in Table 27.1?

L (m)	ΔV (V)	I (A)	R (Ω)	ρ (Ω·m)
0.540	5.22	0.500		
1.028	5.82	0.276		
1.543	5.94	0.187		

58. An electric utility company supplies a customer's house from the main power lines (120 V) with two copper wires, each of which is 50.0 m long and has a resistance of 0.108 Ω per 300 m. (a) Find the voltage at the customer's house for a load current of 110 A. For this load current, find (b) the power that the customer is receiving and (c) the power lost in the copper wires.

59. A straight cylindrical wire lying along the x axis has a length of 0.500 m and a diameter of 0.200 mm. It is made of a material described by Ohm's law with a resistivity of $\rho = 4.00 \times 10^{-8}$ Ω·m. Assume that a potential of 4.00 V is maintained at $x = 0$, and that $V = 0$ at $x = 0.500$ m. Find (a) the electric field $\mathbf{E}$ in the wire, (b) the resistance of the wire, (c) the electric current in the wire, and (d) the current density $\mathbf{J}$ in the wire. Express vectors in vector notation. (e) Show that $\mathbf{E} = \rho\mathbf{J}$.

60. A straight cylindrical wire lying along the x axis has a length L and a diameter d. It is made of a material described by Ohm's law with a resistivity ρ. Assume that a potential V is maintained at $x = 0$, and that $V = 0$ at $x = L$. In terms of L, d, V, ρ, and physical constants, derive expressions for (a) the electric field in the wire, (b) the resistance of the wire, (c) the electric current in the wire, and (d) the current density in the wire. Express vectors in vector notation. (e) Show that $\mathbf{E} = \rho\mathbf{J}$.

61. The potential difference across the filament of a lamp is maintained at a constant level while equilibrium temperature is being reached. It is observed that the steady-state current in the lamp is only one tenth of the current drawn by the lamp when it is first turned on. If the temperature coefficient of resistivity for the lamp at 20.0°C is 0.004 50 (°C)$^{-1}$, and if the resistance increases linearly with increasing temperature, what is the final operating temperature of the filament?

62. The current in a resistor decreases by 3.00 A when the potential difference applied across the resistor decreases from 12.0 V to 6.00 V. Find the resistance of the resistor.

63. An electric car is designed to run off a bank of 12.0-V batteries with a total energy storage of 2.00×10^7 J. (a) If the electric motor draws 8.00 kW, what is the current delivered to the motor? (b) If the electric motor draws 8.00 kW as the car moves at a steady speed of 20.0 m/s, how far will the car travel before it is "out of juice"?

64. Review Problem. When a straight wire is heated, its resistance is given by the expression $R = R_0[1 + \alpha(T - T_0)]$ according to Equation 27.21, where α is the temperature coefficient of resistivity. (a) Show that a more precise result, one that accounts for the fact that the length and area of the wire change when heated, is

$$R = \frac{R_0[1 + \alpha(T - T_0)][1 + \alpha'(T - T_0)]}{[1 + 2\alpha'(T - T_0)]}$$

where α' is the coefficient of linear expansion (see Chapter 19). (b) Compare these two results for a 2.00-m-long copper wire of radius 0.100 mm, first at 20.0°C and then heated to 100.0°C.

65. The temperature coefficients of resistivity in Table 27.1 were determined at a temperature of 20°C. What would they be at 0°C? (*Hint:* The temperature coefficient of resistivity at 20°C satisfies the expression $\rho = \rho_0[1 + \alpha(T - T_0)]$, where ρ_0 is the resistivity of the material at $T_0 = 20°\text{C}$. The temperature coefficient of resistivity α' at 0°C must satisfy the expression $\rho = \rho_0'[1 + \alpha' T]$, where ρ_0' is the resistivity of the material at 0°C.)

66. A resistor is constructed by shaping a material of resistivity ρ into a hollow cylinder of length L and with inner and outer radii r_a and r_b, respectively (Fig. P27.66). In use, the application of a potential difference between the ends of the cylinder produces a current parallel to the axis. (a) Find a general expression for the resistance of such a device in terms of L, ρ, r_a, and r_b. (b) Obtain a numerical value for R when $L = 4.00$ cm, $r_a = 0.500$ cm, $r_b = 1.20$ cm, and $\rho = 3.50 \times 10^5\ \Omega \cdot \text{m}$. (c) Now suppose that the potential difference is applied between the inner and outer surfaces so that the resulting current flows radially outward. Find a general expression for the resistance of the device in terms of L, ρ, r_a, and r_b. (d) Calculate the value of R, using the parameter values given in part (b).

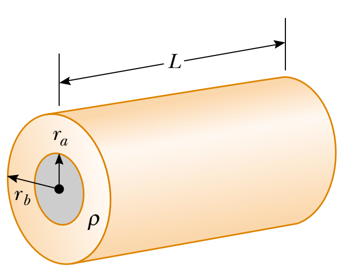


Figure P27.66

67. In a certain stereo system, each speaker has a resistance of 4.00 Ω. The system is rated at 60.0 W in each channel, and each speaker circuit includes a fuse rated at 4.00 A. Is this system adequately protected against overload? Explain your reasoning.

68. A close analogy exists between the flow of energy due to a temperature difference (see Section 20.7) and the flow of electric charge due to a potential difference. The energy dQ and the electric charge dq are both transported by free electrons in the conducting material. Consequently, a good electrical conductor is usually a good thermal conductor as well. Consider a thin conducting slab of thickness dx, area A, and electrical conductivity σ, with a potential difference dV between opposite faces. Show that the current $I = dq/dt$ is given by the equation on the left:

Charge conduction	Analogous thermal conduction (Eq. 20.14)
$\dfrac{dq}{dt} = \sigma A \left\lvert \dfrac{dV}{dx} \right\rvert$	$\dfrac{dQ}{dt} = kA \left\lvert \dfrac{dT}{dx} \right\rvert$

In the analogous thermal conduction equation on the right, the rate of energy flow dQ/dt (in SI units of joules per second) is due to a temperature gradient dT/dx in a material of thermal conductivity k. State analogous rules relating the direction of the electric current to the change in potential and relating the direction of energy flow to the change in temperature.

69. Material with uniform resistivity ρ is formed into a wedge, as shown in Figure P27.69. Show that the resistance between face A and face B of this wedge is

$$R = \rho \frac{L}{w(y_2 - y_1)} \ln\left(\frac{y_2}{y_1}\right)$$

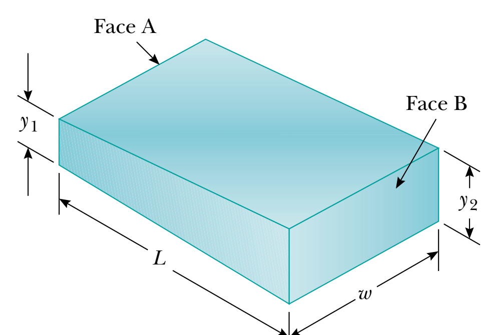


Figure P27.69

70. A material of resistivity ρ is formed into the shape of a truncated cone of altitude h, as shown in Figure P27.70.

The bottom end has a radius b, and the top end has a radius a. Assuming that the current is distributed uniformly over any particular cross-section of the cone so that the current density is not a function of radial position (although it does vary with position along the axis

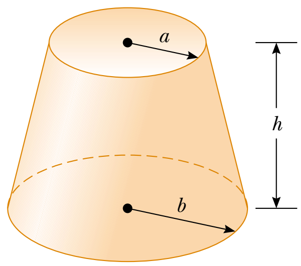


Figure P27.70

of the cone), show that the resistance between the two ends is given by the expression

$$R = \frac{\rho}{\pi}\left(\frac{h}{ab}\right)$$

71. The current–voltage characteristic curve for a semiconductor diode as a function of temperature T is given by the equation

$$I = I_0(e^{e\Delta V/k_B T} - 1)$$

Here, the first symbol e represents the base of the natural logarithm. The second e is the charge on the electron. The k_B is Boltzmann's constant, and T is the absolute temperature. Set up a spreadsheet to calculate I and $R = (\Delta V)/I$ for $\Delta V = 0.400$ V to 0.600 V in increments of 0.005 V. Assume that $I_0 = 1.00$ nA. Plot R versus ΔV for $T = 280$ K, 300 K, and 320 K.

Answers to Quick Quizzes

27.1 d, b = c, a. The current in part (d) is equivalent to two positive charges moving to the left. Parts (b) and (c) each represent four positive charges moving in the same direction because negative charges moving to the left are equivalent to positive charges moving to the right. The current in part (a) is equivalent to five positive charges moving to the right.

27.2 Every portion of the wire carries the same current even though the wire constricts. As the cross-sectional area decreases, the drift velocity must increase in order for the constant current to be maintained, in accordance with Equation 27.4. Equations 27.5 and 27.6 indicate that the current density also increases. An increasing electric field must be causing the increasing current density, as indicated by Equation 27.7. If you were to draw this situation, you would show the electric field lines being compressed into the smaller area, indicating increasing magnitude of the electric field.

27.3 The curvature of the line indicates that the device is nonohmic (that is, its resistance varies with potential difference). Being the definition of resistance, Equation 27.8 still applies, giving different values for R at different points on the curve. The slope of the tangent to the graph line at a point is the reciprocal of the "dynamic resistance" at that point. Note that the resistance of the device (as measured by an ohmmeter) is the reciprocal of the slope of a secant line joining the origin to a particular point on the curve.

27.4 The cable should be as short as possible but still able to reach from one vehicle to another (small ℓ), it should be quite thick (large A), and it should be made of a material with a low resistivity ρ. Referring to Table 27.1, you should probably choose copper or aluminum because the only two materials in the table that have lower ρ values—silver and gold—are prohibitively expensive for your purposes.

27.5 Just after it is turned on. When the filament is at room temperature, its resistance is low, and hence the current is relatively large ($I = \Delta V/R$). As the filament warms up, its resistance increases, and the current decreases. Older lightbulbs often fail just as they are turned on because this large initial current "spike" produces rapid temperature increase and stress on the filament.

27.6 (c). Because the potential difference ΔV is the same across the two bulbs and because the power delivered to a conductor is $\mathcal{P} = I\Delta V$, the 60-W bulb, with its higher power rating, must carry the greater current. The 30-W bulb has the higher resistance because it draws less current at the same potential difference.

27.7 $I_a = I_b > I_c = I_d > I_e = I_f$. The current I_a leaves the positive terminal of the battery and then splits to flow through the two bulbs; thus, $I_a = I_c + I_e$. From Quick Quiz 27.6, we know that the current in the 60-W bulb is greater than that in the 30-W bulb. (Note that all the current does not follow the "path of least resistance," which in this case is through the 60-W bulb.) Because charge does not build up in the bulbs, we know that all the charge flowing into a bulb from the left must flow out on the right; consequently, $I_c = I_d$ and $I_e = I_f$. The two currents leaving the bulbs recombine to form the current back into the battery, $I_f + I_d = I_b$.

PUZZLER

If all these appliances were operating at one time, a circuit breaker would probably be tripped, preventing a potentially dangerous situation. What causes a circuit breaker to trip when too many electrical devices are plugged into one circuit? *(George Semple)*

chapter

28 Direct Current Circuits

Chapter Outline

This chapter is concerned with the analysis of some simple electric circuits that contain batteries, resistors, and capacitors in various combinations. The analysis of these circuits is simplified by the use of two rules known as *Kirchhoff's rules,* which follow from the laws of conservation of energy and conservation of electric charge. Most of the circuits analyzed are assumed to be in *steady state,* which means that the currents are constant in magnitude and direction. In Section 28.4 we discuss circuits in which the current varies with time. Finally, we describe a variety of common electrical devices and techniques for measuring current, potential difference, resistance, and emf.

28.1 ELECTROMOTIVE FORCE

In Section 27.6 we found that a constant current can be maintained in a closed circuit through the use of a source of *emf,* which is a device (such as a battery or generator) that produces an electric field and thus may cause charges to move around a circuit. One can think of a source of emf as a "charge pump." When an electric potential difference exists between two points, the source moves charges "uphill" from the lower potential to the higher. The emf $\mathcal{E}$ describes the work done per unit charge, and hence the SI unit of emf is the volt.

Consider the circuit shown in Figure 28.1, consisting of a battery connected to a resistor. We assume that the connecting wires have no resistance. The positive terminal of the battery is at a higher potential than the negative terminal. If we neglect the internal resistance of the battery, the potential difference across it (called the *terminal voltage*) equals its emf. However, because a real battery always has some internal resistance r, the terminal voltage is not equal to the emf for a battery in a circuit in which there is a current. To understand why this is so, consider the circuit diagram in Figure 28.2a, where the battery of Figure 28.1 is represented by the dashed rectangle containing an emf $\mathcal{E}$ in series with an internal resistance r. Now imagine moving through the battery clockwise from a to b and measuring the electric potential at various locations. As we pass from the negative terminal to the positive terminal, the potential *increases* by an amount $\mathcal{E}$. However, as we move through the resistance r, the potential *decreases* by an amount Ir, where I is the current in the circuit. Thus, the terminal voltage of the battery $\Delta V = V_b - V_a$ is[1]

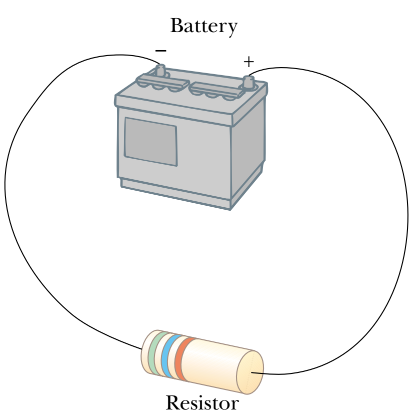
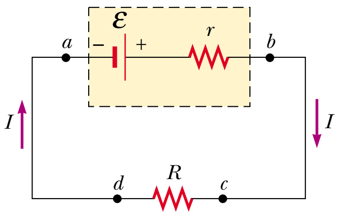


Figure 28.1 A circuit consisting of a resistor connected to the terminals of a battery.

[1] The terminal voltage in this case is less than the emf by an amount Ir. In some situations, the terminal voltage may *exceed* the emf by an amount Ir. This happens when the direction of the current is *opposite* that of the emf, as in the case of charging a battery with another source of emf.



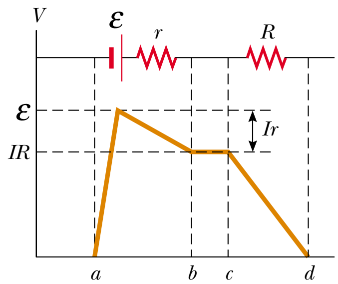


Figure 28.2 (a) Circuit diagram of a source of emf $\mathcal{E}$ (in this case, a battery), of internal resistance r, connected to an external resistor of resistance R. (b) Graphical representation showing how the electric potential changes as the circuit in part (a) is traversed clockwise.

$$\Delta V = \mathcal{E} - Ir \tag{28.1}$$

From this expression, note that $\mathcal{E}$ is equivalent to the **open-circuit voltage**—that is, the *terminal voltage when the current is zero.* The emf is the voltage labeled on a battery—for example, the emf of a D cell is 1.5 V. The actual potential difference between the terminals of the battery depends on the current through the battery, as described by Equation 28.1.

Figure 28.2b is a graphical representation of the changes in electric potential as the circuit is traversed in the clockwise direction. By inspecting Figure 28.2a, we see that the terminal voltage ΔV must equal the potential difference across the external resistance R, often called the **load resistance.** The load resistor might be a simple resistive circuit element, as in Figure 28.1, or it could be the resistance of some electrical device (such as a toaster, an electric heater, or a lightbulb) connected to the battery (or, in the case of household devices, to the wall outlet). The resistor represents a *load* on the battery because the battery must supply energy to operate the device. The potential difference across the load resistance is $\Delta V = IR$. Combining this expression with Equation 28.1, we see that

$$\mathcal{E} = IR + Ir \tag{28.2}$$

Solving for the current gives

$$I = \frac{\mathcal{E}}{R + r} \tag{28.3}$$

This equation shows that the current in this simple circuit depends on both the load resistance R external to the battery and the internal resistance r. If R is much greater than r, as it is in many real-world circuits, we can neglect r.

If we multiply Equation 28.2 by the current I, we obtain

$$I\mathcal{E} = I^2R + I^2r \tag{28.4}$$

This equation indicates that, because power $\mathcal{P} = I\Delta V$ (see Eq. 27.22), the total power output $I\mathcal{E}$ of the battery is delivered to the external load resistance in the amount I^2R and to the internal resistance in the amount I^2r. Again, if $r \ll R$, then most of the power delivered by the battery is transferred to the load resistance.

EXAMPLE 28.1 Terminal Voltage of a Battery

A battery has an emf of 12.0 V and an internal resistance of 0.05 Ω. Its terminals are connected to a load resistance of 3.00 Ω. (a) Find the current in the circuit and the terminal voltage of the battery.

Solution Using first Equation 28.3 and then Equation 28.1, we obtain

$$I = \frac{\mathcal{E}}{R + r} = \frac{12.0\ \text{V}}{3.05\ \Omega} = \boxed{3.93\ \text{A}}$$

$$\Delta V = \mathcal{E} - Ir = 12.0\ \text{V} - (3.93\ \text{A})(0.05\ \Omega) = \boxed{11.8\ \text{V}}$$

To check this result, we can calculate the voltage across the load resistance R:

$$\Delta V = IR = (3.93\ \text{A})(3.00\ \Omega) = 11.8\ \text{V}$$

(b) Calculate the power delivered to the load resistor, the power delivered to the internal resistance of the battery, and the power delivered by the battery.

Solution The power delivered to the load resistor is

$$\mathcal{P}_R = I^2R = (3.93\ \text{A})^2\ (3.00\ \Omega) = \boxed{46.3\ \text{W}}$$

The power delivered to the internal resistance is

$$\mathcal{P}_r = I^2r = (3.93\ \text{A})^2\ (0.05\ \Omega) = \boxed{0.772\ \text{W}}$$

Hence, the power delivered by the battery is the sum of these quantities, or 47.1 W. You should check this result, using the expression $\mathcal{P} = I\mathcal{E}$.

EXAMPLE 28.2 Matching the Load

Show that the maximum power delivered to the load resistance R in Figure 28.2a occurs when the load resistance matches the internal resistance—that is, when $R = r$.

Solution The power delivered to the load resistance is equal to I^2R, where I is given by Equation 28.3:

$$\mathcal{P} = I^2R = \frac{\mathcal{E}^2R}{(R+r)^2}$$

When $\mathcal{P}$ is plotted versus R as in Figure 28.3, we find that $\mathcal{P}$ reaches a maximum value of $\mathcal{E}^2/4r$ at $R = r$. We can also prove this by differentiating $\mathcal{P}$ with respect to R, setting the result equal to zero, and solving for R. The details are left as a problem for you to solve (Problem 57).

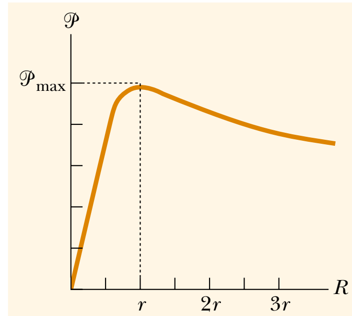


Figure 28.3 Graph of the power $\mathcal{P}$ delivered by a battery to a load resistor of resistance R as a function of R. The power delivered to the resistor is a maximum when the load resistance equals the internal resistance of the battery.

28.2 RESISTORS IN SERIES AND IN PARALLEL

Suppose that you and your friends are at a crowded basketball game in a sports arena and decide to leave early. You have two choices: (1) your whole group can exit through a single door and walk down a long hallway containing several concession stands, each surrounded by a large crowd of people waiting to buy food or souvenirs; or (b) each member of your group can exit through a separate door in the main hall of the arena, where each will have to push his or her way through a single group of people standing by the door. In which scenario will less time be required for your group to leave the arena?

It should be clear that your group will be able to leave faster through the separate doors than down the hallway where each of you has to push through several groups of people. We could describe the groups of people in the hallway as acting in *series,* because each of you must push your way through all of the groups. The groups of people around the doors in the arena can be described as acting in *parallel.* Each member of your group must push through only one group of people, and each member pushes through a *different* group of people. This simple analogy will help us understand the behavior of currents in electric circuits containing more than one resistor.

When two or more resistors are connected together as are the lightbulbs in Figure 28.4a, they are said to be in *series*. Figure 28.4b is the circuit diagram for the lightbulbs, which are shown as resistors, and the battery. In a series connection, all the charges moving through one resistor must also pass through the second resistor. (This is analogous to all members of your group pushing through the crowds in the single hallway of the sports arena.) Otherwise, charge would accumulate between the resistors. Thus,

> for a series combination of resistors, the currents in the two resistors are the same because any charge that passes through R_1 must also pass through R_2.

The potential difference applied across the series combination of resistors will divide between the resistors. In Figure 28.4b, because the voltage drop[2] from a to b

[2] The term *voltage drop* is synonymous with a decrease in electric potential across a resistor and is used often by individuals working with electric circuits.

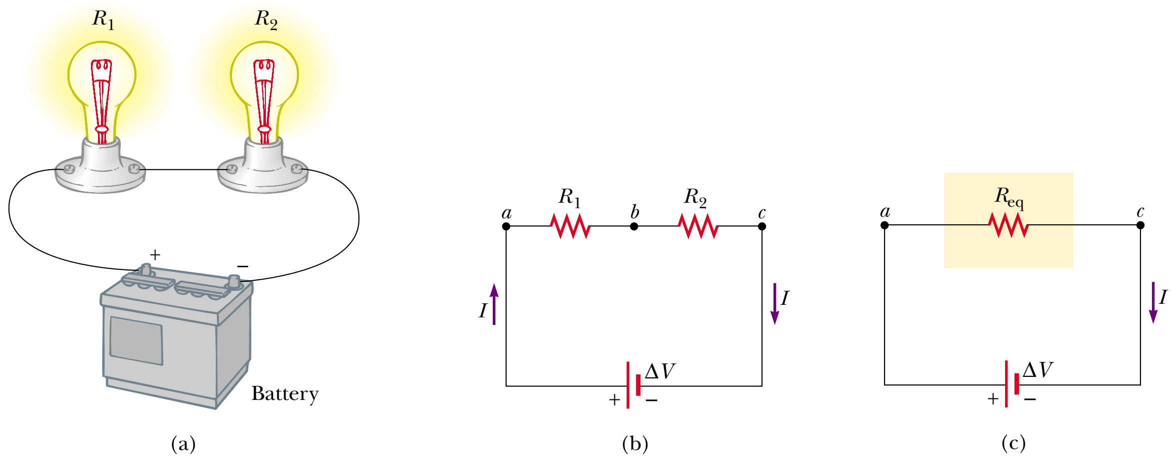


Figure 28.4 (a) A series connection of two resistors R_1 and R_2. The current in R_1 is the same as that in R_2. (b) Circuit diagram for the two-resistor circuit. (c) The resistors replaced with a single resistor having an equivalent resistance $R_{eq} = R_1 + R_2$.

equals IR_1 and the voltage drop from b to c equals IR_2, the voltage drop from a to c is

$$\Delta V = IR_1 + IR_2 = I(R_1 + R_2)$$

Therefore, we can replace the two resistors in series with a single resistor having an *equivalent resistance* R_{eq}, where

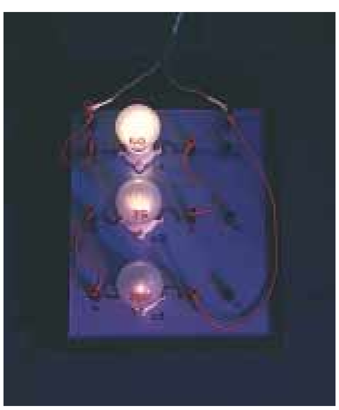
$$R_{eq} = R_1 + R_2 \qquad \textbf{(28.5)}$$

The resistance R_{eq} is equivalent to the series combination $R_1 + R_2$ in the sense that the circuit current is unchanged when R_{eq} replaces $R_1 + R_2$.

The equivalent resistance of three or more resistors connected in series is

$$R_{eq} = R_1 + R_2 + R_3 + \cdots \qquad \textbf{(28.6)}$$

This relationship indicates that **the equivalent resistance of a series connection of resistors is always greater than any individual resistance.**



A series connection of three lightbulbs, all rated at 120 V but having power ratings of 60 W, 75 W, and 200 W. Why are the intensities of the bulbs different? Which bulb has the greatest resistance? How would their relative intensities differ if they were connected in parallel?

Quick Quiz 28.1

If a piece of wire is used to connect points b and c in Figure 28.4b, does the brightness of bulb R_1 increase, decrease, or stay the same? What happens to the brightness of bulb R_2?

Now consider two resistors connected in *parallel,* as shown in Figure 28.5. When the current I reaches point a in Figure 28.5b, called a *junction,* it splits into two parts, with I_1 going through R_1 and I_2 going through R_2. A **junction** is any point in a circuit where a current can split (just as your group might split up and leave the arena through several doors, as described earlier.) This split results in less current in each individual resistor than the current leaving the battery. Because charge must be conserved, the current I that enters point a must equal the total current leaving that point:

$$I = I_1 + I_2$$

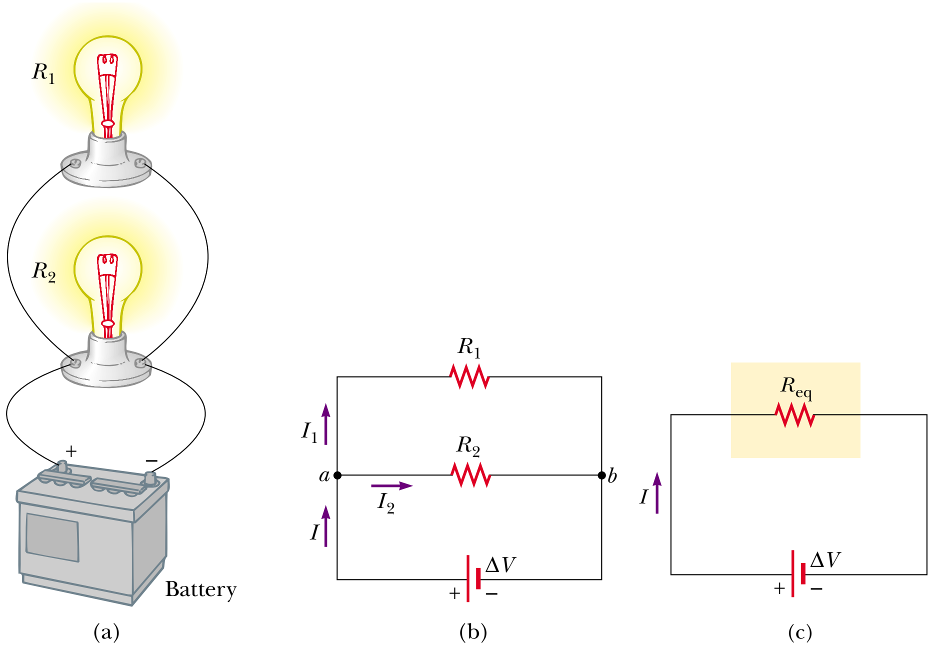
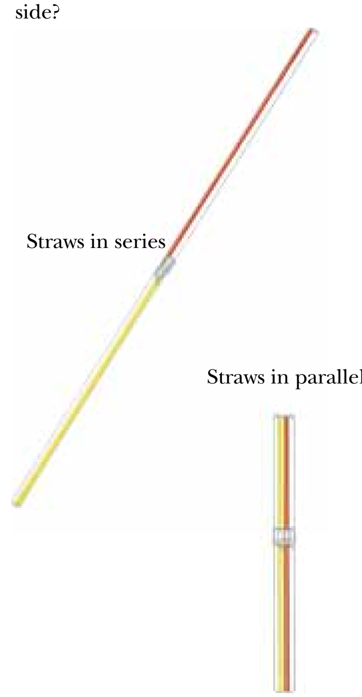


Figure 28.5 (a) A parallel connection of two resistors R_1 and R_2. The potential difference across R_1 is the same as that across R_2. (b) Circuit diagram for the two-resistor circuit. (c) The resistors replaced with a single resistor having an equivalent resistance $R_{eq} = (R_1^{-1} + R_2^{-1})^{-1}$.

QuickLab

Tape one pair of drinking straws end to end, and tape a second pair side by side. Which pair is easier to blow through? What would happen if you were comparing three straws taped end to end with three taped side by side?



As can be seen from Figure 28.5, both resistors are connected directly across the terminals of the battery. Thus,

when resistors are connected in parallel, the potential differences across them are the same.

Because the potential differences across the resistors are the same, the expression $\Delta V = IR$ gives

$$I = I_1 + I_2 = \frac{\Delta V}{R_1} + \frac{\Delta V}{R_2} = \Delta V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{\Delta V}{R_{eq}}$$

From this result, we see that the equivalent resistance of two resistors in parallel is given by

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} \tag{28.7}$$

or

$$R_{eq} = \frac{1}{\dfrac{1}{R_1} + \dfrac{1}{R_2}}$$

An extension of this analysis to three or more resistors in parallel gives

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \tag{28.8}$$

The equivalent resistance of several resistors in parallel

Three lightbulbs having power ratings of 25 W, 75 W, and 150 W, connected in parallel to a voltage source of about 100 V. All bulbs are rated at the same voltage. Why do the intensities differ? Which bulb draws the most current? Which has the least resistance?

We can see from this expression that **the equivalent resistance of two or more resistors connected in parallel is always less than the least resistance in the group.**

Household circuits are always wired such that the appliances are connected in parallel. Each device operates independently of the others so that if one is switched off, the others remain on. In addition, the devices operate on the same voltage.

Quick Quiz 28.2

Assume that the battery of Figure 28.1 has zero internal resistance. If we add a second resistor in series with the first, does the current in the battery increase, decrease, or stay the same? How about the potential difference across the battery terminals? Would your answers change if the second resistor were connected in parallel to the first one?

Quick Quiz 28.3

Are automobile headlights wired in series or in parallel? How can you tell?

EXAMPLE 28.3 Find the Equivalent Resistance

Four resistors are connected as shown in Figure 28.6a. (a) Find the equivalent resistance between points a and c.

Solution The combination of resistors can be reduced in steps, as shown in Figure 28.6. The 8.0-Ω and 4.0-Ω resistors are in series; thus, the equivalent resistance between a and b is 12 Ω (see Eq. 28.5). The 6.0-Ω and 3.0-Ω resistors are in parallel, so from Equation 28.7 we find that the equivalent resistance from b to c is 2.0 Ω. Hence, the equivalent resistance from a to c is 14 Ω.

(b) What is the current in each resistor if a potential difference of 42 V is maintained between a and c?

Solution The currents in the 8.0-Ω and 4.0-Ω resistors are the same because they are in series. In addition, this is the same as the current that would exist in the 14-Ω equivalent resistor subject to the 42-V potential difference. Therefore, using Equation 27.8 ($R = \Delta V/I$) and the results from part (a), we obtain

$$I = \frac{\Delta V_{ac}}{R_{eq}} = \frac{42\text{ V}}{14\ \Omega} = 3.0\text{ A}$$

This is the current in the 8.0-Ω and 4.0-Ω resistors. When this 3.0-A current enters the junction at b, however, it splits, with part passing through the 6.0-Ω resistor (I_1) and part through the 3.0-Ω resistor (I_2). Because the potential difference is ΔV_{bc} across each of these resistors (since they are in parallel), we see that $(6.0\ \Omega)I_1 = (3.0\ \Omega)I_2$, or $I_2 = 2I_1$. Using this result and the fact that $I_1 + I_2 = 3.0$ A, we find that $I_1 = 1.0$ A and $I_2 = 2.0$ A. We could have guessed this at the start by noting that the current through the 3.0-Ω resistor has to be twice that through the 6.0-Ω resistor, in view of their relative resistances and the fact that the same voltage is applied to each of them.

As a final check of our results, note that $\Delta V_{bc} = (6.0\ \Omega)I_1 = (3.0\ \Omega)I_2 = 6.0\text{ V}$ and $\Delta V_{ab} = (12\ \Omega)I = 36\text{ V}$; therefore, $\Delta V_{ac} = \Delta V_{ab} + \Delta V_{bc} = 42$ V, as it must.

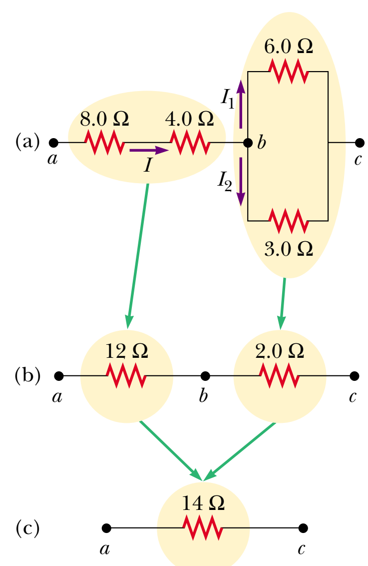


Figure 28.6

EXAMPLE 28.4 Three Resistors in Parallel

Three resistors are connected in parallel as shown in Figure 28.7. A potential difference of 18 V is maintained between points *a* and *b*. (a) Find the current in each resistor.

Solution The resistors are in parallel, and so the potential difference across each must be 18 V. Applying the relationship $\Delta V = IR$ to each resistor gives

$$I_1 = \frac{\Delta V}{R_1} = \frac{18\text{ V}}{3.0\ \Omega} = \boxed{6.0\text{ A}}$$

$$I_2 = \frac{\Delta V}{R_2} = \frac{18\text{ V}}{6.0\ \Omega} = \boxed{3.0\text{ A}}$$

$$I_3 = \frac{\Delta V}{R_3} = \frac{18\text{ V}}{9.0\ \Omega} = \boxed{2.0\text{ A}}$$

(b) Calculate the power delivered to each resistor and the total power delivered to the combination of resistors.

Solution We apply the relationship $\mathcal{P} = (\Delta V)^2/R$ to each resistor and obtain

$$\mathcal{P}_1 = \frac{\Delta V^2}{R_1} = \frac{(18\text{ V})^2}{3.0\ \Omega} = \boxed{110\text{ W}}$$

$$\mathcal{P}_2 = \frac{\Delta V^2}{R_2} = \frac{(18\text{ V})^2}{6.0\ \Omega} = \boxed{54\text{ W}}$$

$$\mathcal{P}_3 = \frac{\Delta V^2}{R_3} = \frac{(18\text{ V})^2}{9.0\ \Omega} = \boxed{36\text{ W}}$$

This shows that the smallest resistor receives the most power. Summing the three quantities gives a total power of 200 W.

(c) Calculate the equivalent resistance of the circuit.

Solution We can use Equation 28.8 to find R_{eq}:

$$\frac{1}{R_{eq}} = \frac{1}{3.0\ \Omega} + \frac{1}{6.0\ \Omega} + \frac{1}{9.0\ \Omega}$$

$$= \frac{6}{18\ \Omega} + \frac{3}{18\ \Omega} + \frac{2}{18\ \Omega} = \frac{11}{18\ \Omega}$$

$$R_{eq} = \frac{18\ \Omega}{11} = \boxed{1.6\ \Omega}$$

Exercise Use R_{eq} to calculate the total power delivered by the battery.

Answer 200 W.

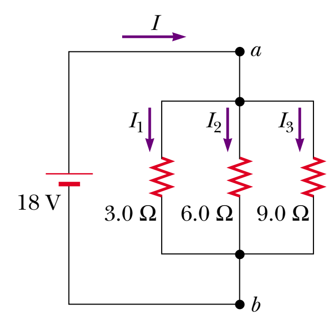


Figure 28.7 Three resistors connected in parallel. The voltage across each resistor is 18 V.

EXAMPLE 28.5 Finding R_{eq} by Symmetry Arguments

Consider five resistors connected as shown in Figure 28.8a. Find the equivalent resistance between points *a* and *b*.

Solution In this type of problem, it is convenient to assume a current entering junction *a* and then apply symmetry

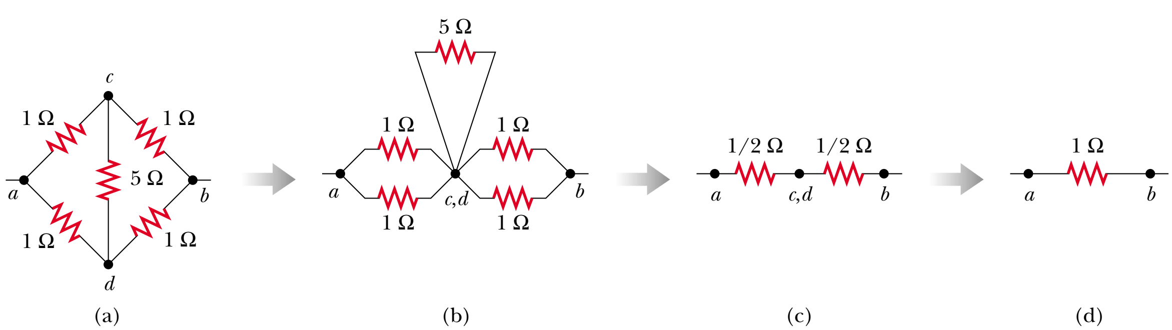


Figure 28.8 Because of the symmetry in this circuit, the 5-Ω resistor does not contribute to the resistance between points *a* and *b* and therefore can be disregarded when we calculate the equivalent resistance.

arguments. Because of the symmetry in the circuit (all 1-Ω resistors in the outside loop), the currents in branches *ac* and *ad* must be equal; hence, the electric potentials at points *c* and *d* must be equal. This means that $\Delta V_{cd} = 0$ and, as a result, points *c* and *d* may be connected together without affecting the circuit, as in Figure 28.8b. Thus, the 5-Ω resistor may be removed from the circuit and the remaining circuit then reduced as in Figures 28.8c and d. From this reduction we see that the equivalent resistance of the combination is 1 Ω. Note that the result is 1 Ω regardless of the value of the resistor connected between *c* and *d*.

CONCEPTUAL EXAMPLE 28.6 Operation of a Three-Way Lightbulb

Figure 28.9 illustrates how a three-way lightbulb is constructed to provide three levels of light intensity. The socket of the lamp is equipped with a three-way switch for selecting different light intensities. The bulb contains two filaments. When the lamp is connected to a 120-V source, one filament receives 100 W of power, and the other receives 75 W. Explain how the two filaments are used to provide three different light intensities.

Solution The three light intensities are made possible by applying the 120 V to one filament alone, to the other filament alone, or to the two filaments in parallel. When switch S_1 is closed and switch S_2 is opened, current passes only through the 75-W filament. When switch S_1 is open and switch S_2 is closed, current passes only through the 100-W filament. When both switches are closed, current passes through both filaments, and the total power is 175 W.

If the filaments were connected in series and one of them were to break, no current could pass through the bulb, and the bulb would give no illumination, regardless of the switch position. However, with the filaments connected in parallel, if one of them (for example, the 75-W filament) breaks, the bulb will still operate in two of the switch positions as current passes through the other (100-W) filament.

Exercise Determine the resistances of the two filaments and their parallel equivalent resistance.

Answer 144 Ω, 192 Ω, 82.3 Ω.

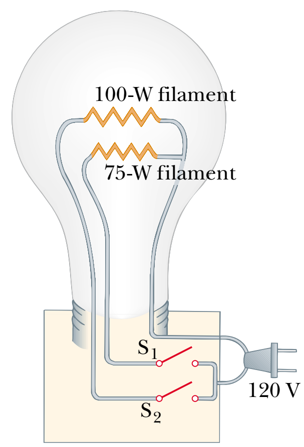


Figure 28.9 A three-way lightbulb.

APPLICATION Strings of Lights

Strings of lights are used for many ornamental purposes, such as decorating Christmas trees. Over the years, both parallel and series connections have been used for multilight strings powered by 120 V.[3] Series-wired bulbs are safer than parallel-wired bulbs for indoor Christmas-tree use because series-wired bulbs operate with less light per bulb and at a lower temperature. However, if the filament of a single bulb fails (or if the bulb is removed from its socket), all the lights on the string are extinguished. The popularity of series-wired light strings diminished because troubleshooting a failed bulb was a tedious, time-consuming chore that involved trial-and-error substitution of a good bulb in each socket along the string until the defective bulb was found.

In a parallel-wired string, each bulb operates at 120 V. By design, the bulbs are brighter and hotter than those on a series-wired string. As a result, these bulbs are inherently more dangerous (more likely to start a fire, for instance), but if one bulb in a parallel-wired string fails or is removed, the rest of the bulbs continue to glow. (A 25-bulb string of 4-W bulbs results in a power of 100 W; the total power becomes substantial when several strings are used.)

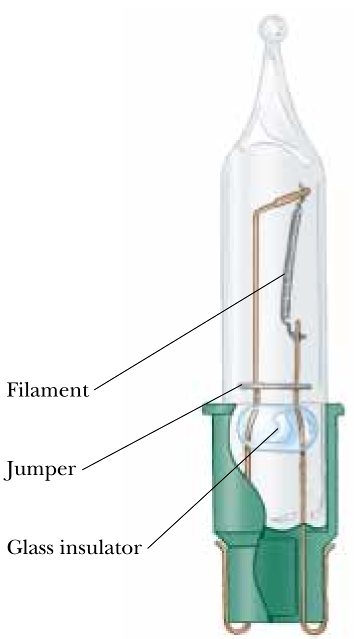
A new design was developed for so-called "miniature" lights wired in series, to prevent the failure of one bulb from extinguishing the entire string. The solution is to create a connection (called a jumper) across the filament after it fails. (If an alternate connection existed across the filament before

[3] These and other household devices, such as the three-way lightbulb in Conceptual Example 28.6 and the kitchen appliances shown in this chapter's Puzzler, actually operate on alternating current (ac), to be introduced in Chapter 33.

it failed, each bulb would represent a parallel circuit; in this circuit, the current would flow through the alternate connection, forming a short circuit, and the bulb would not glow.) When the filament breaks in one of these miniature lightbulbs, 120 V appears across the bulb because no current is present in the bulb and therefore no drop in potential occurs across the other bulbs. Inside the lightbulb, a small loop covered by an insulating material is wrapped around the filament leads. An arc burns the insulation and connects the filament leads when 120 V appears across the bulb—that is, when the filament fails. This "short" now completes the circuit through the bulb even though the filament is no longer active (Fig. 28.10).

Suppose that all the bulbs in a 50-bulb miniature-light string are operating. A 2.4-V potential drop occurs across each bulb because the bulbs are in series. The power input to this style of bulb is 0.34 W, so the total power supplied to the string is only 17 W. We calculate the filament resistance at the operating temperature to be $(2.4\text{ V})^2/(0.34\text{ W}) = 17\ \Omega$. When the bulb fails, the resistance across its terminals is reduced to zero because of the alternate jumper connection mentioned in the preceding paragraph. All the other bulbs not only stay on but glow more brightly because the total resistance of the string is reduced and consequently the current in each bulb increases.

Let us assume that the operating resistance of a bulb remains at 17 Ω even though its temperature rises as a result of the increased current. If one bulb fails, the potential drop across each of the remaining bulbs increases to 2.45 V, the current increases from 0.142 A to 0.145 A, and the power increases to 0.354 W. As more lights fail, the current keeps rising, the filament of each bulb operates at a higher temperature, and the lifetime of the bulb is reduced. It is therefore a good idea to check for failed (nonglowing) bulbs in such a series-wired string and replace them as soon as possible, in order to maximize the lifetimes of all the bulbs.



(a)

(b)

Figure 28.10 (a) Schematic diagram of a modern "miniature" holiday lightbulb, with a jumper connection to provide a current path if the filament breaks. (b) A Christmas-tree lightbulb.

28.3 KIRCHHOFF'S RULES

13.4 As we saw in the preceding section, we can analyze simple circuits using the expression $\Delta V = IR$ and the rules for series and parallel combinations of resistors. Very often, however, it is not possible to reduce a circuit to a single loop. The procedure for analyzing more complex circuits is greatly simplified if we use two principles called **Kirchhoff's rules:**

1. The sum of the currents entering any junction in a circuit must equal the sum of the currents leaving that junction:

$$\sum I_{\text{in}} = \sum I_{\text{out}} \tag{28.9}$$

2. The sum of the potential differences across all elements around any closed circuit loop must be zero:

$$\sum_{\substack{\text{closed}\\ \text{loop}}} \Delta V = 0 \qquad \textbf{(28.10)}$$

Gustav Kirchhoff (1824–1887) Kirchhoff, a professor at Heidelberg, Germany, and Robert Bunsen invented the spectroscope and founded the science of spectroscopy, which we shall study in Chapter 40. They discovered the elements cesium and rubidium and invented astronomical spectroscopy. Kirchhoff formulated another Kirchhoff's rule, namely, "a cool substance will absorb light of the same wavelengths that it emits when hot." *(AIP ESVA/W. F. Meggers Collection)*

Kirchhoff's first rule is a statement of conservation of electric charge. All current that enters a given point in a circuit must leave that point because charge cannot build up at a point. If we apply this rule to the junction shown in Figure 28.11a, we obtain

$$I_1 = I_2 + I_3$$

Figure 28.11b represents a mechanical analog of this situation, in which water flows through a branched pipe having no leaks. The flow rate into the pipe equals the total flow rate out of the two branches on the right.

Kirchhoff's second rule follows from the law of conservation of energy. Let us imagine moving a charge around the loop. When the charge returns to the starting point, the charge–circuit system must have the same energy as when the charge started from it. The sum of the increases in energy in some circuit elements must equal the sum of the decreases in energy in other elements. The potential energy decreases whenever the charge moves through a potential drop $-IR$ across a resistor or whenever it moves in the reverse direction through a source of emf. The potential energy increases whenever the charge passes through a battery from the negative terminal to the positive terminal. Kirchhoff's second rule applies only for circuits in which an electric potential is defined at each point; this criterion may not be satisfied if changing electromagnetic fields are present, as we shall see in Chapter 31.

In justifying our claim that Kirchhoff's second rule is a statement of conservation of energy, we imagined carrying a charge around a loop. When applying this rule, we imagine *traveling* around the loop and consider changes in *electric potential,* rather than the changes in *potential energy* described in the previous paragraph. You should note the following sign conventions when using the second rule:

- Because charges move from the high-potential end of a resistor to the low-potential end, if a resistor is traversed in the direction of the current, the change in potential ΔV across the resistor is $-IR$ (Fig. 28.12a).
- If a resistor is traversed in the direction *opposite* the current, the change in potential ΔV across the resistor is $+IR$ (Fig. 28.12b).
- If a source of emf (assumed to have zero internal resistance) is traversed in the direction of the emf (from $-$ to $+$), the change in potential ΔV is $+\boldsymbol{\mathcal{E}}$ (Fig. 28.12c). The emf of the battery increases the electric potential as we move through it in this direction.
- If a source of emf (assumed to have zero internal resistance) is traversed in the direction opposite the emf (from $+$ to $-$), the change in potential ΔV is $-\boldsymbol{\mathcal{E}}$ (Fig. 28.12d). In this case the emf of the battery reduces the electric potential as we move through it.

Limitations exist on the numbers of times you can usefully apply Kirchhoff's rules in analyzing a given circuit. You can use the junction rule as often as you need, so long as each time you write an equation you include in it a current that has not been used in a preceding junction-rule equation. In general, the number of times you can use the junction rule is one fewer than the number of junction

QuickLab

Draw an arbitrarily shaped closed loop that does not cross over itself. Label five points on the loop *a*, *b*, *c*, *d*, and *e*, and assign a random number to each point. Now start at *a* and work your way around the loop, calculating the difference between each pair of adjacent numbers. Some of these differences will be positive, and some will be negative. Add the differences together, making sure you accurately keep track of the algebraic signs. What is the sum of the differences all the way around the loop?

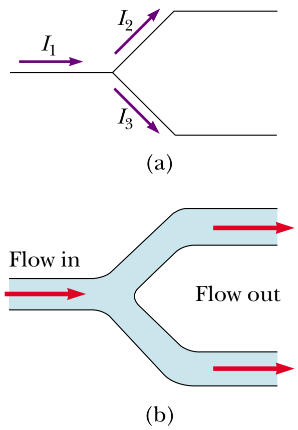


Figure 28.11 (a) Kirchhoff's junction rule. Conservation of charge requires that all current entering a junction must leave that junction. Therefore, $I_1 = I_2 + I_3$. (b) A mechanical analog of the junction rule: the amount of water flowing out of the branches on the right must equal the amount flowing into the single branch on the left.

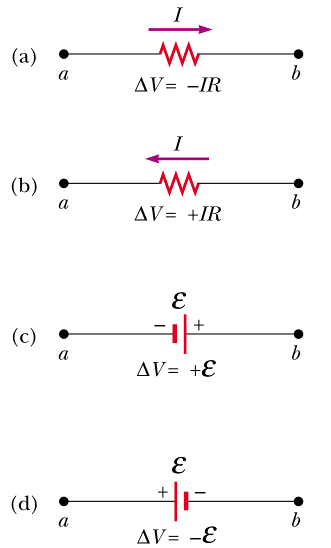


Figure 28.12 Rules for determining the potential changes across a resistor and a battery. (The battery is assumed to have no internal resistance.) Each circuit element is traversed from left to right.

points in the circuit. You can apply the loop rule as often as needed, so long as a new circuit element (resistor or battery) or a new current appears in each new equation. In general, **in order to solve a particular circuit problem, the number of independent equations you need to obtain from the two rules equals the number of unknown currents.**

Complex networks containing many loops and junctions generate great numbers of independent linear equations and a correspondingly great number of unknowns. Such situations can be handled formally through the use of matrix algebra. Computer programs can also be written to solve for the unknowns.

The following examples illustrate how to use Kirchhoff's rules. In all cases, it is assumed that the circuits have reached steady-state conditions—that is, the currents in the various branches are constant. Any capacitor **acts as an open circuit;** that is, the current in the branch containing the capacitor is zero under steady-state conditions.

Problem-Solving Hints

Kirchhoff's Rules

- Draw a circuit diagram, and label all the known and unknown quantities. You must assign a *direction* to the current in each branch of the circuit. Do not be alarmed if you guess the direction of a current incorrectly; your result will be negative, but *its magnitude will be correct.* Although the assignment of current directions is arbitrary, you must adhere rigorously to the assigned directions when applying Kirchhoff's rules.
- Apply the junction rule to any junctions in the circuit that provide new relationships among the various currents.

- Apply the loop rule to as many loops in the circuit as are needed to solve for the unknowns. To apply this rule, you must correctly identify the change in potential as you imagine crossing each element in traversing the closed loop (either clockwise or counterclockwise). Watch out for errors in sign!
- Solve the equations simultaneously for the unknown quantities.

EXAMPLE 28.7 A Single-Loop Circuit

A single-loop circuit contains two resistors and two batteries, as shown in Figure 28.13. (Neglect the internal resistances of the batteries.) (a) Find the current in the circuit.

Solution We do not need Kirchhoff's rules to analyze this simple circuit, but let us use them anyway just to see how they are applied. There are no junctions in this single-loop circuit; thus, the current is the same in all elements. Let us assume that the current is clockwise, as shown in Figure 28.13. Traversing the circuit in the clockwise direction, starting at *a*, we see that $a \rightarrow b$ represents a potential change of $+\mathcal{E}_1$, $b \rightarrow c$ represents a potential change of $-IR_1$, $c \rightarrow d$ represents a potential change of $-\mathcal{E}_2$, and $d \rightarrow a$ represents a potential change of $-IR_2$. Applying Kirchhoff's loop rule gives

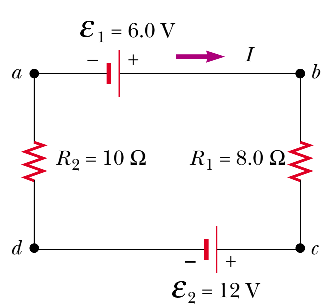


Figure 28.13 A series circuit containing two batteries and two resistors, where the polarities of the batteries are in opposition.

$$\sum \Delta V = 0$$

$$\mathcal{E}_1 - IR_1 - \mathcal{E}_2 - IR_2 = 0$$

Solving for I and using the values given in Figure 28.13, we obtain

$$I = \frac{\mathcal{E}_1 - \mathcal{E}_2}{R_1 + R_2} = \frac{6.0\text{ V} - 12\text{ V}}{8.0\ \Omega + 10\ \Omega} = \boxed{-0.33\text{ A}}$$

The negative sign for I indicates that the direction of the current is opposite the assumed direction.

(b) What power is delivered to each resistor? What power is delivered by the 12-V battery?

Solution

$$\mathcal{P}_1 = I^2R_1 = (0.33\text{ A})^2(8.0\ \Omega) = \boxed{0.87\text{ W}}$$

$$\mathcal{P}_2 = I^2R_2 = (0.33\text{ A})^2(10\ \Omega) = \boxed{1.1\text{ W}}$$

Hence, the total power delivered to the resistors is $\mathcal{P}_1 + \mathcal{P}_2 = 2.0$ W.

The 12-V battery delivers power $I\mathcal{E}_2 = 4.0$ W. Half of this power is delivered to the two resistors, as we just calculated. The other half is delivered to the 6-V battery, which is being charged by the 12-V battery. If we had included the internal resistances of the batteries in our analysis, some of the power would appear as internal energy in the batteries; as a result, we would have found that less power was being delivered to the 6-V battery.

EXAMPLE 28.8 Applying Kirchhoff's Rules

Find the currents I_1, I_2, and I_3 in the circuit shown in Figure 28.14.

Solution Notice that we cannot reduce this circuit to a simpler form by means of the rules of adding resistances in series and in parallel. We must use Kirchhoff's rules to analyze this circuit. We arbitrarily choose the directions of the currents as labeled in Figure 28.14. Applying Kirchhoff's junction rule to junction *c* gives

$$(1) \qquad I_1 + I_2 = I_3$$

We now have one equation with three unknowns—I_1, I_2, and I_3. There are three loops in the circuit—*abcda, befcb,* and *aefda*. We therefore need only two loop equations to determine the unknown currents. (The third loop equation would give no new information.) Applying Kirchhoff's loop rule to loops *abcda* and *befcb* and traversing these loops clockwise, we obtain the expressions

$$(2) \quad abcda \quad 10\text{ V} - (6\ \Omega)I_1 - (2\ \Omega)I_3 = 0$$

$$(3) \quad befcb \quad -14\text{ V} + (6\ \Omega)I_1 - 10\text{ V} - (4\ \Omega)I_2 = 0$$

Note that in loop *befcb* we obtain a positive value when traversing the 6-Ω resistor because our direction of travel is opposite the assumed direction of I_1.

Expressions (1), (2), and (3) represent three independent equations with three unknowns. Substituting Equation (1) into Equation (2) gives

$$10\text{ V} - (6\ \Omega)I_1 - (2\ \Omega)\ (I_1 + I_2) = 0$$

$$(4) \qquad 10\text{ V} = (8\ \Omega)I_1 + (2\ \Omega)I_2$$

Dividing each term in Equation (3) by 2 and rearranging gives

$$(5) \qquad -12\text{ V} = -(3\ \Omega)I_1 + (2\ \Omega)I_2$$

Subtracting Equation (5) from Equation (4) eliminates I_2, giving

$$22\text{ V} = (11\ \Omega)I_1$$

$$I_1 = \boxed{2\text{ A}}$$

Using this value of I_1 in Equation (5) gives a value for I_2:

$$(2\ \Omega)I_2 = (3\ \Omega)I_1 - 12\text{ V} = (3\ \Omega)\ (2\text{ A}) - 12\text{ V} = -6\text{ V}$$

$$I_2 = \boxed{-3\text{ A}}$$

Finally,

$$I_3 = I_1 + I_2 = \boxed{-1\text{ A}}$$

The fact that I_2 and I_3 are both negative indicates only that the currents are opposite the direction we chose for them. However, the numerical values are correct. What would have happened had we left the current directions as labeled in Figure 28.14 but traversed the loops in the opposite direction?

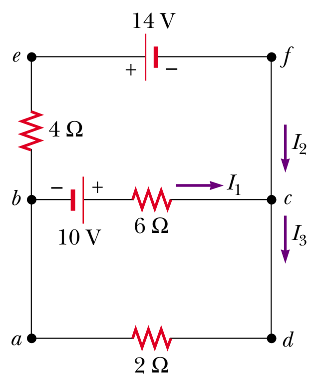


Figure 28.14 A circuit containing three loops.

Exercise Find the potential difference between points b and c.

Answer 2 V.

EXAMPLE 28.9 A Multiloop Circuit

(a) Under steady-state conditions, find the unknown currents I_1, I_2, and I_3 in the multiloop circuit shown in Figure 28.15.

Solution First note that because the capacitor represents an open circuit, there is no current between *g* and *b* along path *ghab* under steady-state conditions. Therefore, when the charges associated with I_1 reach point *g*, they all go through the 8.00-V battery to point *b*; hence, $I_{gb} = I_1$. Labeling the currents as shown in Figure 28.15 and applying Equation 28.9 to junction *c*, we obtain

$$(1) \quad I_1 + I_2 = I_3$$

Equation 28.10 applied to loops *defcd* and *cfgbc*, traversed clockwise, gives

$$(2)\ \ defcd \qquad 4.00\text{ V} - (3.00\ \Omega)I_2 - (5.00\ \Omega)I_3 = 0$$

$$(3)\ \ cfgbc \qquad (3.00\ \Omega)I_2 - (5.00\ \Omega)I_1 + 8.00\text{ V} = 0$$

From Equation (1) we see that $I_1 = I_3 - I_2$, which, when substituted into Equation (3), gives

$$(4) \quad (8.00\ \Omega)I_2 - (5.00\ \Omega)I_3 + 8.00\text{ V} = 0$$

Subtracting Equation (4) from Equation (2), we eliminate I_3 and find that

$$I_2 = -\frac{4.00\text{ V}}{11.0\ \Omega} = \boxed{-0.364\text{ A}}$$

Because our value for I_2 is negative, we conclude that the direction of I_2 is from *c* to *f* through the 3.00-Ω resistor. Despite

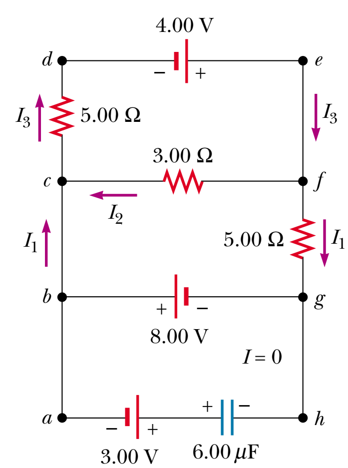


Figure 28.15 A multiloop circuit. Kirchhoff's loop rule can be applied to *any* closed loop, including the one containing the capacitor.

this interpretation of the direction, however, we must continue to use this negative value for I_2 in subsequent calculations because our equations were established with our original choice of direction.

Using $I_2 = -0.364$ A in Equations (3) and (1) gives

$$I_1 = \boxed{1.38\text{ A}} \qquad I_3 = \boxed{1.02\text{ A}}$$

(b) What is the charge on the capacitor?

Solution We can apply Kirchhoff's loop rule to loop *bghab* (or any other loop that contains the capacitor) to find the potential difference ΔV_{cap} across the capacitor. We enter this potential difference in the equation without reference to a sign convention because the charge on the capacitor depends only on the magnitude of the potential difference. Moving clockwise around this loop, we obtain

$$-8.00\text{ V} + \Delta V_{\text{cap}} - 3.00\text{ V} = 0$$

$$\Delta V_{\text{cap}} = 11.0\text{ V}$$

Because $Q = C\,\Delta V_{\text{cap}}$ (see Eq. 26.1), the charge on the capacitor is

$$Q = (6.00\ \mu\text{F})(11.0\text{ V}) = \boxed{66.0\ \mu\text{C}}$$

Why is the left side of the capacitor positively charged?

Exercise Find the voltage across the capacitor by traversing any other loop.

Answer 11.0 V.

Exercise Reverse the direction of the 3.00-V battery and answer parts (a) and (b) again.

Answer (a) $I_1 = 1.38$ A, $I_2 = -0.364$ A, $I_3 = 1.02$ A; (b) 30 μC.

28.4 *RC* CIRCUITS

So far we have been analyzing steady-state circuits, in which the current is constant. In circuits containing capacitors, the current may vary in time. A circuit containing a series combination of a resistor and a capacitor is called an ***RC* circuit.**

Charging a Capacitor

Let us assume that the capacitor in Figure 28.16 is initially uncharged. There is no current while switch S is open (Fig. 28.16b). If the switch is closed at $t = 0$, however, charge begins to flow, setting up a current in the circuit, and the capacitor begins to charge.[4] Note that during charging, charges do not jump across the capacitor plates because the gap between the plates represents an open circuit. Instead, charge is transferred between each plate and its connecting wire due to the electric field established in the wires by the battery, until the capacitor is fully charged. As the plates become charged, the potential difference across the capacitor increases. The value of the maximum charge depends on the voltage of the battery. Once the maximum charge is reached, the current in the circuit is zero because the potential difference across the capacitor matches that supplied by the battery.

To analyze this circuit quantitatively, let us apply Kirchhoff's loop rule to the circuit after the switch is closed. Traversing the loop clockwise gives

$$\mathcal{E} - \frac{q}{C} - IR = 0 \qquad \textbf{(28.11)}$$

where q/C is the potential difference across the capacitor and IR is the potential

[4] In previous discussions of capacitors, we assumed a steady-state situation, in which no current was present in any branch of the circuit containing a capacitor. Now we are considering the case *before* the steady-state condition is realized; in this situation, charges are moving and a current exists in the wires connected to the capacitor.

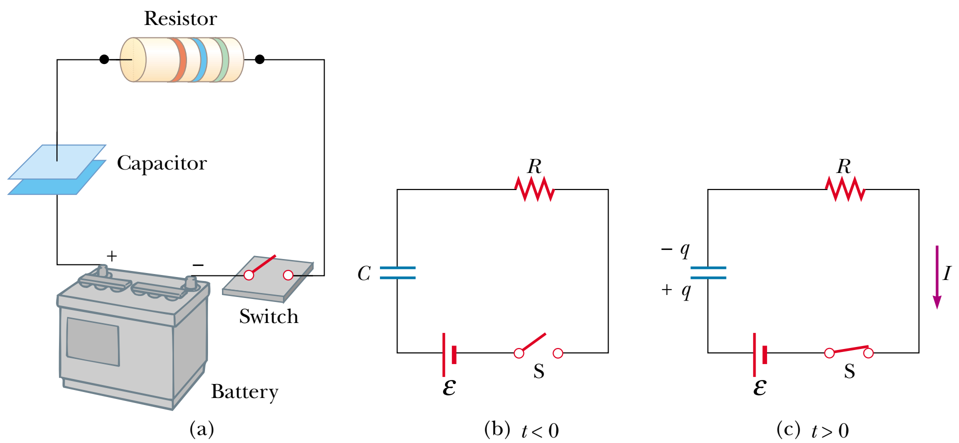


Figure 28.16 (a) A capacitor in series with a resistor, switch, and battery. (b) Circuit diagram representing this system at time $t < 0$, before the switch is closed. (c) Circuit diagram at time $t > 0$, after the switch has been closed.

difference across the resistor. We have used the sign conventions discussed earlier for the signs on $\mathcal{E}$ and IR. For the capacitor, notice that we are traveling in the direction from the positive plate to the negative plate; this represents a decrease in potential. Thus, we use a negative sign for this voltage in Equation 28.11. Note that q and I are *instantaneous* values that depend on time (as opposed to steady-state values) as the capacitor is being charged.

We can use Equation 28.11 to find the initial current in the circuit and the maximum charge on the capacitor. At the instant the switch is closed ($t = 0$), the charge on the capacitor is zero, and from Equation 28.11 we find that the initial current in the circuit I_0 is a maximum and is equal to

$$I_0 = \frac{\mathcal{E}}{R} \qquad \text{(current at } t = 0\text{)} \tag{28.12}$$

Maximum current

At this time, the potential difference from the battery terminals appears entirely across the resistor. Later, when the capacitor is charged to its maximum value Q, charges cease to flow, the current in the circuit is zero, and the potential difference from the battery terminals appears entirely across the capacitor. Substituting $I = 0$ into Equation 28.11 gives the charge on the capacitor at this time:

$$Q = C\mathcal{E} \qquad \text{(maximum charge)} \tag{28.13}$$

Maximum charge on the capacitor

To determine analytical expressions for the time dependence of the charge and current, we must solve Equation 28.11—a single equation containing two variables, q and I. The current in all parts of the series circuit must be the same. Thus, the current in the resistance R must be the same as the current flowing out of and into the capacitor plates. This current is equal to the time rate of change of the charge on the capacitor plates. Thus, we substitute $I = dq/dt$ into Equation 28.11 and rearrange the equation:

$$\frac{dq}{dt} = \frac{\mathcal{E}}{R} - \frac{q}{RC}$$

To find an expression for q, we first combine the terms on the right-hand side:

$$\frac{dq}{dt} = \frac{C\mathcal{E}}{RC} - \frac{q}{RC} = -\frac{q - C\mathcal{E}}{RC}$$

Now we multiply by dt and divide by $q - C\mathcal{E}$ to obtain

$$\frac{dq}{q - C\mathcal{E}} = -\frac{1}{RC}\, dt$$

Integrating this expression, using the fact that $q = 0$ at $t = 0$, we obtain

$$\int_0^q \frac{dq}{q - C\mathcal{E}} = -\frac{1}{RC}\int_0^t dt$$

$$\ln\left(\frac{q - C\mathcal{E}}{-C\mathcal{E}}\right) = -\frac{t}{RC}$$

From the definition of the natural logarithm, we can write this expression as

Charge versus time for a capacitor being charged

$$q(t) = C\mathcal{E}\,(1 - e^{-t/RC}) = Q(1 - e^{-t/RC}) \qquad \textbf{(28.14)}$$

where e is the base of the natural logarithm and we have made the substitution $C\mathcal{E} = Q$ from Equation 28.13.

We can find an expression for the charging current by differentiating Equation 28.14 with respect to time. Using $I = dq/dt$, we find that

Current versus time for a charging capacitor

$$I(t) = \frac{\mathcal{E}}{R}\, e^{-t/RC} \qquad \textbf{(28.15)}$$

Plots of capacitor charge and circuit current versus time are shown in Figure 28.17. Note that the charge is zero at $t = 0$ and approaches the maximum value $C\mathcal{E}$ as $t \rightarrow \infty$. The current has its maximum value $I_0 = \mathcal{E}/R$ at $t = 0$ and decays exponentially to zero as $t \rightarrow \infty$. The quantity RC, which appears in the exponents of Equations 28.14 and 28.15, is called the **time constant** τ of the circuit. It represents the time it takes the current to decrease to $1/e$ of its initial value; that is, in a time τ, $I = e^{-1}I_0 = 0.368I_0$. In a time 2τ, $I = e^{-2}I_0 = 0.135I_0$, and so forth. Likewise, in a time τ, the charge increases from zero to $C\mathcal{E}\,(1 - e^{-1}) = 0.632C\mathcal{E}$.

The following dimensional analysis shows that τ has the units of time:

$$[\tau] = [RC] = \left[\frac{\Delta V}{I} \times \frac{Q}{\Delta V}\right] = \left[\frac{Q}{Q/\Delta t}\right] = [\Delta t] = \mathrm{T}$$

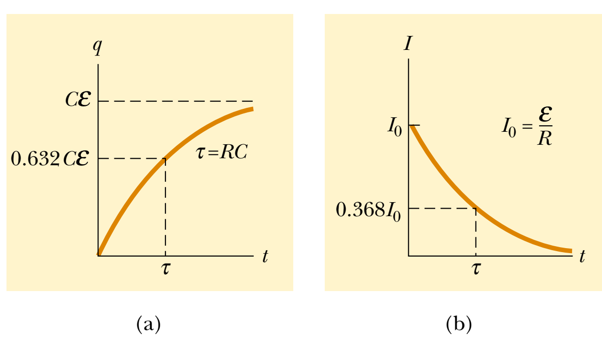


Figure 28.17 (a) Plot of capacitor charge versus time for the circuit shown in Figure 28.16. After a time interval equal to one time constant τ has passed, the charge is 63.2% of the maximum value $C\mathcal{E}$. The charge approaches its maximum value as t approaches infinity. (b) Plot of current versus time for the circuit shown in Figure 28.16. The current has its maximum value $I_0 = \mathcal{E}/R$ at $t = 0$ and decays to zero exponentially as t approaches infinity. After a time interval equal to one time constant τ has passed, the current is 36.8% of its initial value.

Because $\tau = RC$ has units of time, the combination t/RC is dimensionless, as it must be in order to be an exponent of e in Equations 28.14 and 28.15.

The energy output of the battery as the capacitor is fully charged is $Q\mathcal{E} = C\mathcal{E}^2$. After the capacitor is fully charged, the energy stored in the capacitor is $\frac{1}{2}Q\mathcal{E} = \frac{1}{2}C\mathcal{E}^2$, which is just half the energy output of the battery. It is left as a problem (Problem 60) to show that the remaining half of the energy supplied by the battery appears as internal energy in the resistor.

Discharging a Capacitor

Now let us consider the circuit shown in Figure 28.18, which consists of a capacitor carrying an initial charge Q, a resistor, and a switch. The *initial* charge Q is not the same as the *maximum* charge Q in the previous discussion, unless the discharge occurs after the capacitor is fully charged (as described earlier). When the switch is open, a potential difference Q/C exists across the capacitor and there is zero potential difference across the resistor because $I = 0$. If the switch is closed at $t = 0$, the capacitor begins to discharge through the resistor. At some time t during the discharge, the current in the circuit is I and the charge on the capacitor is q (Fig. 28.18b). The circuit in Figure 28.18 is the same as the circuit in Figure 28.16 except for the absence of the battery. Thus, we eliminate the emf $\mathcal{E}$ from Equation 28.11 to obtain the appropriate loop equation for the circuit in Figure 28.18:

$$-\frac{q}{C} - IR = 0 \tag{28.16}$$

When we substitute $I = dq/dt$ into this expression, it becomes

$$-R\frac{dq}{dt} = \frac{q}{C}$$

$$\frac{dq}{q} = -\frac{1}{RC}\,dt$$

Integrating this expression, using the fact that $q = Q$ at $t = 0$, gives

$$\int_Q^q \frac{dq}{q} = -\frac{1}{RC}\int_0^t dt$$

$$\ln\left(\frac{q}{Q}\right) = -\frac{t}{RC}$$

$$q(t) = Qe^{-t/RC} \tag{28.17}$$

Charge versus time for a discharging capacitor

Differentiating this expression with respect to time gives the instantaneous current as a function of time:

$$I(t) = \frac{dq}{dt} = \frac{d}{dt}(Qe^{-t/RC}) = -\frac{Q}{RC}e^{-t/RC} \tag{28.18}$$

Current versus time for a discharging capacitor

where $Q/RC = I_0$ is the initial current. The negative sign indicates that the current direction now that the capacitor is discharging is opposite the current direction when the capacitor was being charged. (Compare the current directions in Figs. 28.16c and 28.18b.) We see that both the charge on the capacitor and the current decay exponentially at a rate characterized by the time constant $\tau = RC$.

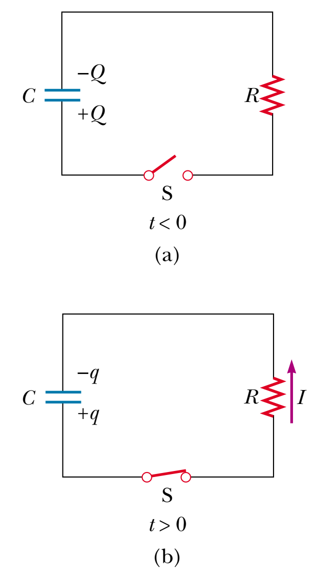


Figure 28.18 (a) A charged capacitor connected to a resistor and a switch, which is open at $t < 0$. (b) After the switch is closed, a current that decreases in magnitude with time is set up in the direction shown, and the charge on the capacitor decreases exponentially with time.

CONCEPTUAL EXAMPLE 28.10 Intermittent Windshield Wipers

Many automobiles are equipped with windshield wipers that can operate intermittently during a light rainfall. How does the operation of such wipers depend on the charging and discharging of a capacitor?

Solution The wipers are part of an *RC* circuit whose time constant can be varied by selecting different values of *R* through a multiposition switch. As it increases with time, the voltage across the capacitor reaches a point at which it triggers the wipers and discharges, ready to begin another charging cycle. The time interval between the individual sweeps of the wipers is determined by the value of the time constant.

EXAMPLE 28.11 Charging a Capacitor in an *RC* Circuit

An uncharged capacitor and a resistor are connected in series to a battery, as shown in Figure 28.19. If $\mathcal{E} = 12.0$ V, $C = 5.00\ \mu$F, and $R = 8.00 \times 10^5\ \Omega$, find the time constant of the circuit, the maximum charge on the capacitor, the maximum current in the circuit, and the charge and current as functions of time.

Solution The time constant of the circuit is $\tau = RC = (8.00 \times 10^5\ \Omega)(5.00 \times 10^{-6}\ \text{F}) = 4.00$ s. The maximum charge on the capacitor is $Q = C\mathcal{E} = (5.00\ \mu\text{F})(12.0\ \text{V}) = 60.0\ \mu$C. The maximum current in the circuit is $I_0 = \mathcal{E}/R = (12.0\ \text{V})/(8.00 \times 10^5\ \Omega) = 15.0\ \mu$A. Using these values and Equations 28.14 and 28.15, we find that

$$q(t) = \boxed{(60.0\ \mu\text{C})(1 - e^{-t/4.00\ \text{s}})}$$

$$I(t) = \boxed{(15.0\ \mu\text{A})\ e^{-t/4.00\ \text{s}}}$$

Graphs of these functions are provided in Figure 28.20.

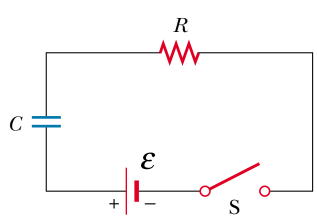


Figure 28.19 The switch of this series *RC* circuit, open for times $t < 0$, is closed at $t = 0$.

Exercise Calculate the charge on the capacitor and the current in the circuit after one time constant has elapsed.

Answer 37.9 μC, 5.52 μA.

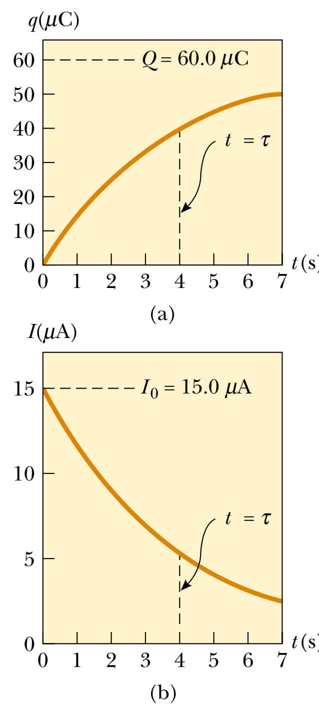


Figure 28.20 Plots of (a) charge versus time and (b) current versus time for the *RC* circuit shown in Figure 28.19, with $\mathcal{E} = 12.0$ V, $R = 8.00 \times 10^5\ \Omega$, and $C = 5.00\ \mu$F.

EXAMPLE 28.12 Discharging a Capacitor in an *RC* Circuit

Consider a capacitor of capacitance *C* that is being discharged through a resistor of resistance *R*, as shown in Figure 28.18. (a) After how many time constants is the charge on the capacitor one-fourth its initial value?

Solution The charge on the capacitor varies with time according to Equation 28.17, $q(t) = Qe^{-t/RC}$. To find the time it takes q to drop to one-fourth its initial value, we substitute $q(t) = Q/4$ into this expression and solve for t:

$$\frac{Q}{4} = Qe^{-t/RC}$$

$$\tfrac{1}{4} = e^{-t/RC}$$

Taking logarithms of both sides, we find

$$-\ln 4 = -\frac{t}{RC}$$

$$t = RC(\ln 4) = 1.39RC = \boxed{1.39\tau}$$

(b) The energy stored in the capacitor decreases with time as the capacitor discharges. After how many time constants is this stored energy one-fourth its initial value?

Solution Using Equations 26.11 ($U = Q^2/2C$) and 28.17, we can express the energy stored in the capacitor at any time t as

$$U = \frac{q^2}{2C} = \frac{(Qe^{-t/RC})^2}{2C} = \frac{Q^2}{2C}e^{-2t/RC} = U_0e^{-2t/RC}$$

where $U_0 = Q^2/2C$ is the initial energy stored in the capacitor. As in part (a), we now set $U = U_0/4$ and solve for t:

$$\frac{U_0}{4} = U_0e^{-2t/RC}$$

$$\tfrac{1}{4} = e^{-2t/RC}$$

Again, taking logarithms of both sides and solving for t gives

$$t = \tfrac{1}{2}RC(\ln 4) = 0.693RC = \boxed{0.693\tau}$$

Exercise After how many time constants is the current in the circuit one-half its initial value?

Answer $0.693RC = 0.693\tau$.

EXAMPLE 28.13 Energy Delivered to a Resistor

A 5.00-μF capacitor is charged to a potential difference of 800 V and then discharged through a 25.0-kΩ resistor. How much energy is delivered to the resistor in the time it takes to fully discharge the capacitor?

Solution We shall solve this problem in two ways. The first way is to note that the initial energy in the circuit equals the energy stored in the capacitor, $C\mathcal{E}^2/2$ (see Eq. 26.11). Once the capacitor is fully discharged, the energy stored in it is zero. Because energy is conserved, the initial energy stored in the capacitor is transformed into internal energy in the resistor. Using the given values of C and $\mathcal{E}$, we find

$$\text{Energy} = \tfrac{1}{2}C\mathcal{E}^2 = \tfrac{1}{2}(5.00 \times 10^{-6}\ \text{F})(800\ \text{V})^2 = \boxed{1.60\ \text{J}}$$

The second way, which is more difficult but perhaps more instructive, is to note that as the capacitor discharges through the resistor, the rate at which energy is delivered to the resistor is given by I^2R, where I is the instantaneous current given by Equation 28.18. Because power is defined as the time rate of change of energy, we conclude that the energy delivered to the resistor must equal the time integral of $I^2R\,dt$:

$$\text{Energy} = \int_0^\infty I^2R\,dt = \int_0^\infty (I_0e^{-t/RC})^2\,R\,dt$$

To evaluate this integral, we note that the initial current I_0 is equal to $\mathcal{E}/R$ and that all parameters except t are constant. Thus, we find

$$(1) \qquad \text{Energy} = \frac{\mathcal{E}^2}{R}\int_0^\infty e^{-2t/RC}\,dt$$

This integral has a value of $RC/2$; hence, we find

$$\text{Energy} = \tfrac{1}{2}C\mathcal{E}^2$$

which agrees with the result we obtained using the simpler approach, as it must. Note that we can use this second approach to find the total energy delivered to the resistor at *any* time after the switch is closed by simply replacing the upper limit in the integral with that specific value of t.

Exercise Show that the integral in Equation (1) has the value $RC/2$.

Optional Section

28.5 ELECTRICAL INSTRUMENTS

The Ammeter

A device that measures current is called an **ammeter.** The current to be measured must pass directly through the ammeter, so the ammeter must be connected in se-

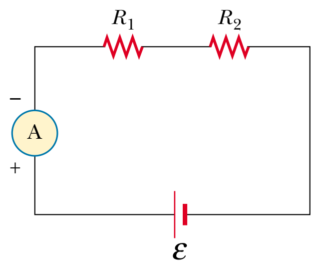


Figure 28.21 Current can be measured with an ammeter connected in series with the resistor and battery of a circuit. An ideal ammeter has zero resistance.

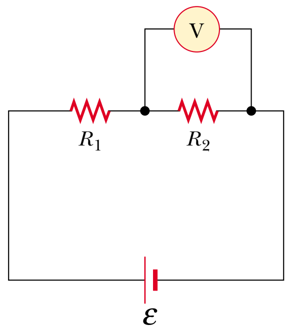


Figure 28.22 The potential difference across a resistor can be measured with a voltmeter connected in parallel with the resistor. An ideal voltmeter has infinite resistance.

ries with other elements in the circuit, as shown in Figure 28.21. When using an ammeter to measure direct currents, you must be sure to connect it so that current enters the instrument at the positive terminal and exits at the negative terminal.

Ideally, an ammeter should have zero resistance so that the current being measured is not altered. In the circuit shown in Figure 28.21, this condition requires that the resistance of the ammeter be much less than $R_1 + R_2$. Because any ammeter always has some internal resistance, the presence of the ammeter in the circuit slightly reduces the current from the value it would have in the meter's absence.

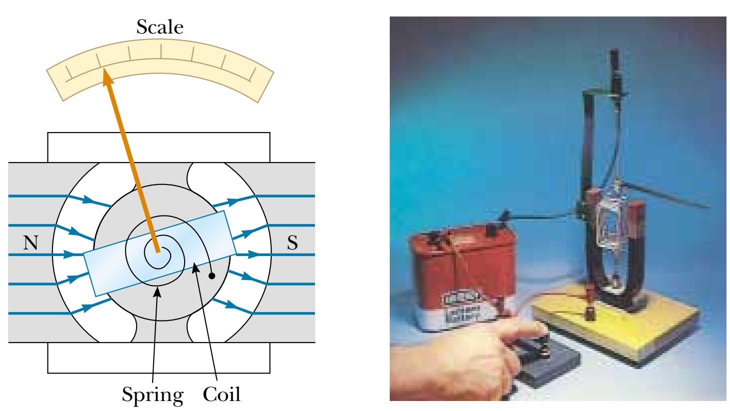
The Voltmeter

A device that measures potential difference is called a **voltmeter.** The potential difference between any two points in a circuit can be measured by attaching the terminals of the voltmeter between these points without breaking the circuit, as shown in Figure 28.22. The potential difference across resistor R_2 is measured by connecting the voltmeter in parallel with R_2. Again, it is necessary to observe the polarity of the instrument. The positive terminal of the voltmeter must be connected to the end of the resistor that is at the higher potential, and the negative terminal to the end of the resistor at the lower potential.

An ideal voltmeter has infinite resistance so that no current passes through it. In Figure 28.22, this condition requires that the voltmeter have a resistance much greater than R_2. In practice, if this condition is not met, corrections should be made for the known resistance of the voltmeter.

The Galvanometer

The **galvanometer** is the main component in analog ammeters and voltmeters. Figure 28.23a illustrates the essential features of a common type called the *D'Arsonval galvanometer.* It consists of a coil of wire mounted so that it is free to rotate on a pivot in a magnetic field provided by a permanent magnet. The basic op-



(a) (b)

Figure 28.23 (a) The principal components of a D'Arsonval galvanometer. When the coil situated in a magnetic field carries a current, the magnetic torque causes the coil to twist. The angle through which the coil rotates is proportional to the current in the coil because of the counteracting torque of the spring. (b) A large-scale model of a galvanometer movement. Why does the coil rotate about the vertical axis after the switch is closed?

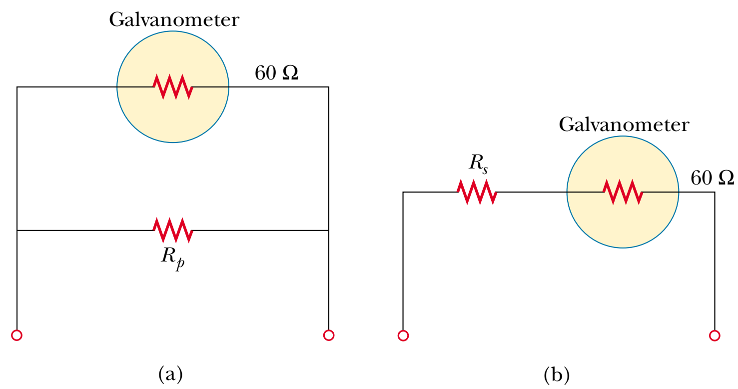


Figure 28.24 (a) When a galvanometer is to be used as an ammeter, a shunt resistor R_p is connected in parallel with the galvanometer. (b) When the galvanometer is used as a voltmeter, a resistor R_s is connected in series with the galvanometer.

eration of the galvanometer makes use of the fact that a torque acts on a current loop in the presence of a magnetic field (Chapter 29). The torque experienced by the coil is proportional to the current through it: the larger the current, the greater the torque and the more the coil rotates before the spring tightens enough to stop the rotation. Hence, the deflection of a needle attached to the coil is proportional to the current. Once the instrument is properly calibrated, it can be used in conjunction with other circuit elements to measure either currents or potential differences.

A typical off-the-shelf galvanometer is often not suitable for use as an ammeter, primarily because it has a resistance of about 60 Ω. An ammeter resistance this great considerably alters the current in a circuit. You can understand this by considering the following example: The current in a simple series circuit containing a 3-V battery and a 3-Ω resistor is 1 A. If you insert a 60-Ω galvanometer in this circuit to measure the current, the total resistance becomes 63 Ω and the current is reduced to 0.048 A!

A second factor that limits the use of a galvanometer as an ammeter is the fact that a typical galvanometer gives a full-scale deflection for currents of the order of 1 mA or less. Consequently, such a galvanometer cannot be used directly to measure currents greater than this value. However, it can be converted to a useful ammeter by placing a shunt resistor R_p in parallel with the galvanometer, as shown in Figure 28.24a. The value of R_p must be much less than the galvanometer resistance so that most of the current to be measured passes through the shunt resistor.

A galvanometer can also be used as a voltmeter by adding an external resistor R_s in series with it, as shown in Figure 28.24b. In this case, the external resistor must have a value much greater than the resistance of the galvanometer to ensure that the galvanometer does not significantly alter the voltage being measured.

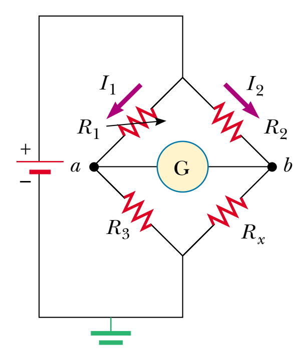


Figure 28.25 Circuit diagram for a Wheatstone bridge, an instrument used to measure an unknown resistance R_x in terms of known resistances R_1, R_2, and R_3. When the bridge is balanced, no current is present in the galvanometer. The arrow superimposed on the circuit symbol for resistor R_1 indicates that the value of this resistor can be varied by the person operating the bridge.

The Wheatstone Bridge

An unknown resistance value can be accurately measured using a circuit known as a **Wheatstone bridge** (Fig. 28.25). This circuit consists of the unknown resistance R_x, three known resistances R_1, R_2, and R_3 (where R_1 is a calibrated variable resistor), a galvanometer, and a battery. The known resistor R_1 is varied until the galvanometer reading is zero—that is, until there is no current from a to b. Under this condition the bridge is said to be balanced. Because the electric potential at

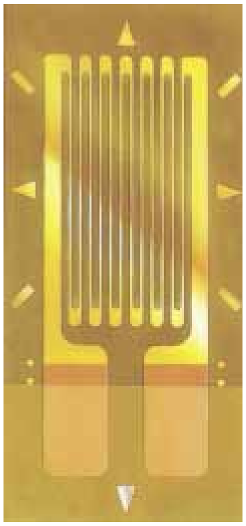
point a must equal the potential at point b when the bridge is balanced, the potential difference across R_1 must equal the potential difference across R_2. Likewise, the potential difference across R_3 must equal the potential difference across R_x. From these considerations we see that

$$(1) \qquad I_1R_1 = I_2R_2$$

$$(2) \qquad I_1R_3 = I_2R_x$$

Dividing Equation (1) by Equation (2) eliminates the currents, and solving for R_x, we find that

$$R_x = \frac{R_2R_3}{R_1} \tag{28.19}$$



The strain gauge, a device used for experimental stress analysis, consists of a thin coiled wire bonded to a flexible plastic backing. The gauge measures stresses by detecting changes in the resistance of the coil as the strip bends. Resistance measurements are made with this device as one element of a Wheatstone bridge. Strain gauges are commonly used in modern electronic balances to measure the masses of objects.

A number of similar devices also operate on the principle of null measurement (that is, adjustment of one circuit element to make the galvanometer read zero). One example is the capacitance bridge used to measure unknown capacitances. These devices do not require calibrated meters and can be used with any voltage source.

Wheatstone bridges are not useful for resistances above $10^5\ \Omega$, but modern electronic instruments can measure resistances as high as $10^{12}\ \Omega$. Such instruments have an extremely high resistance between their input terminals. For example, input resistances of $10^{10}\ \Omega$ are common in most digital multimeters, which are devices that are used to measure voltage, current, and resistance (Fig. 28.26).

Figure 28.26 Voltages, currents, and resistances are frequently measured with digital multimeters like this one.

The Potentiometer

A **potentiometer** is a circuit that is used to measure an unknown emf $\mathcal{E}_x$ by comparison with a known emf. In Figure 28.27, point d represents a sliding contact that is used to vary the resistance (and hence the potential difference) between points a and d. The other required components are a galvanometer, a battery of known emf $\mathcal{E}_0$, and a battery of unknown emf $\mathcal{E}_x$.

With the currents in the directions shown in Figure 28.27, we see from Kirchhoff's junction rule that the current in the resistor R_x is $I - I_x$, where I is the current in the left branch (through the battery of emf $\mathcal{E}_0$) and I_x is the current in the right branch. Kirchhoff's loop rule applied to loop *abcda* traversed clockwise gives

$$-\mathcal{E}_x + (I - I_x)R_x = 0$$

Because current I_x passes through it, the galvanometer displays a nonzero reading. The sliding contact at d is now adjusted until the galvanometer reads zero (indicating a balanced circuit and that the potentiometer is another null-measurement device). Under this condition, the current in the galvanometer is zero, and the potential difference between a and d must equal the unknown emf $\mathcal{E}_x$:

$$\mathcal{E}_x = IR_x$$

Next, the battery of unknown emf is replaced by a standard battery of known emf $\mathcal{E}_s$, and the procedure is repeated. If R_s is the resistance between a and d when balance is achieved this time, then

$$\mathcal{E}_s = IR_s$$

where it is assumed that I remains the same. Combining this expression with the preceding one, we see that

$$\mathcal{E}_x = \frac{R_x}{R_s}\,\mathcal{E}_s \tag{28.20}$$

If the resistor is a wire of resistivity ρ, its resistance can be varied by using the sliding contact to vary the length L, indicating how much of the wire is part of the circuit. With the substitutions $R_s = \rho L_s/A$ and $R_x = \rho L_x/A$, Equation 28.20 becomes

$$\mathcal{E}_x = \frac{L_x}{L_s}\mathcal{E}_s \qquad (28.21)$$

where L_x is the resistor length when the battery of unknown emf $\mathcal{E}_x$ is in the circuit and L_s is the resistor length when the standard battery is in the circuit.

The sliding-wire circuit of Figure 28.27 without the unknown emf and the galvanometer is sometimes called a *voltage divider.* This circuit makes it possible to tap into any desired smaller portion of the emf $\mathcal{E}_0$ by adjusting the length of the resistor.

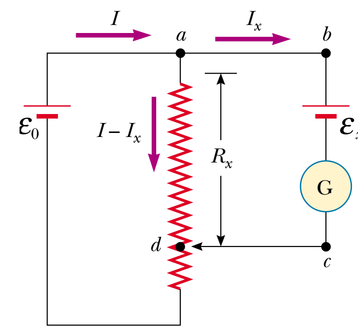


Figure 28.27 Circuit diagram for a potentiometer. The circuit is used to measure an unknown emf $\mathcal{E}_x$.

Optional Section

28.6 HOUSEHOLD WIRING AND ELECTRICAL SAFETY

Household circuits represent a practical application of some of the ideas presented in this chapter. In our world of electrical appliances, it is useful to understand the power requirements and limitations of conventional electrical systems and the safety measures that prevent accidents.

In a conventional installation, the utility company distributes electric power to individual homes by means of a pair of wires, with each home connected in parallel to these wires. One wire is called the *live wire,*[5] as illustrated in Figure 28.28, and the other is called the *neutral wire.* The potential difference between these two wires is about 120 V. This voltage alternates in time, with the neutral wire connected to ground and the potential of the live wire oscillating relative to ground. Much of what we have learned so far for the constant-emf situation (direct current) can also be applied to the alternating current that power companies supply to businesses and households. (Alternating voltage and current are discussed in Chapter 33.)

A meter is connected in series with the live wire entering the house to record the household's usage of electricity. After the meter, the wire splits so that there are several separate circuits in parallel distributed throughout the house. Each circuit contains a circuit breaker (or, in older installations, a fuse). The wire and circuit breaker for each circuit are carefully selected to meet the current demands for that circuit. If a circuit is to carry currents as large as 30 A, a heavy wire and an appropriate circuit breaker must be selected to handle this current. A circuit used to power only lamps and small appliances often requires only 15 A. Each circuit has its own circuit breaker to accommodate various load conditions.

As an example, consider a circuit in which a toaster oven, a microwave oven, and a coffee maker are connected (corresponding to R_1, R_2, and R_3 in Figure 28.28 and as shown in the chapter-opening photograph). We can calculate the current drawn by each appliance by using the expression $\mathcal{P} = I\Delta V$. The toaster oven, rated at 1 000 W, draws a current of 1 000 W/120 V = 8.33 A. The microwave oven, rated at 1 300 W, draws 10.8 A, and the coffee maker, rated at 800 W, draws 6.67 A. If the three appliances are operated simultaneously, they draw a total cur-

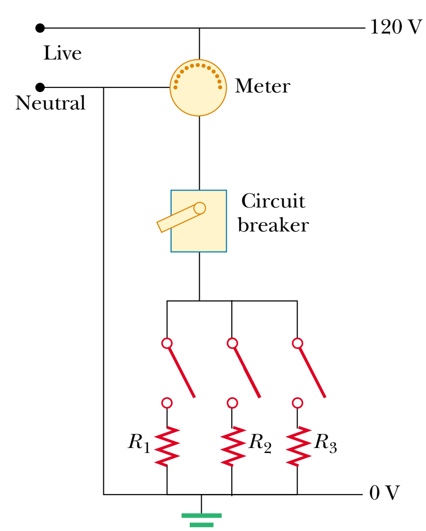


Figure 28.28 Wiring diagram for a household circuit. The resistances represent appliances or other electrical devices that operate with an applied voltage of 120 V.

[5] *Live wire* is a common expression for a conductor whose electric potential is above or below ground potential.

Figure 28.29 A power connection for a 240-V appliance.

Figure 28.30 A three-pronged power cord for a 120-V appliance.

rent of 25.8 A. Therefore, the circuit should be wired to handle at least this much current. If the rating of the circuit breaker protecting the circuit is too small—say, 20 A—the breaker will be tripped when the third appliance is turned on, preventing all three appliances from operating. To avoid this situation, the toaster oven and coffee maker can be operated on one 20-A circuit and the microwave oven on a separate 20-A circuit.

Many heavy-duty appliances, such as electric ranges and clothes dryers, require 240 V for their operation (Fig. 28.29). The power company supplies this voltage by providing a third wire that is 120 V below ground potential. The potential difference between this live wire and the other live wire (which is 120 V above ground potential) is 240 V. An appliance that operates from a 240-V line requires half the current of one operating from a 120-V line; therefore, smaller wires can be used in the higher-voltage circuit without overheating.

Electrical Safety

When the live wire of an electrical outlet is connected directly to ground, the circuit is completed and a short-circuit condition exists. A *short circuit* occurs when almost zero resistance exists between two points at different potentials; this results in a very large current. When this happens accidentally, a properly operating circuit breaker opens the circuit and no damage is done. However, a person in contact with ground can be electrocuted by touching the live wire of a frayed cord or other exposed conductor. An exceptionally good (although very dangerous) ground contact is made when the person either touches a water pipe (normally at ground potential) or stands on the ground with wet feet. The latter situation represents a good ground because normal, nondistilled water is a conductor because it contains a large number of ions associated with impurities. This situation should be avoided at all cost.

Electric shock can result in fatal burns, or it can cause the muscles of vital organs, such as the heart, to malfunction. The degree of damage to the body depends on the magnitude of the current, the length of time it acts, the part of the body touched by the live wire, and the part of the body through which the current passes. Currents of 5 mA or less cause a sensation of shock but ordinarily do little or no damage. If the current is larger than about 10 mA, the muscles contract and the person may be unable to release the live wire. If a current of about 100 mA passes through the body for only a few seconds, the result can be fatal. Such a large current paralyzes the respiratory muscles and prevents breathing. In some cases, currents of about 1 A through the body can produce serious (and sometimes fatal) burns. In practice, no contact with live wires is regarded as safe whenever the voltage is greater than 24 V.

Many 120-V outlets are designed to accept a three-pronged power cord such as the one shown in Figure 28.30. (This feature is required in all new electrical installations.) One of these prongs is the live wire at a nominal potential of 120 V. The second, called the “neutral,” is nominally at 0 V and carries current to ground. The third, round prong is a safety ground wire that normally carries no current but is both grounded and connected directly to the casing of the appliance. If the live wire is accidentally shorted to the casing (which can occur if the wire insulation wears off), most of the current takes the low-resistance path through the appliance to ground. In contrast, if the casing of the appliance is not properly grounded and a short occurs, anyone in contact with the appliance experiences an electric shock because the body provides a low-resistance path to ground.

Special power outlets called *ground-fault interrupters* (GFIs) are now being used in kitchens, bathrooms, basements, exterior outlets, and other hazardous areas of new homes. These devices are designed to protect persons from electric shock by sensing small currents (≈ 5 mA) leaking to ground. (The principle of their operation is described in Chapter 31.) When an excessive leakage current is detected, the current is shut off in less than 1 ms.

Quick Quiz 28.4

Is a circuit breaker wired in series or in parallel with the device it is protecting?

SUMMARY

The **emf** of a battery is equal to the voltage across its terminals when the current is zero. That is, the emf is equivalent to the **open-circuit voltage** of the battery.

The **equivalent resistance** of a set of resistors connected in **series** is

$$R_{eq} = R_1 + R_2 + R_3 + \cdots \tag{28.6}$$

The **equivalent resistance** of a set of resistors connected in **parallel** is

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \tag{28.8}$$

If it is possible to combine resistors into series or parallel equivalents, the preceding two equations make it easy to determine how the resistors influence the rest of the circuit.

Circuits involving more than one loop are conveniently analyzed with the use of **Kirchhoff's rules:**

1. The sum of the currents entering any junction in an electric circuit must equal the sum of the currents leaving that junction:

$$\sum I_{\text{in}} = \sum I_{\text{out}} \tag{28.9}$$

2. The sum of the potential differences across all elements around any circuit loop must be zero:

$$\sum_{\substack{\text{closed}\\ \text{loop}}} \Delta V = 0 \tag{28.10}$$

The first rule is a statement of conservation of charge; the second is equivalent to a statement of conservation of energy.

When a resistor is traversed in the direction of the current, the change in potential ΔV across the resistor is $-IR$. When a resistor is traversed in the direction opposite the current, $\Delta V = +IR$. When a source of emf is traversed in the direction of the emf (negative terminal to positive terminal), the change in potential is $+\mathcal{E}$. When a source of emf is traversed opposite the emf (positive to negative), the change in potential is $-\mathcal{E}$. The use of these rules together with Equations 28.9 and 28.10 allows you to analyze electric circuits.

If a capacitor is charged with a battery through a resistor of resistance R, the charge on the capacitor and the current in the circuit vary in time according to

the expressions

$$q(t) = Q(1 - e^{-t/RC}) \tag{28.14}$$

$$I(t) = \frac{\mathcal{E}}{R} e^{-t/RC} \tag{28.15}$$

where $Q = C\mathcal{E}$ is the maximum charge on the capacitor. The product RC is called the **time constant** τ of the circuit. If a charged capacitor is discharged through a resistor of resistance R, the charge and current decrease exponentially in time according to the expressions

$$q(t) = Qe^{-t/RC} \tag{28.17}$$

$$I(t) = -\frac{Q}{RC} e^{-t/RC} \tag{28.18}$$

where Q is the initial charge on the capacitor and $Q/RC = I_0$ is the initial current in the circuit. Equations 28.14, 28.15, 28.17, and 28.18 permit you to analyze the current and potential differences in an RC circuit and the charge stored in the circuit's capacitor.

QUESTIONS

1. Explain the difference between load resistance in a circuit and internal resistance in a battery.

2. Under what condition does the potential difference across the terminals of a battery equal its emf? Can the terminal voltage ever exceed the emf? Explain.

3. Is the direction of current through a battery always from the negative terminal to the positive one? Explain.

4. How would you connect resistors so that the equivalent resistance is greater than the greatest individual resistance? Give an example involving three resistors.

5. How would you connect resistors so that the equivalent resistance is less than the least individual resistance? Give an example involving three resistors.

6. Given three lightbulbs and a battery, sketch as many different electric circuits as you can.

7. Which of the following are the same for each resistor in a series connection—potential difference, current, power?

8. Which of the following are the same for each resistor in a parallel connection—potential difference, current, power?

9. What advantage might there be in using two identical resistors in parallel connected in series with another identical parallel pair, rather than just using a single resistor?

10. An incandescent lamp connected to a 120-V source with a short extension cord provides more illumination than the same lamp connected to the same source with a very long extension cord. Explain why.

11. When can the potential difference across a resistor be positive?

12. In Figure 28.15, suppose the wire between points g and h is replaced by a 10-Ω resistor. Explain why this change does not affect the currents calculated in Example 28.9.

13. Describe what happens to the lightbulb shown in Figure Q28.13 after the switch is closed. Assume that the capacitor has a large capacitance and is initially uncharged, and assume that the light illuminates when connected directly across the battery terminals.

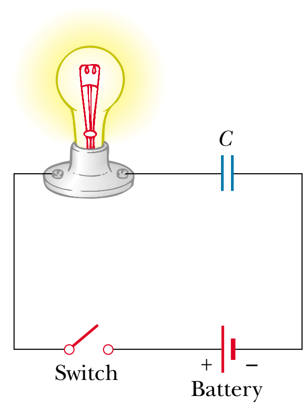


Figure Q28.13

14. What are the internal resistances of an ideal ammeter? of an ideal voltmeter? Do real meters ever attain these ideals?

15. Although the internal resistances of all sources of emf were neglected in the treatment of the potentiometer (Section 28.5), it is really not necessary to make this assumption. Explain why internal resistances play no role in the measurement of $\mathcal{E}_x$.

16. Why is it dangerous to turn on a light when you are in the bathtub?
17. Suppose you fall from a building, and on your way down you grab a high-voltage wire. Assuming that you are hanging from the wire, will you be electrocuted? If the wire then breaks, should you continue to hold onto an end of the wire as you fall?
18. What advantage does 120-V operation offer over 240 V? What are its disadvantages compared with 240 V?
19. When electricians work with potentially live wires, they often use the backs of their hands or fingers to move the wires. Why do you suppose they employ this technique?
20. What procedure would you use to try to save a person who is "frozen" to a live high-voltage wire without endangering your own life?
21. If it is the current through the body that determines the seriousness of a shock, why do we see warnings of high *voltage* rather than high *current* near electrical equipment?
22. Suppose you are flying a kite when it strikes a high-voltage wire. What factors determine how great a shock you receive?
23. A series circuit consists of three identical lamps that are connected to a battery as shown in Figure Q28.23. When switch S is closed, what happens (a) to the intensities of lamps A and B, (b) to the intensity of lamp C, (c) to the current in the circuit, and (d) to the voltage across the three lamps? (e) Does the power delivered to the circuit increase, decrease, or remain the same?

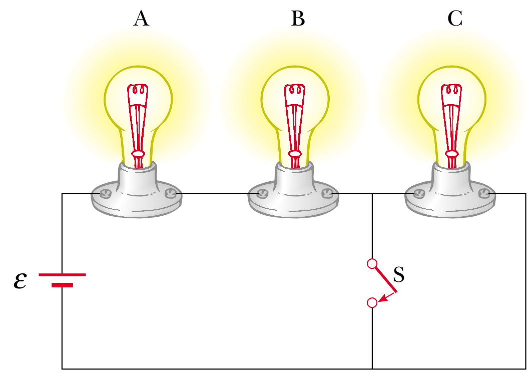


Figure Q28.23

24. If your car's headlights are on when you start the ignition, why do they dim while the car is starting?
25. A ski resort consists of a few chair lifts and several interconnected downhill runs on the side of a mountain, with a lodge at the bottom. The lifts are analogous to batteries, and the runs are analogous to resistors. Describe how two runs can be in series. Describe how three runs can be in parallel. Sketch a junction of one lift and two runs. State Kirchhoff's junction rule for ski resorts. One of the skiers, who happens to be carrying an altimeter, stops to warm up her toes each time she passes the lodge. State Kirchhoff's loop rule for altitude.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
☐ = paired numerical/symbolic problems

Section 28.1 Electromotive Force

WEB 1. A battery has an emf of 15.0 V. The terminal voltage of the battery is 11.6 V when it is delivering 20.0 W of power to an external load resistor *R*. (a) What is the value of *R*? (b) What is the internal resistance of the battery?

2. (a) What is the current in a 5.60-Ω resistor connected to a battery that has a 0.200-Ω internal resistance if the terminal voltage of the battery is 10.0 V? (b) What is the emf of the battery?

3. Two 1.50-V batteries—with their positive terminals in the same direction—are inserted in series into the barrel of a flashlight. One battery has an internal resistance of 0.255 Ω, the other an internal resistance of 0.153 Ω. When the switch is closed, a current of 600 mA occurs in the lamp. (a) What is the lamp's resistance? (b) What percentage of the power from the batteries appears in the batteries themselves, as represented by an increase in temperature?

4. An automobile battery has an emf of 12.6 V and an internal resistance of 0.080 0 Ω. The headlights have a total resistance of 5.00 Ω (assumed constant). What is the potential difference across the headlight bulbs (a) when they are the only load on the battery and (b) when the starter motor, which takes an additional 35.0 A from the battery, is operated?

Section 28.2 Resistors in Series and in Parallel

5. The current in a loop circuit that has a resistance of R_1 is 2.00 A. The current is reduced to 1.60 A when an additional resistor $R_2 = 3.00\ \Omega$ is added in series with R_1. What is the value of R_1?

6. (a) Find the equivalent resistance between points *a* and *b* in Figure P28.6. (b) Calculate the current in each resistor if a potential difference of 34.0 V is applied between points *a* and *b*.

7. A television repairman needs a 100-Ω resistor to repair a malfunctioning set. He is temporarily out of resistors

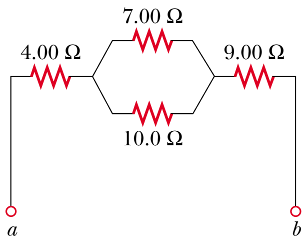


Figure P28.6

of this value. All he has in his toolbox are a 500-Ω resistor and two 250-Ω resistors. How can he obtain the desired resistance using the resistors he has on hand?

8. A lightbulb marked "75 W [at] 120 V" is screwed into a socket at one end of a long extension cord in which each of the two conductors has a resistance of 0.800 Ω. The other end of the extension cord is plugged into a 120-V outlet. Draw a circuit diagram, and find the actual power delivered to the bulb in this circuit.

WEB **9.** Consider the circuit shown in Figure P28.9. Find (a) the current in the 20.0-Ω resistor and (b) the potential difference between points *a* and *b*.

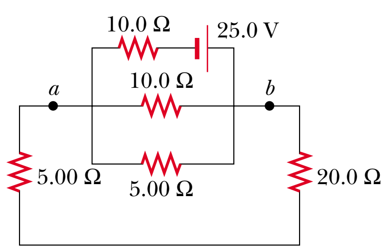


Figure P28.9

10. Four copper wires of equal length are connected in series. Their cross-sectional areas are 1.00 cm^2, 2.00 cm^2, 3.00 cm^2, and 5.00 cm^2. If a voltage of 120 V is applied to the arrangement, what is the voltage across the 2.00-cm^2 wire?

11. Three 100-Ω resistors are connected as shown in Figure P28.11. The maximum power that can safely be delivered to any one resistor is 25.0 W. (a) What is the maximum voltage that can be applied to the terminals *a* and *b*? (b) For the voltage determined in part (a), what is the power delivered to each resistor? What is the total power delivered?

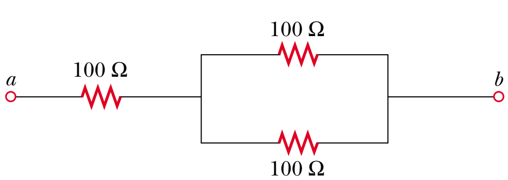


Figure P28.11

12. Using only three resistors—2.00 Ω, 3.00 Ω, and 4.00 Ω—find 17 resistance values that can be obtained with various combinations of one or more resistors. Tabulate the combinations in order of increasing resistance.

13. The current in a circuit is tripled by connecting a 500-Ω resistor in parallel with the resistance of the circuit. Determine the resistance of the circuit in the absence of the 500-Ω resistor.

14. The power delivered to the top part of the circuit shown in Figure P28.14 does not depend on whether the switch is opened or closed. If R = 1.00 Ω, what is R'? Neglect the internal resistance of the voltage source.

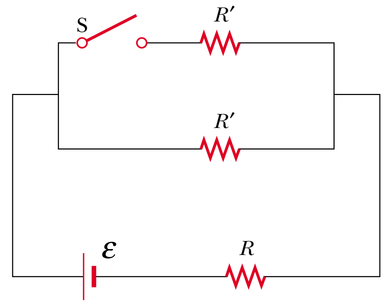


Figure P28.14

15. Calculate the power delivered to each resistor in the circuit shown in Figure P28.15.

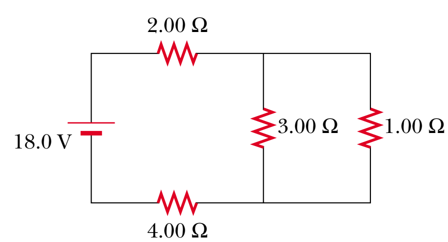


Figure P28.15

16. Two resistors connected in series have an equivalent resistance of 690 Ω. When they are connected in parallel, their equivalent resistance is 150 Ω. Find the resistance of each resistor.

17. In Figures 28.4 and 28.5, let R_1 = 11.0 Ω, let R_2 = 22.0 Ω, and let the battery have a terminal voltage of 33.0 V. (a) In the parallel circuit shown in Figure 28.5, which resistor uses more power? (b) Verify that the sum of the power (I^2R) used by each resistor equals the power supplied by the battery ($I\Delta V$). (c) In the series circuit, which resistor uses more power? (d) Verify that the sum of the power (I^2R) used by each resistor equals

the power supplied by the battery ($\mathcal{P} = I\Delta V$).
(e) Which circuit configuration uses more power?

Section 28.3 Kirchhoff's Rules

Note: The currents are not necessarily in the direction shown for some circuits.

18. The ammeter shown in Figure P28.18 reads 2.00 A. Find I_1, I_2, and $\mathcal{E}$.

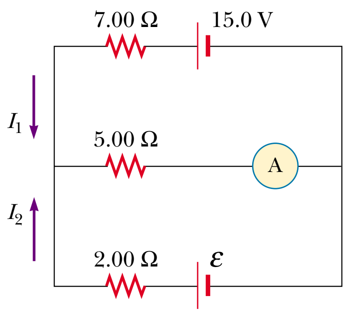


Figure P28.18

WEB **19.** Determine the current in each branch of the circuit shown in Figure P28.19.

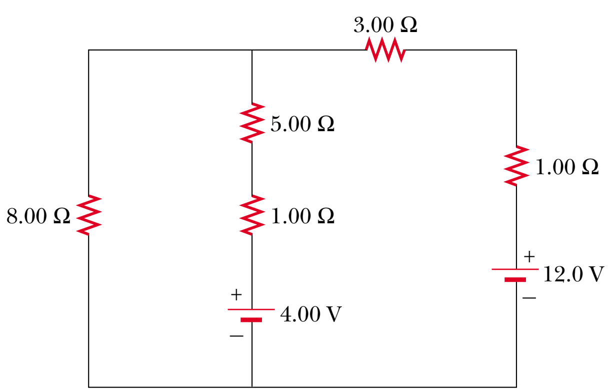


Figure P28.19 Problems 19, 20, and 21.

20. In Figure P28.19, show how to add just enough ammeters to measure every different current that is flowing. Show how to add just enough voltmeters to measure the potential difference across each resistor and across each battery.

21. The circuit considered in Problem 19 and shown in Figure P28.19 is connected for 2.00 min. (a) Find the energy supplied by each battery. (b) Find the energy delivered to each resistor. (c) Find the total amount of energy converted from chemical energy in the battery to internal energy in the circuit resistance.

22. (a) Using Kirchhoff's rules, find the current in each resistor shown in Figure P28.22 and (b) find the potential difference between points *c* and *f*. Which point is at the higher potential?

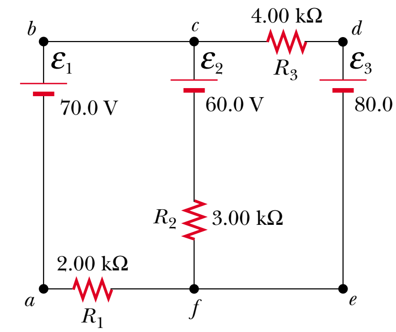


Figure P28.22

23. If $R = 1.00$ kΩ and $\mathcal{E} = 250$ V in Figure P28.23, determine the direction and magnitude of the current in the horizontal wire between *a* and *e*.

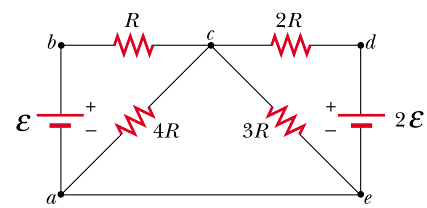


Figure P28.23

24. In the circuit of Figure P28.24, determine the current in each resistor and the voltage across the 200-Ω resistor.

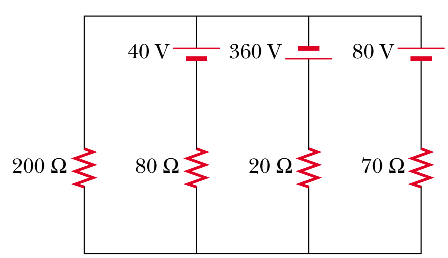


Figure P28.24

25. A dead battery is charged by connecting it to the live battery of another car with jumper cables (Fig. P28.25). Determine the current in the starter and in the dead battery.

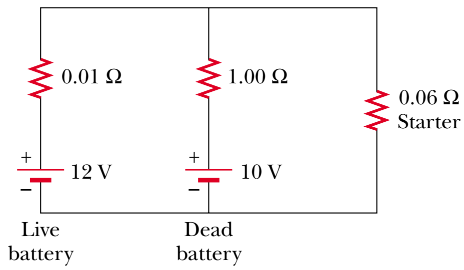


Figure P28.25

26. For the network shown in Figure P28.26, show that the resistance $R_{ab} = \frac{27}{17}\ \Omega$.

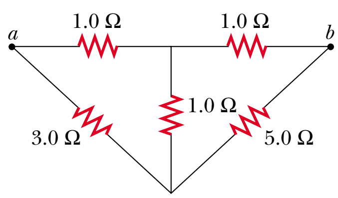


Figure P28.26

27. For the circuit shown in Figure P28.27, calculate (a) the current in the 2.00-Ω resistor and (b) the potential difference between points a and b.

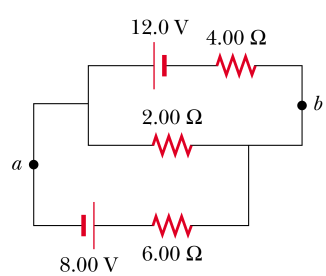


Figure P28.27

28. Calculate the power delivered to each of the resistors shown in Figure P28.28.

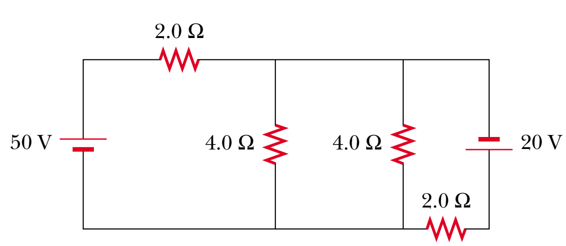


Figure P28.28

Section 28.4 *RC* Circuits

WEB 29. Consider a series RC circuit (see Fig. 28.16) for which $R = 1.00\ \text{M}\Omega$, $C = 5.00\ \mu\text{F}$, and $\mathcal{E} = 30.0$ V. Find (a) the time constant of the circuit and (b) the maximum charge on the capacitor after the switch is closed. (c) If the switch is closed at $t = 0$, find the current in the resistor 10.0 s later.

30. A 2.00-nF capacitor with an initial charge of 5.10 μC is discharged through a 1.30-kΩ resistor. (a) Calculate the current through the resistor 9.00 μs after the resistor is connected across the terminals of the capacitor. (b) What charge remains on the capacitor after 8.00 μs? (c) What is the maximum current in the resistor?

31. A fully charged capacitor stores energy U_0. How much energy remains when its charge has decreased to half its original value?

32. In the circuit of Figure P28.32, switch S has been open for a long time. It is then suddenly closed. Determine the time constant (a) before the switch is closed and (b) after the switch is closed. (c) If the switch is closed at $t = 0$, determine the current through it as a function of time.

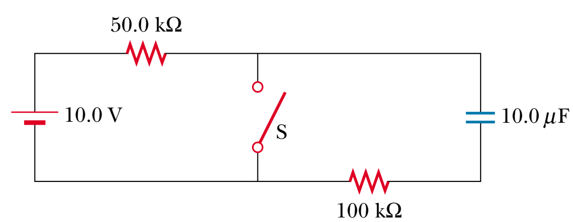


Figure P28.32

33. The circuit shown in Figure P28.33 has been connected for a long time. (a) What is the voltage across the capacitor? (b) If the battery is disconnected, how long does it take the capacitor to discharge to one-tenth its initial voltage?

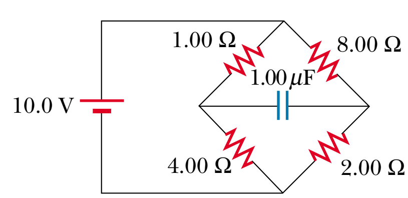


Figure P28.33

34. A 4.00-MΩ resistor and a 3.00-μF capacitor are connected in series with a 12.0-V power supply. (a) What is the time constant for the circuit? (b) Express the current in the circuit and the charge on the capacitor as functions of time.

35. Dielectric materials used in the manufacture of capacitors are characterized by conductivities that are small but not zero. Therefore, a charged capacitor slowly loses its charge by "leaking" across the dielectric. If a certain 3.60-μF capacitor leaks charge such that the potential difference decreases to half its initial value in 4.00 s, what is the equivalent resistance of the dielectric?

36. Dielectric materials used in the manufacture of capacitors are characterized by conductivities that are small but not zero. Therefore, a charged capacitor slowly loses its charge by "leaking" across the dielectric. If a capacitor having capacitance C leaks charge such that the potential difference decreases to half its initial value in a time t, what is the equivalent resistance of the dielectric?

37. A capacitor in an RC circuit is charged to 60.0% of its maximum value in 0.900 s. What is the time constant of the circuit?

(Optional)

Section 28.5 Electrical Instruments

38. A typical galvanometer, which requires a current of 1.50 mA for full-scale deflection and has a resistance of 75.0 Ω, can be used to measure currents of much greater values. A relatively small shunt resistor is wired in parallel with the galvanometer (refer to Fig. 28.24a) so that an operator can measure large currents without causing damage to the galvanometer. Most of the current then flows through the shunt resistor. Calculate the value of the shunt resistor that enables the galvanometer to be used to measure a current of 1.00 A at full-scale deflection. (*Hint:* Use Kirchhoff's rules.)

$\boxed{39.}$ The galvanometer described in the preceding problem can be used to measure voltages. In this case a large resistor is wired in series with the galvanometer in a way similar to that shown in Figure 28.24b. This arrangement, in effect, limits the current that flows through the galvanometer when large voltages are applied. Most of the potential drop occurs across the resistor placed in series. Calculate the value of the resistor that enables the galvanometer to measure an applied voltage of 25.0 V at full-scale deflection.

40. A galvanometer with a full-scale sensitivity of 1.00 mA requires a 900-Ω series resistor to make a voltmeter reading full scale when 1.00 V is measured across the terminals. What series resistor is required to make the same galvanometer into a 50.0-V (full-scale) voltmeter?

$\boxed{41.}$ Assume that a galvanometer has an internal resistance of 60.0 Ω and requires a current of 0.500 mA to produce full-scale deflection. What resistance must be connected in parallel with the galvanometer if the combination is to serve as an ammeter that has a full-scale deflection for a current of 0.100 A?

42. A Wheatstone bridge of the type shown in Figure 28.25 is used to make a precise measurement of the resistance of a wire connector. If $R_3 = 1.00\ \text{k}\Omega$ and the bridge is balanced by adjusting R_1 such that $R_1 = 2.50R_2$, what is R_x?

43. Consider the case in which the Wheatstone bridge shown in Figure 28.25 is unbalanced. Calculate the current through the galvanometer when $R_x = R_3 = 7.00\ \Omega$, $R_2 = 21.0\ \Omega$, and $R_1 = 14.0\ \Omega$. Assume that the voltage across the bridge is 70.0 V, and neglect the galvanometer's resistance.

44. **Review Problem.** A Wheatstone bridge can be used to measure the strain $(\Delta L/L_i)$ of a wire (see Section 12.4), where L_i is the length before stretching, L is the length after stretching, and $\Delta L = L - L_i$. Let $\alpha = \Delta L/L_i$. Show that the resistance is $R = R_i(1 + 2\alpha + \alpha^2)$ for any length, where $R_i = \rho L_i/A_i$. Assume that the resistivity and volume of the wire stay constant.

45. Consider the potentiometer circuit shown in Figure 28.27. If a standard battery with an emf of 1.018 6 V is used in the circuit and the resistance between a and d is 36.0 Ω, the galvanometer reads zero. If the standard battery is replaced by an unknown emf, the galvanometer reads zero when the resistance is adjusted to 48.0 Ω. What is the value of the emf?

46. *Meter loading.* Work this problem to five-digit precision. Refer to Figure P28.46. (a) When a 180.00-Ω resistor is put across a battery with an emf of 6.000 0 V and an internal resistance of 20.000 Ω, what current flows in the resistor? What will be the potential difference across it? (b) Suppose now that an ammeter with a resistance of 0.500 00 Ω and a voltmeter with a resistance of

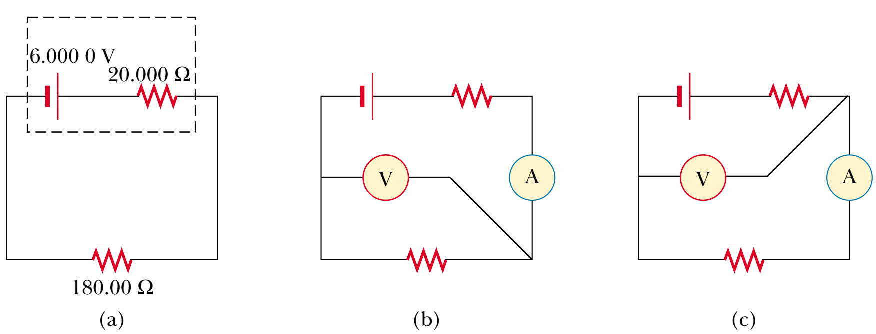


Figure P28.46

20 000 Ω are added to the circuit, as shown in Figure P28.46b. Find the reading of each. (c) One terminal of one wire is moved, as shown in Figure P28.46c. Find the new meter readings.

(Optional)
Section 28.6 Household Wiring and Electrical Safety

WEB **47.** An electric heater is rated at 1 500 W, a toaster at 750 W, and an electric grill at 1 000 W. The three appliances are connected to a common 120-V circuit. (a) How much current does each draw? (b) Is a 25.0-A circuit breaker sufficient in this situation? Explain your answer.

48. An 8.00-ft extension cord has two 18-gauge copper wires, each with a diameter of 1.024 mm. What is the I^2R loss in this cord when it carries a current of (a) 1.00 A? (b) 10.0 A?

49. Sometimes aluminum wiring has been used instead of copper for economic reasons. According to the National Electrical Code, the maximum allowable current for 12-gauge copper wire with rubber insulation is 20 A. What should be the maximum allowable current in a 12-gauge aluminum wire if it is to have the same I^2R loss per unit length as the copper wire?

50. Turn on your desk lamp. Pick up the cord with your thumb and index finger spanning its width. (a) Compute an order-of-magnitude estimate for the current that flows through your hand. You may assume that at a typical instant the conductor inside the lamp cord next to your thumb is at potential $\sim 10^2$ V and that the conductor next to your index finger is at ground potential (0 V). The resistance of your hand depends strongly on the thickness and moisture content of the outer layers of your skin. Assume that the resistance of your hand between fingertip and thumb tip is $\sim 10^4$ Ω. You may model the cord as having rubber insulation. State the other quantities you measure or estimate and their values. Explain your reasoning. (b) Suppose that your body is isolated from any other charges or currents. In order-of-magnitude terms, describe the potential of your thumb where it contacts the cord and the potential of your finger where it touches the cord.

ADDITIONAL PROBLEMS

51. Four 1.50-V AA batteries in series are used to power a transistor radio. If the batteries can provide a total charge of 240 C, how long will they last if the radio has a resistance of 200 Ω?

52. A battery has an emf of 9.20 V and an internal resistance of 1.20 Ω. (a) What resistance across the battery will extract from it a power of 12.8 W? (b) a power of 21.2 W?

53. Calculate the potential difference between points a and b in Figure P28.53, and identify which point is at the higher potential.

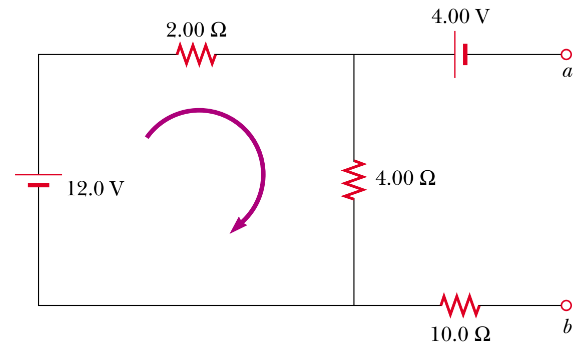


Figure P28.53

54. A 10.0-μF capacitor is charged by a 10.0-V battery through a resistance R. The capacitor reaches a potential difference of 4.00 V at a time 3.00 s after charging begins. Find R.

55. When two unknown resistors are connected in series with a battery, 225 W is delivered to the combination with a total current of 5.00 A. For the same total current, 50.0 W is delivered when the resistors are connected in parallel. Determine the values of the two resistors.

56. When two unknown resistors are connected in series with a battery, a total power $\mathcal{P}_s$ is delivered to the combination with a total current of I. For the same total current, a total power $\mathcal{P}_p$ is delivered when the resistors are connected in parallel. Determine the values of the two resistors.

57. A battery has an emf $\mathcal{E}$ and internal resistance r. A variable resistor R is connected across the terminals of the battery. Determine the value of R such that (a) the potential difference across the terminals is a maximum, (b) the current in the circuit is a maximum, (c) the power delivered to the resistor is a maximum.

58. A power supply has an open-circuit voltage of 40.0 V and an internal resistance of 2.00 Ω. It is used to charge two storage batteries connected in series, each having an emf of 6.00 V and internal resistance of 0.300 Ω. If the charging current is to be 4.00 A, (a) what additional resistance should be added in series? (b) Find the power delivered to the internal resistance of the supply, the I^2R loss in the batteries, and the power delivered to the added series resistance. (c) At what rate is the chemical energy in the batteries increasing?

59. The value of a resistor R is to be determined using the ammeter-voltmeter setup shown in Figure P28.59. The ammeter has a resistance of 0.500 Ω, and the voltmeter has a resistance of 20 000 Ω. Within what range of actual values of R will the measured values be correct, to within 5.00%, if the measurement is made using (a) the circuit shown in Figure P28.59a? (b) the circuit shown in Figure P28.59b?

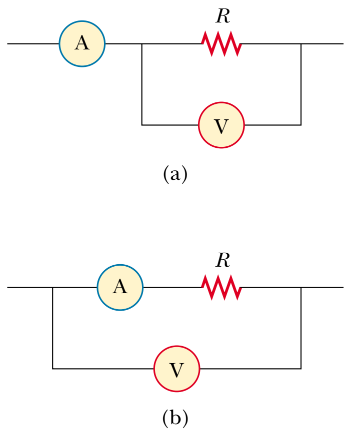


Figure P28.59

60. A battery is used to charge a capacitor through a resistor, as shown in Figure 28.16. Show that half the energy supplied by the battery appears as internal energy in the resistor and that half is stored in the capacitor.

61. The values of the components in a simple series RC circuit containing a switch (Fig. 28.16) are $C = 1.00\ \mu\text{F}$, $R = 2.00 \times 10^6\ \Omega$, and $\boldsymbol{\mathcal{E}} = 10.0$ V. At the instant 10.0 s after the switch is closed, calculate (a) the charge on the capacitor, (b) the current in the resistor, (c) the rate at which energy is being stored in the capacitor, and (d) the rate at which energy is being delivered by the battery.

62. The switch in Figure P28.62a closes when $\Delta V_c > 2\Delta V/3$ and opens when $\Delta V_c < \Delta V/3$. The voltmeter reads a voltage as plotted in Figure P28.62b. What is the period T of the waveform in terms of R_A, R_B, and C?

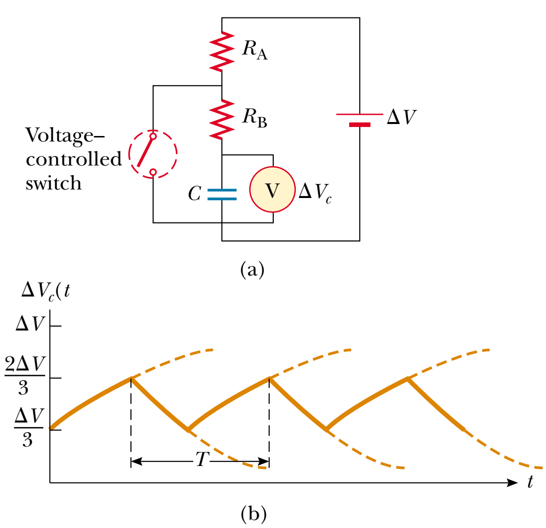


Figure P28.62

63. Three 60.0-W, 120-V lightbulbs are connected across a 120-V power source, as shown in Figure P28.63. Find (a) the total power delivered to the three bulbs and (b) the voltage across each. Assume that the resistance of each bulb conforms to Ohm's law (even though in reality the resistance increases markedly with current).

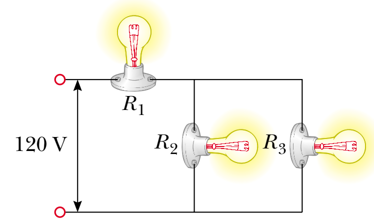


Figure P28.63

64. Design a multirange voltmeter capable of full-scale deflection for 20.0 V, 50.0 V, and 100 V. Assume that the meter movement is a galvanometer that has a resistance of 60.0 Ω and gives a full-scale deflection for a current of 1.00 mA.

65. Design a multirange ammeter capable of full-scale deflection for 25.0 mA, 50.0 mA, and 100 mA. Assume that the meter movement is a galvanometer that has a resistance of 25.0 Ω and gives a full-scale deflection for 1.00 mA.

66. A particular galvanometer serves as a 2.00-V full-scale voltmeter when a 2 500-Ω resistor is connected in series with it. It serves as a 0.500-A full-scale ammeter when a 0.220-Ω resistor is connected in parallel with it. Determine the internal resistance of the galvanometer and the current required to produce full-scale deflection.

67. In Figure P28.67, suppose that the switch has been closed for a length of time sufficiently long for the capacitor to become fully charged. (a) Find the steady-state current in each resistor. (b) Find the charge Q on the capacitor. (c) The switch is opened at $t = 0$. Write an equation for the current I_{R_2} in R_2 as a function of time, and (d) find the time that it takes for the charge on the capacitor to fall to one-fifth its initial value.

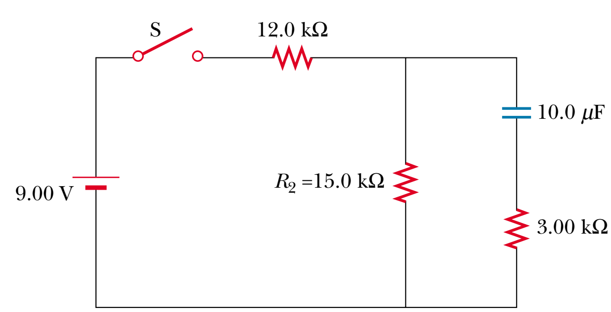


Figure P28.67

68. The circuit shown in Figure P28.68 is set up in the laboratory to measure an unknown capacitance C with the use of a voltmeter of resistance $R = 10.0\ \text{M}\Omega$ and a battery whose emf is 6.19 V. The data given in the table below are the measured voltages across the capacitor as a function of time, where $t = 0$ represents the time at which the switch is opened. (a) Construct a graph of $\ln(\mathcal{E}/\Delta V)$ versus t, and perform a linear least-squares fit to the data. (b) From the slope of your graph, obtain a value for the time constant of the circuit and a value for the capacitance.

ΔV (V)	t (s)	$\ln(\mathcal{E}/\Delta V)$
6.19	0	
5.55	4.87	
4.93	11.1	
4.34	19.4	
3.72	30.8	
3.09	46.6	
2.47	67.3	
1.83	102.2	

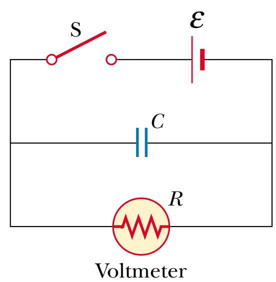


Figure P28.68

69. (a) Using symmetry arguments, show that the current through any resistor in the configuration of Figure P28.69 is either $I/3$ or $I/6$. All resistors have the same resistance r. (b) Show that the equivalent resistance between points a and b is $(5/6)r$.

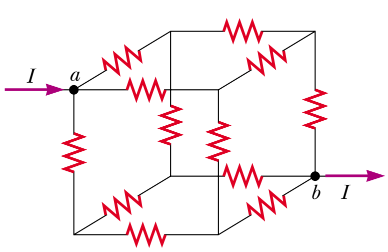


Figure P28.69

70. The student engineer of a campus radio station wishes to verify the effectiveness of the lightning rod on the antenna mast (Fig. P28.70). The unknown resistance R_x is between points C and E. Point E is a true ground but is inaccessible for direct measurement since this stratum is several meters below the Earth's surface. Two identical rods are driven into the ground at A and B, introducing an unknown resistance R_y. The procedure is as follows. Measure resistance R_1 between points A and B, then connect A and B with a heavy conducting wire and measure resistance R_2 between points A and C. (a) Derive a formula for R_x in terms of the observable resistances R_1 and R_2. (b) A satisfactory ground resistance would be $R_x < 2.00\ \Omega$. Is the grounding of the station adequate if measurements give $R_1 = 13.0\ \Omega$ and $R_2 = 6.00\ \Omega$?

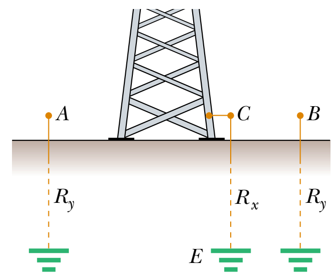


Figure P28.70

71. Three 2.00-Ω resistors are connected as shown in Figure P28.71. Each can withstand a maximum power of 32.0 W without becoming excessively hot. Determine the maximum power that can be delivered to the combination of resistors.

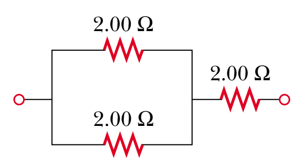


Figure P28.71

72. The circuit in Figure P28.72 contains two resistors, $R_1 = 2.00\ \text{k}\Omega$ and $R_2 = 3.00\ \text{k}\Omega$, and two capacitors, $C_1 = 2.00\ \mu\text{F}$ and $C_2 = 3.00\ \mu\text{F}$, connected to a battery with emf $\mathcal{E} = 120$ V. If no charges exist on the capacitors before switch S is closed, determine the charges q_1 and q_2 on capacitors C_1 and C_2, respectively, after the switch is closed. (*Hint:* First reconstruct the circuit so that it becomes a simple RC circuit containing a single resistor and single capacitor in series, connected to the battery, and then determine the total charge q stored in the equivalent circuit.)

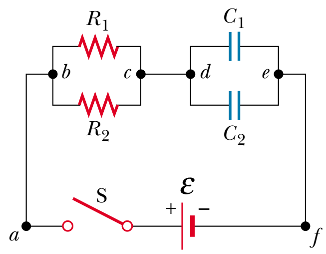


Figure P28.72

73. Assume that you have a battery of emf $\mathcal{E}$ and three identical lightbulbs, each having constant resistance R. What is the total power from the battery if the bulbs are connected (a) in series? (b) in parallel? (c) For which connection do the bulbs shine the brightest?

Answers to Quick Quizzes

28.1 Bulb R_1 becomes brighter. Connecting b to c "shorts out" bulb R_2 and changes the total resistance of the circuit from $R_1 + R_2$ to just R_1. Because the resistance has decreased (and the potential difference supplied by the battery does not change), the current through the battery increases. This means that the current through bulb R_1 increases, and bulb R_1 glows more brightly. Bulb R_2 goes out because the new piece of wire provides an almost resistance-free path for the current; hence, essentially zero current exists in bulb R_2.

28.2 Adding another series resistor increases the total resistance of the circuit and thus reduces the current in the battery. The potential difference across the battery terminals would increase because the reduced current results in a smaller voltage decrease across the internal resistance.

If the second resistor were connected in parallel, the total resistance of the circuit would decrease, and an increase in current through the battery would result. The potential difference across the terminals would decrease because the increased current results in a greater voltage decrease across the internal resistance.

28.3 They must be in parallel because if one burns out, the other continues to operate. If they were in series, one failed headlamp would interrupt the current throughout the entire circuit, including the other headlamp.

28.4 Because the circuit breaker trips and opens the circuit when the current in that circuit exceeds a certain preset value, it must be in series to sense the appropriate current (see Fig. 28.28).

PUZZLER

Aurora Borealis, the Northern Lights, photographed near Fairbanks, Alaska. Such beautiful auroral displays are a common sight in far northern or southern latitudes, but they are quite rare in the middle latitudes. What causes these shimmering curtains of light, and why are they usually visible only near the Earth's North and South poles? *(George Lepp/Tony Stone Images)*

chapter

29

Magnetic Fields

Chapter Outline

Many historians of science believe that the compass, which uses a magnetic needle, was used in China as early as the 13th century B.C., its invention being of Arabic or Indian origin. The early Greeks knew about magnetism as early as 800 B.C. They discovered that the stone magnetite (Fe_3O_4) attracts pieces of iron. Legend ascribes the name *magnetite* to the shepherd Magnes, the nails of whose shoes and the tip of whose staff stuck fast to chunks of magnetite while he pastured his flocks.

In 1269 a Frenchman named Pierre de Maricourt mapped out the directions taken by a needle placed at various points on the surface of a spherical natural magnet. He found that the directions formed lines that encircled the sphere and passed through two points diametrically opposite each other, which he called the *poles* of the magnet. Subsequent experiments showed that every magnet, regardless of its shape, has two poles, called *north* and *south* poles, that exert forces on other magnetic poles just as electric charges exert forces on one another. That is, like poles repel each other, and unlike poles attract each other.

The poles received their names because of the way a magnet behaves in the presence of the Earth's magnetic field. If a bar magnet is suspended from its midpoint and can swing freely in a horizontal plane, it will rotate until its north pole points to the Earth's geographic North Pole and its south pole points to the Earth's geographic South Pole.[1] (The same idea is used in the construction of a simple compass.)

An electromagnet is used to move tons of scrap metal.

In 1600 William Gilbert (1540–1603) extended de Maricourt's experiments to a variety of materials. Using the fact that a compass needle orients in preferred directions, he suggested that the Earth itself is a large permanent magnet. In 1750 experimenters used a torsion balance to show that magnetic poles exert attractive or repulsive forces on each other and that these forces vary as the inverse square of the distance between interacting poles. Although the force between two magnetic poles is similar to the force between two electric charges, there is an important difference. Electric charges can be isolated (witness the electron and proton), whereas **a single magnetic pole has never been isolated.** That is, **magnetic poles are always found in pairs.** All attempts thus far to detect an isolated magnetic pole have been unsuccessful. No matter how many times a permanent magnet is cut in two, each piece always has a north and a south pole. (There is some theoretical basis for speculating that magnetic *monopoles*—isolated north or south poles—may exist in nature, and attempts to detect them currently make up an active experimental field of investigation.)

Hans Christian Oersted
Danish physicist (1777–1851)
(North Wind Picture Archives)

The relationship between magnetism and electricity was discovered in 1819 when, during a lecture demonstration, the Danish scientist Hans Christian Oersted found that an electric current in a wire deflected a nearby compass needle.[2] Shortly thereafter, André Ampère (1775–1836) formulated quantitative laws for calculating the magnetic force exerted by one current-carrying electrical conductor on another. He also suggested that on the atomic level, electric current loops are responsible for *all* magnetic phenomena.

In the 1820s, further connections between electricity and magnetism were demonstrated by Faraday and independently by Joseph Henry (1797–1878). They

[1] Note that the Earth's geographic North Pole is magnetically a south pole, whereas its geographic South Pole is magnetically a north pole. Because *opposite* magnetic poles attract each other, the pole on a magnet that is attracted to the Earth's geographic North Pole is the magnet's *north* pole and the pole attracted to the Earth's geographic South Pole is the magnet's *south* pole.

[2] The same discovery was reported in 1802 by an Italian jurist, Gian Dominico Romognosi, but was overlooked, probably because it was published in the newspaper *Gazetta de Trentino* rather than in a scholarly journal.

showed that an electric current can be produced in a circuit either by moving a magnet near the circuit or by changing the current in a nearby circuit. These observations demonstrate that a changing magnetic field creates an electric field. Years later, theoretical work by Maxwell showed that the reverse is also true: A changing electric field creates a magnetic field.

A similarity between electric and magnetic effects has provided methods of making permanent magnets. In Chapter 23 we learned that when rubber and wool are rubbed together, both become charged—one positively and the other negatively. In an analogous fashion, one can magnetize an unmagnetized piece of iron by stroking it with a magnet. Magnetism can also be induced in iron (and other materials) by other means. For example, if a piece of unmagnetized iron is placed near (but not touching) a strong magnet, the unmagnetized piece eventually becomes magnetized.

This chapter examines the forces that act on moving charges and on current-carrying wires in the presence of a magnetic field. The source of the magnetic field itself is described in Chapter 30.

QuickLab

If iron or steel is left in a weak magnetic field (such as that due to the Earth) long enough, it can become magnetized. Use a compass to see if you can detect a magnetic field near a steel file cabinet, cast iron radiator, or some other piece of ferrous metal that has been in one position for several years.

29.1 THE MAGNETIC FIELD

12.2

In our study of electricity, we described the interactions between charged objects in terms of electric fields. Recall that an electric field surrounds any stationary or moving electric charge. In addition to an electric field, the region of space surrounding any *moving* electric charge also contains a magnetic field, as we shall see in Chapter 30. A magnetic field also surrounds any magnetic substance.

These refrigerator magnets are similar to a series of very short bar magnets placed end to end. If you slide the back of one refrigerator magnet in a circular path across the back of another one, you can feel a vibration as the two series of north and south poles move across each other.

Historically, the symbol **B** has been used to represent a magnetic field, and this is the notation we use in this text. The direction of the magnetic field **B** at any location is the direction in which a compass needle points at that location. Figure 29.1 shows how the magnetic field of a bar magnet can be traced with the aid of a compass. Note that the magnetic field lines outside the magnet point away from north poles and toward south poles. One can display magnetic field patterns of a bar magnet using small iron filings, as shown in Figure 29.2.

We can define a magnetic field **B** at some point in space in terms of the magnetic force $\mathbf{F}_B$ that the field exerts on a test object, for which we use a charged particle moving with a velocity $\mathbf{v}$. For the time being, let us assume that no electric or gravitational fields are present at the location of the test object. Experiments on various charged particles moving in a magnetic field give the following results:

- The magnitude F_B of the magnetic force exerted on the particle is proportional to the charge q and to the speed v of the particle.

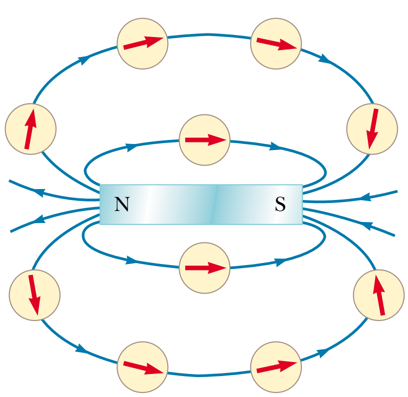


Figure 29.1 Compass needles can be used to trace the magnetic field lines of a bar magnet.

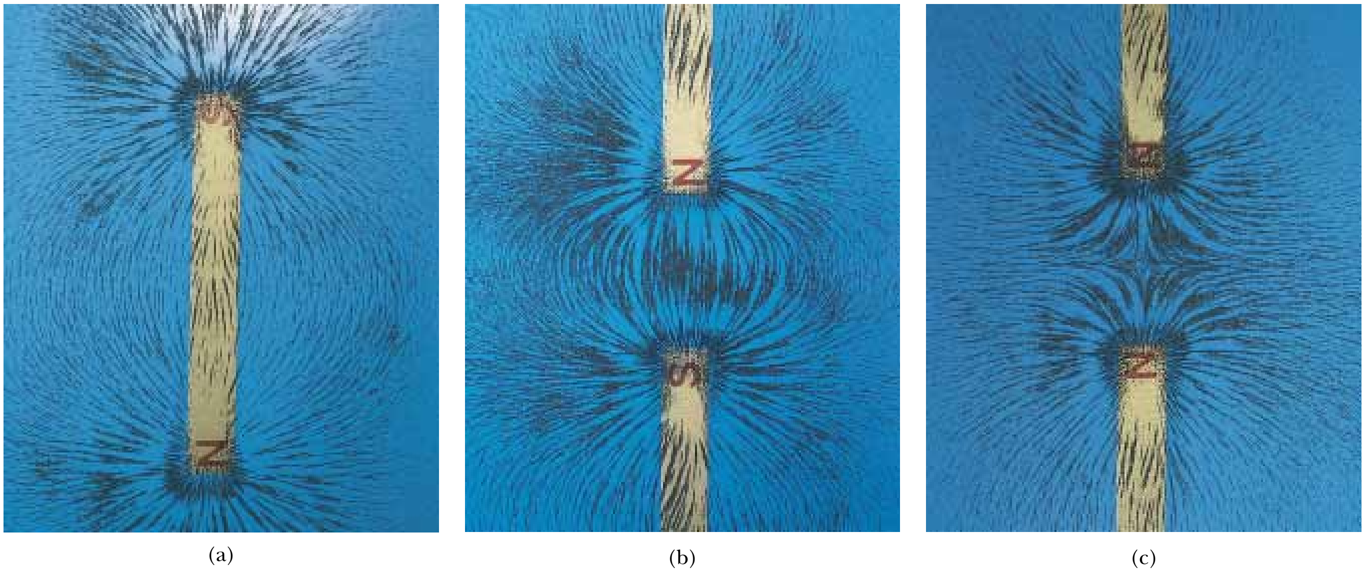


Figure 29.2 (a) Magnetic field pattern surrounding a bar magnet as displayed with iron filings. (b) Magnetic field pattern between *unlike* poles of two bar magnets. (c) Magnetic field pattern between *like* poles of two bar magnets.

- The magnitude and direction of $\mathbf{F}_B$ depend on the velocity of the particle and on the magnitude and direction of the magnetic field $\mathbf{B}$.
- When a charged particle moves parallel to the magnetic field vector, the magnetic force acting on the particle is zero.
- When the particle's velocity vector makes any angle $\theta \neq 0$ with the magnetic field, the magnetic force acts in a direction perpendicular to both $\mathbf{v}$ and $\mathbf{B}$; that is, $\mathbf{F}_B$ is perpendicular to the plane formed by $\mathbf{v}$ and $\mathbf{B}$ (Fig. 29.3a).

Properties of the magnetic force on a charge moving in a magnetic field **B**

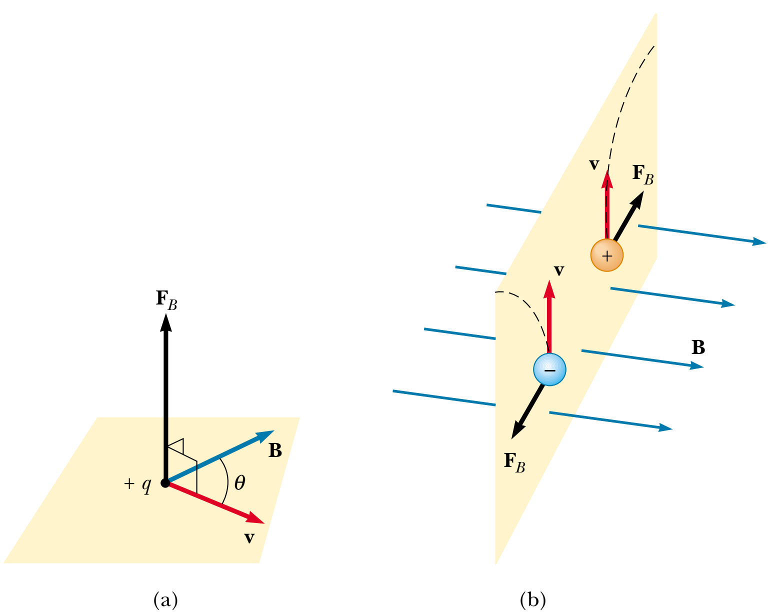


Figure 29.3 The direction of the magnetic force $\mathbf{F}_B$ acting on a charged particle moving with a velocity $\mathbf{v}$ in the presence of a magnetic field $\mathbf{B}$. (a) The magnetic force is perpendicular to both $\mathbf{v}$ and $\mathbf{B}$. (b) Oppositely directed magnetic forces $\mathbf{F}_B$ are exerted on two oppositely charged particles moving at the same velocity in a magnetic field.

The blue-white arc in this photograph indicates the circular path followed by an electron beam moving in a magnetic field. The vessel contains gas at very low pressure, and the beam is made visible as the electrons collide with the gas atoms, which then emit visible light. The magnetic field is produced by two coils (not shown). The apparatus can be used to measure the ratio e/m_e for the electron.

- The magnetic force exerted on a positive charge is in the direction opposite the direction of the magnetic force exerted on a negative charge moving in the same direction (Fig. 29.3b).
- The magnitude of the magnetic force exerted on the moving particle is proportional to $\sin\theta$, where θ is the angle the particle's velocity vector makes with the direction of $\mathbf{B}$.

We can summarize these observations by writing the magnetic force in the form

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B} \tag{29.1}$$

where the direction of $\mathbf{F}_B$ is in the direction of $\mathbf{v} \times \mathbf{B}$ if q is positive, which by definition of the cross product (see Section 11.2) is perpendicular to both $\mathbf{v}$ and $\mathbf{B}$. We can regard this equation as an operational definition of the magnetic field at some point in space. That is, the magnetic field is defined in terms of the force acting on a moving charged particle.

Figure 29.4 reviews the right-hand rule for determining the direction of the cross product $\mathbf{v} \times \mathbf{B}$. You point the four fingers of your right hand along the direction of $\mathbf{v}$ with the palm facing $\mathbf{B}$ and curl them toward $\mathbf{B}$. The extended thumb, which is at a right angle to the fingers, points in the direction of $\mathbf{v} \times \mathbf{B}$. Because

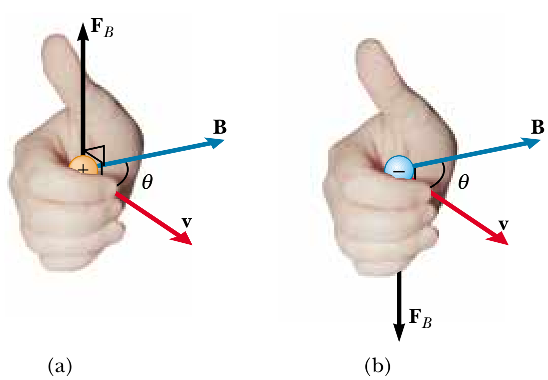


Figure 29.4 The right-hand rule for determining the direction of the magnetic force $\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$ acting on a particle with charge q moving with a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$. The direction of $\mathbf{v} \times \mathbf{B}$ is the direction in which the thumb points. (a) If q is positive, $\mathbf{F}_B$ is upward. (b) If q is negative, $\mathbf{F}_B$ is downward, antiparallel to the direction in which the thumb points.

$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$, $\mathbf{F}_B$ is in the direction of $\mathbf{v} \times \mathbf{B}$ if q is positive (Fig. 29.4a) and opposite the direction of $\mathbf{v} \times \mathbf{B}$ if q is negative (Fig. 29.4b). (If you need more help understanding the cross product, you should review pages 333 to 334, including Fig. 11.8.)

The magnitude of the magnetic force is

Magnitude of the magnetic force on a charged particle moving in a magnetic field

$$F_B = |q|vB \sin \theta \tag{29.2}$$

where θ is the smaller angle between $\mathbf{v}$ and $\mathbf{B}$. From this expression, we see that F is zero when $\mathbf{v}$ is parallel or antiparallel to $\mathbf{B}$ ($\theta = 0$ or $180°$) and maximum ($F_{B,\,\max} = |q|vB$) when $\mathbf{v}$ is perpendicular to $\mathbf{B}$ ($\theta = 90°$).

Quick Quiz 29.1

What is the maximum work that a constant magnetic field $\mathbf{B}$ can perform on a charge q moving through the field with velocity $\mathbf{v}$?

There are several important differences between electric and magnetic forces:

Differences between electric and magnetic forces

- The electric force acts in the direction of the electric field, whereas the magnetic force acts perpendicular to the magnetic field.
- The electric force acts on a charged particle regardless of whether the particle is moving, whereas the magnetic force acts on a charged particle only when the particle is in motion.
- The electric force does work in displacing a charged particle, whereas the magnetic force associated with a steady magnetic field does no work when a particle is displaced.

From the last statement and on the basis of the work–kinetic energy theorem, we conclude that the kinetic energy of a charged particle moving through a magnetic field cannot be altered by the magnetic field alone. In other words,

when a charged particle moves with a velocity $\mathbf{v}$ through a magnetic field, the field can alter the direction of the velocity vector but cannot change the speed or kinetic energy of the particle.

A magnetic field cannot change the speed of a particle

From Equation 29.2, we see that the SI unit of magnetic field is the newton per coulomb-meter per second, which is called the **tesla** (T):

$$1\ \text{T} = \frac{\text{N}}{\text{C} \cdot \text{m/s}}$$

Because a coulomb per second is defined to be an ampere, we see that

$$1\ \text{T} = 1\ \frac{\text{N}}{\text{A} \cdot \text{m}}$$

A non-SI magnetic-field unit in common use, called the *gauss* (G), is related to the tesla through the conversion $1\ \text{T} = 10^4\ \text{G}$. Table 29.1 shows some typical values of magnetic fields.

Quick Quiz 29.2

The north-pole end of a bar magnet is held near a positively charged piece of plastic. Is the plastic attracted, repelled, or unaffected by the magnet?

TABLE 29.1 Some Approximate Magnetic Field Magnitudes

Source of Field	Field Magnitude (T)
Strong superconducting laboratory magnet	30
Strong conventional laboratory magnet	2
Medical MRI unit	1.5
Bar magnet	10^{-2}
Surface of the Sun	10^{-2}
Surface of the Earth	0.5×10^{-4}
Inside human brain (due to nerve impulses)	10^{-13}

EXAMPLE 29.1 An Electron Moving in a Magnetic Field

An electron in a television picture tube moves toward the front of the tube with a speed of 8.0×10^6 m/s along the x axis (Fig. 29.5). Surrounding the neck of the tube are coils of wire that create a magnetic field of magnitude 0.025 T, directed at an angle of 60° to the x axis and lying in the xy plane. Calculate the magnetic force on and acceleration of the electron.

Solution Using Equation 29.2, we can find the magnitude of the magnetic force:

$$F_B = |q|vB\sin\theta$$
$$= (1.6 \times 10^{-19}\text{ C})(8.0 \times 10^6\text{ m/s})(0.025\text{ T})(\sin 60°)$$
$$= \boxed{2.8 \times 10^{-14}\text{ N}}$$

Because $\mathbf{v} \times \mathbf{B}$ is in the positive z direction (from the right-hand rule) and the charge is negative, $\mathbf{F}_B$ is in the negative z direction.

The mass of the electron is 9.11×10^{-31} kg, and so its acceleration is

$$a = \frac{F_B}{m_e} = \frac{2.8 \times 10^{-14}\text{ N}}{9.11 \times 10^{-31}\text{ kg}} = \boxed{3.1 \times 10^{16}\text{ m/s}^2}$$

in the negative z direction.

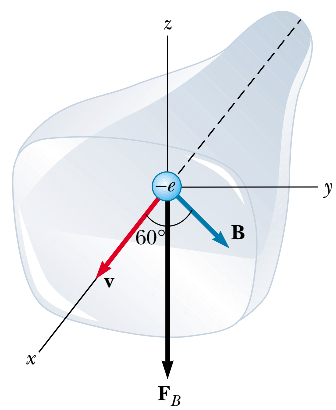


Figure 29.5 The magnetic force $\mathbf{F}_B$ acting on the electron is in the negative z direction when $\mathbf{v}$ and $\mathbf{B}$ lie in the xy plane.

29.2 MAGNETIC FORCE ACTING ON A CURRENT-CARRYING CONDUCTOR

12.3 If a magnetic force is exerted on a single charged particle when the particle moves through a magnetic field, it should not surprise you that a current-carrying wire also experiences a force when placed in a magnetic field. This follows from the fact that the current is a collection of many charged particles in motion; hence, the resultant force exerted by the field on the wire is the vector sum of the individual forces exerted on all the charged particles making up the current. The force exerted on the particles is transmitted to the wire when the particles collide with the atoms making up the wire.

Before we continue our discussion, some explanation of the notation used in this book is in order. To indicate the direction of $\mathbf{B}$ in illustrations, we sometimes present perspective views, such as those in Figures 29.5, 29.6a, and 29.7. In flat il-

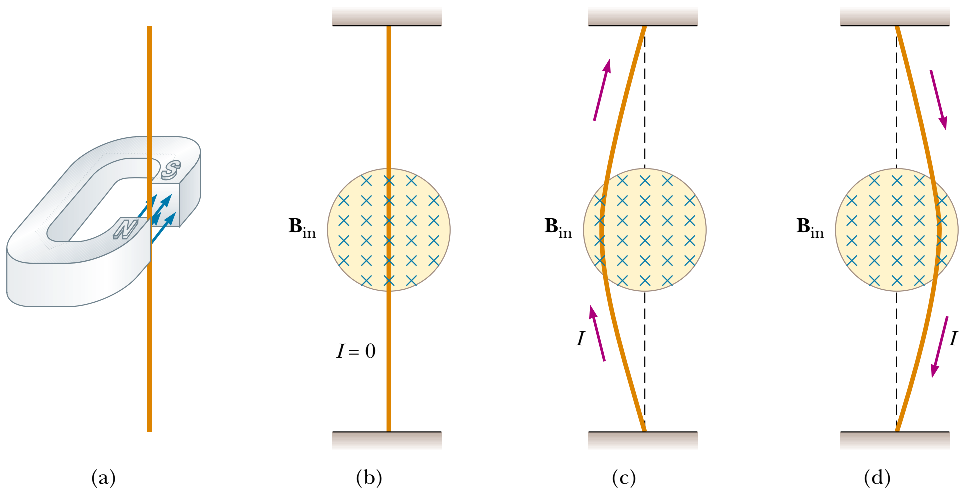


Figure 29.6 (a) A wire suspended vertically between the poles of a magnet. (b) The setup shown in part (a) as seen looking at the south pole of the magnet, so that the magnetic field (blue crosses) is directed into the page. When there is no current in the wire, it remains vertical. (c) When the current is upward, the wire deflects to the left. (d) When the current is downward, the wire deflects to the right.

lustrations, such as in Figure 29.6b to d, we depict a magnetic field directed into the page with blue crosses, which represent the tails of arrows shot perpendicularly and away from you. In this case, we call the field $\mathbf{B}_{in}$, where the subscript "in" indicates "into the page." If $\mathbf{B}$ is perpendicular and directed out of the page, we use a series of blue dots, which represent the tips of arrows coming toward you (see Fig. P29.56). In this case, we call the field $\mathbf{B}_{out}$. If $\mathbf{B}$ lies in the plane of the page, we use a series of blue field lines with arrowheads, as shown in Figure 29.8.

One can demonstrate the magnetic force acting on a current-carrying conductor by hanging a wire between the poles of a magnet, as shown in Figure 29.6a. For ease in visualization, part of the horseshoe magnet in part (a) is removed to show the end face of the south pole in parts (b), (c), and (d) of Figure 29.6. The magnetic field is directed into the page and covers the region within the shaded circles. When the current in the wire is zero, the wire remains vertical, as shown in Figure 29.6b. However, when a current directed upward flows in the wire, as shown in Figure 29.6c, the wire deflects to the left. If we reverse the current, as shown in Figure 29.6d, the wire deflects to the right.

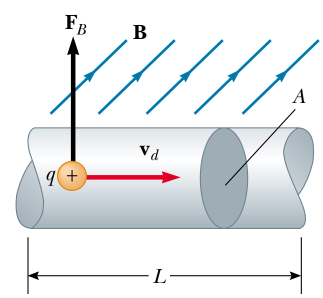


Figure 29.7 A segment of a current-carrying wire located in a magnetic field $\mathbf{B}$. The magnetic force exerted on each charge making up the current is $q\mathbf{v}_d \times \mathbf{B}$, and the net force on the segment of length L is $I\mathbf{L} \times \mathbf{B}$.

Let us quantify this discussion by considering a straight segment of wire of length L and cross-sectional area A, carrying a current I in a uniform magnetic field $\mathbf{B}$, as shown in Figure 29.7. The magnetic force exerted on a charge q moving with a drift velocity $\mathbf{v}_d$ is $q\mathbf{v}_d \times \mathbf{B}$. To find the total force acting on the wire, we multiply the force $q\mathbf{v}_d \times \mathbf{B}$ exerted on one charge by the number of charges in the segment. Because the volume of the segment is AL, the number of charges in the segment is nAL, where n is the number of charges per unit volume. Hence, the total magnetic force on the wire of length L is

$$\mathbf{F}_B = (q\mathbf{v}_d \times \mathbf{B})nAL$$

We can write this expression in a more convenient form by noting that, from Equation 27.4, the current in the wire is $I = nqv_dA$. Therefore,

$$\mathbf{F}_B = I\mathbf{L} \times \mathbf{B} \tag{29.3}$$

Force on a segment of a wire in a uniform magnetic field

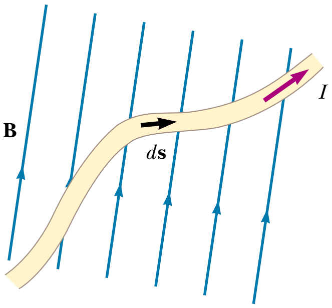


Figure 29.8 A wire segment of arbitrary shape carrying a current I in a magnetic field $\mathbf{B}$ experiences a magnetic force. The force on any segment $d\mathbf{s}$ is $I\,d\mathbf{s} \times \mathbf{B}$ and is directed out of the page. You should use the right-hand rule to confirm this force direction.

where $\mathbf{L}$ is a vector that points in the direction of the current I and has a magnitude equal to the length L of the segment. Note that this expression applies only to a straight segment of wire in a uniform magnetic field.

Now let us consider an arbitrarily shaped wire segment of uniform cross-section in a magnetic field, as shown in Figure 29.8. It follows from Equation 29.3 that the magnetic force exerted on a small segment of vector length $d\mathbf{s}$ in the presence of a field $\mathbf{B}$ is

$$d\mathbf{F}_B = I\,d\mathbf{s} \times \mathbf{B} \tag{29.4}$$

where $d\mathbf{F}_B$ is directed out of the page for the directions assumed in Figure 29.8. We can consider Equation 29.4 as an alternative definition of $\mathbf{B}$. That is, we can define the magnetic field $\mathbf{B}$ in terms of a measurable force exerted on a current element, where the force is a maximum when $\mathbf{B}$ is perpendicular to the element and zero when $\mathbf{B}$ is parallel to the element.

To calculate the total force $\mathbf{F}_B$ acting on the wire shown in Figure 29.8, we integrate Equation 29.4 over the length of the wire:

$$\mathbf{F}_B = I\int_a^b d\mathbf{s} \times \mathbf{B} \tag{29.5}$$

where a and b represent the end points of the wire. When this integration is carried out, the magnitude of the magnetic field and the direction the field makes with the vector $d\mathbf{s}$ (in other words, with the orientation of the element) may differ at different points.

Now let us consider two special cases involving Equation 29.5. In both cases, the magnetic field is taken to be constant in magnitude and direction.

Case 1 A curved wire carries a current I and is located in a uniform magnetic field $\mathbf{B}$, as shown in Figure 29.9a. Because the field is uniform, we can take $\mathbf{B}$ outside the integral in Equation 29.5, and we obtain

$$\mathbf{F}_B = I\left(\int_a^b d\mathbf{s}\right) \times \mathbf{B} \tag{29.6}$$

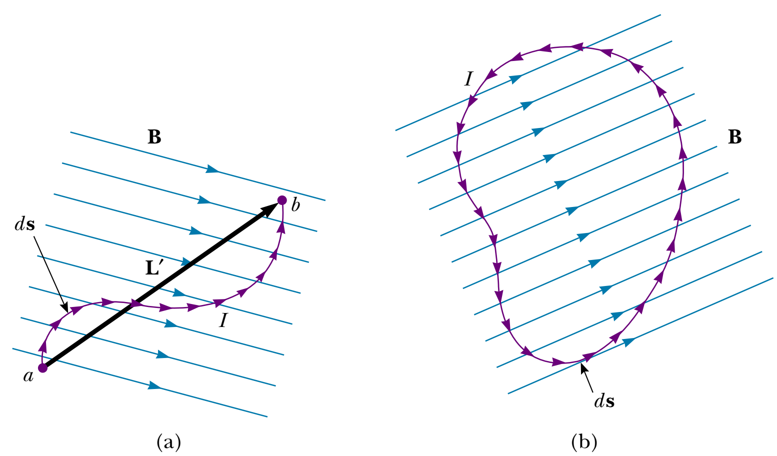


Figure 29.9 (a) A curved wire carrying a current I in a uniform magnetic field. The total magnetic force acting on the wire is equivalent to the force on a straight wire of length L' running between the ends of the curved wire. (b) A current-carrying loop of arbitrary shape in a uniform magnetic field. The net magnetic force on the loop is zero.

But the quantity $\int_a^b d\mathbf{s}$ represents the *vector sum* of all the length elements from a to b. From the law of vector addition, the sum equals the vector $\mathbf{L}'$, directed from a to b. Therefore, Equation 29.6 reduces to

$$\mathbf{F}_B = I\mathbf{L}' \times \mathbf{B} \qquad \textbf{(29.7)}$$

Case 2 An arbitrarily shaped closed loop carrying a current I is placed in a uniform magnetic field, as shown in Figure 29.9b. We can again express the force acting on the loop in the form of Equation 29.6, but this time we must take the vector sum of the length elements $d\mathbf{s}$ over the entire loop:

$$\mathbf{F}_B = I\left(\oint d\mathbf{s}\right) \times \mathbf{B}$$

Because the set of length elements forms a closed polygon, the vector sum must be zero. This follows from the graphical procedure for adding vectors by the polygon method. Because $\oint d\mathbf{s} = 0$, we conclude that $\mathbf{F}_B = 0$:

The net magnetic force acting on any closed current loop in a uniform magnetic field is zero.

EXAMPLE 29.2 Force on a Semicircular Conductor

A wire bent into a semicircle of radius R forms a closed circuit and carries a current I. The wire lies in the xy plane, and a uniform magnetic field is directed along the positive y axis, as shown in Figure 29.10. Find the magnitude and direction of the magnetic force acting on the straight portion of the wire and on the curved portion.

Solution The force $\mathbf{F}_1$ acting on the straight portion has a magnitude $F_1 = ILB = 2IRB$ because $L = 2R$ and the wire is oriented perpendicular to $\mathbf{B}$. The direction of $\mathbf{F}_1$ is out of the page because $\mathbf{L} \times \mathbf{B}$ is along the positive z axis. (That is, $\mathbf{L}$ is to the right, in the direction of the current; thus, according to the rule of cross products, $\mathbf{L} \times \mathbf{B}$ is out of the page in Fig. 29.10.)

To find the force $\mathbf{F}_2$ acting on the curved part, we first write an expression for the force $d\mathbf{F}_2$ on the length element $d\mathbf{s}$ shown in Figure 29.10. If θ is the angle between $\mathbf{B}$ and $d\mathbf{s}$, then the magnitude of $d\mathbf{F}_2$ is

$$dF_2 = I\,|d\mathbf{s} \times \mathbf{B}| = IB \sin\theta\, ds$$

To integrate this expression, we must express ds in terms of θ. Because $s = R\theta$, we have $ds = R\,d\theta$, and we can make this substitution for dF_2:

$$dF_2 = IRB \sin\theta\, d\theta$$

To obtain the total force F_2 acting on the curved portion, we can integrate this expression to account for contributions from all elements $d\mathbf{s}$. Note that the direction of the force on every element is the same: into the page (because $d\mathbf{s} \times \mathbf{B}$ is into the page). Therefore, the resultant force $\mathbf{F}_2$ on the curved wire must also be into the page. Integrating our expression for dF_2 over the limits $\theta = 0$ to $\theta = \pi$ (that is, the entire semicircle) gives

$$F_2 = IRB\int_0^{\pi} \sin\theta\, d\theta = IRB\Big[-\cos\theta\Big]_0^{\pi}$$

$$= -IRB(\cos\pi - \cos 0) = -IRB(-1 - 1) = 2IRB$$

Because $\mathbf{F}_2$, with a magnitude of $2IRB$, is directed into the page and because $\mathbf{F}_1$, with a magnitude of $2IRB$, is directed out of the page, the net force on the closed loop is zero. This result is consistent with Case 2 described earlier.

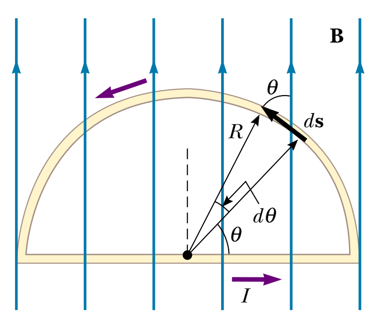


Figure 29.10 The net force acting on a closed current loop in a uniform magnetic field is zero. In the setup shown here, the force on the straight portion of the loop is $2IRB$ and directed out of the page, and the force on the curved portion is $2IRB$ directed into the page.

Quick Quiz 29.3

The four wires shown in Figure 29.11 all carry the same current from point A to point B through the same magnetic field. Rank the wires according to the magnitude of the magnetic force exerted on them, from greatest to least.

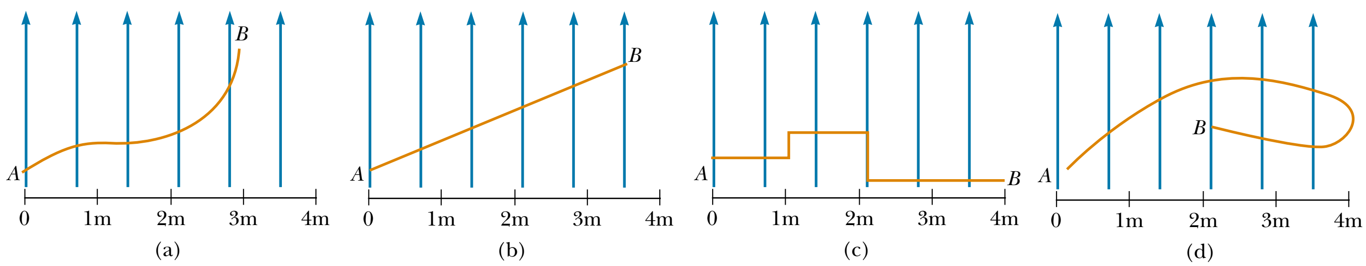


Figure 29.11 Which wire experiences the greatest magnetic force?

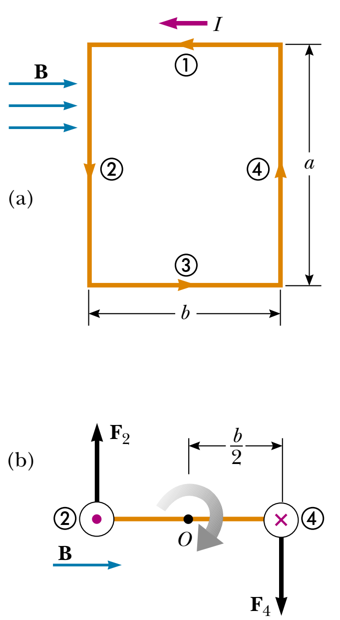


Figure 29.12 (a) Overhead view of a rectangular current loop in a uniform magnetic field. No forces are acting on sides ① and ③ because these sides are parallel to **B**. Forces are acting on sides ② and ④, however. (b) Edge view of the loop sighting down sides ② and ④ shows that the forces $\mathbf{F}_2$ and $\mathbf{F}_4$ exerted on these sides create a torque that tends to twist the loop clockwise. The purple dot in the left circle represents current in wire ② coming toward you; the purple cross in the right circle represents current in wire ④ moving away from you.

29.3 TORQUE ON A CURRENT LOOP IN A UNIFORM MAGNETIC FIELD

In the previous section, we showed how a force is exerted on a current-carrying conductor placed in a magnetic field. With this as a starting point, we now show that a torque is exerted on any current loop placed in a magnetic field. The results of this analysis will be of great value when we discuss motors in Chapter 31.

Consider a rectangular loop carrying a current I in the presence of a uniform magnetic field directed parallel to the plane of the loop, as shown in Figure 29.12a. No magnetic forces act on sides ① and ③ because these wires are parallel to the field; hence, $\mathbf{L} \times \mathbf{B} = 0$ for these sides. However, magnetic forces do act on sides ② and ④ because these sides are oriented perpendicular to the field. The magnitude of these forces is, from Equation 29.3,

$$F_2 = F_4 = IaB$$

The direction of $\mathbf{F}_2$, the force exerted on wire ② is out of the page in the view shown in Figure 29.12a, and that of $\mathbf{F}_4$, the force exerted on wire ④, is into the page in the same view. If we view the loop from side ③ and sight along sides ② and ④, we see the view shown in Figure 29.12b, and the two forces $\mathbf{F}_2$ and $\mathbf{F}_4$ are directed as shown. Note that the two forces point in opposite directions but are *not* directed along the same line of action. If the loop is pivoted so that it can rotate about point O, these two forces produce about O a torque that rotates the loop clockwise. The magnitude of this torque τ_{max} is

$$\tau_{\text{max}} = F_2 \frac{b}{2} + F_4 \frac{b}{2} = (IaB)\frac{b}{2} + (IaB)\frac{b}{2} = IabB$$

where the moment arm about O is $b/2$ for each force. Because the area enclosed by the loop is $A = ab$, we can express the maximum torque as

$$\tau_{\text{max}} = IAB \qquad \textbf{(29.8)}$$

Remember that this maximum-torque result is valid only when the magnetic field is parallel to the plane of the loop. The sense of the rotation is clockwise when viewed from side ③, as indicated in Figure 29.12b. If the current direction were re-

versed, the force directions would also reverse, and the rotational tendency would be counterclockwise.

Now let us suppose that the uniform magnetic field makes an angle $\theta < 90°$ with a line perpendicular to the plane of the loop, as shown in Figure 29.13a. For convenience, we assume that $\mathbf{B}$ is perpendicular to sides ① and ③. In this case, the magnetic forces $\mathbf{F}_2$ and $\mathbf{F}_4$ exerted on sides ② and ④ cancel each other and produce no torque because they pass through a common origin. However, the forces acting on sides ① and ③, $\mathbf{F}_1$ and $\mathbf{F}_3$, form a couple and hence produce a torque about *any point*. Referring to the end view shown in Figure 29.13b, we note that the moment arm of $\mathbf{F}_1$ about the point O is equal to $(a/2)\sin\theta$. Likewise, the moment arm of $\mathbf{F}_3$ about O is also $(a/2)\sin\theta$. Because $F_1 = F_3 = IbB$, the net torque about O has the magnitude

$$\begin{aligned}\tau &= F_1\frac{a}{2}\sin\theta + F_3\frac{a}{2}\sin\theta \\ &= IbB\left(\frac{a}{2}\sin\theta\right) + IbB\left(\frac{a}{2}\sin\theta\right) = IabB\sin\theta \\ &= IAB\sin\theta\end{aligned}$$

where $A = ab$ is the area of the loop. This result shows that the torque has its maximum value IAB when the field is perpendicular to the normal to the plane of the loop $(\theta = 90°)$, as we saw when discussing Figure 29.12, and that it is zero when the field is parallel to the normal to the plane of the loop $(\theta = 0)$. As we see in Figure 29.13, the loop tends to rotate in the direction of decreasing values of θ (that is, such that the area vector $\mathbf{A}$ rotates toward the direction of the magnetic field).

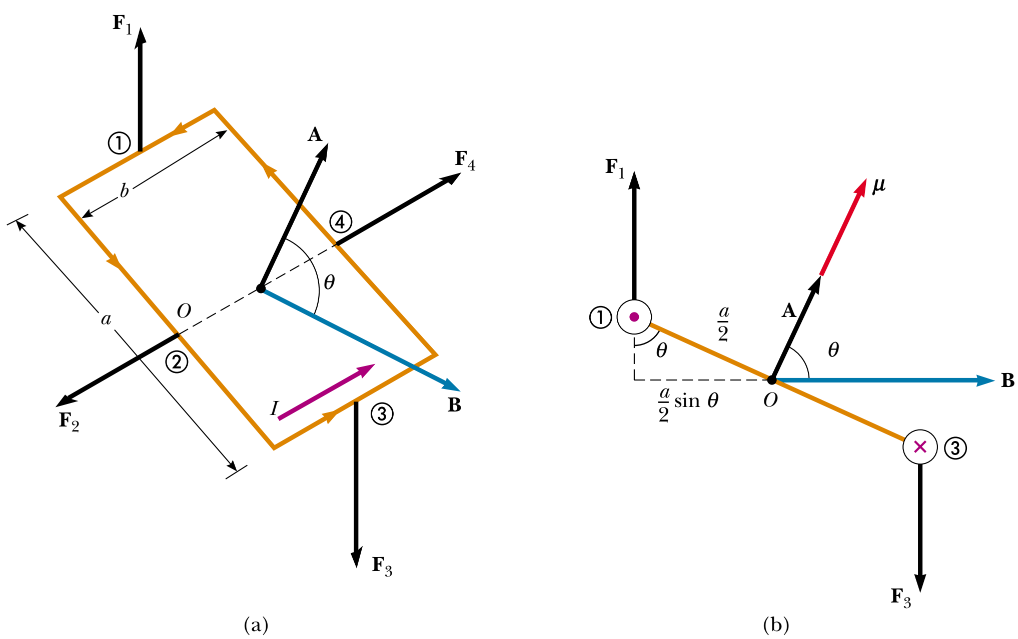


Figure 29.13 (a) A rectangular current loop in a uniform magnetic field. The area vector $\mathbf{A}$ perpendicular to the plane of the loop makes an angle θ with the field. The magnetic forces exerted on sides ② and ④ cancel, but the forces exerted on sides ① and ③ create a torque on the loop. (b) Edge view of the loop sighting down sides ① and ③.

Quick Quiz 29.4

Describe the forces on the rectangular current loop shown in Figure 29.13 if the magnetic field is directed as shown but increases in magnitude going from left to right.

A convenient expression for the torque exerted on a loop placed in a uniform magnetic field **B** is

Torque on a current loop

$$\boldsymbol{\tau} = I\mathbf{A} \times \mathbf{B} \tag{29.9}$$

where **A**, the vector shown in Figure 29.13, is perpendicular to the plane of the loop and has a magnitude equal to the area of the loop. We determine the direction of **A** using the right-hand rule described in Figure 29.14. When you curl the fingers of your right hand in the direction of the current in the loop, your thumb points in the direction of **A**. The product $I\mathbf{A}$ is defined to be the **magnetic dipole moment** $\boldsymbol{\mu}$ (often simply called the "magnetic moment") of the loop:

Magnetic dipole moment of a current loop

$$\boldsymbol{\mu} = I\mathbf{A} \tag{29.10}$$

The SI unit of magnetic dipole moment is ampere–meter^2 ($\mathrm{A \cdot m^2}$). Using this definition, we can express the torque exerted on a current-carrying loop in a magnetic field **B** as

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B} \tag{29.11}$$

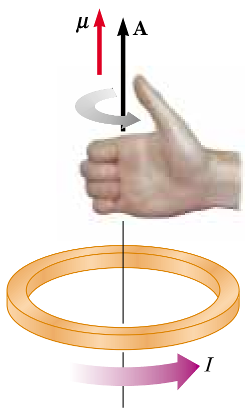


Figure 29.14 Right-hand rule for determining the direction of the vector **A**. The direction of the magnetic moment $\boldsymbol{\mu}$ is the same as the direction of **A**.

Note that this result is analogous to Equation 26.18, $\boldsymbol{\tau} = \mathbf{p} \times \mathbf{E}$, for the torque exerted on an electric dipole in the presence of an electric field **E**, where **p** is the electric dipole moment.

Although we obtained the torque for a particular orientation of **B** with respect to the loop, the equation $\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B}$ is valid for any orientation. Furthermore, although we derived the torque expression for a rectangular loop, the result is valid for a loop of any shape.

If a coil consists of N turns of wire, each carrying the same current and enclosing the same area, the total magnetic dipole moment of the coil is N times the magnetic dipole moment for one turn. The torque on an N-turn coil is N times that on a one-turn coil. Thus, we write $\boldsymbol{\tau} = N\boldsymbol{\mu}_{\text{loop}} \times \mathbf{B} = \boldsymbol{\mu}_{\text{coil}} \times \mathbf{B}$.

In Section 26.6, we found that the potential energy of an electric dipole in an electric field is given by $U = -\mathbf{p} \cdot \mathbf{E}$. This energy depends on the orientation of the dipole in the electric field. Likewise, the potential energy of a magnetic dipole in a magnetic field depends on the orientation of the dipole in the magnetic field and is given by

$$U = -\boldsymbol{\mu} \cdot \mathbf{B} \tag{29.12}$$

From this expression, we see that a magnetic dipole has its lowest energy $U_{\text{min}} = -\mu B$ when $\boldsymbol{\mu}$ points in the same direction as **B**. The dipole has its highest energy $U_{\text{max}} = +\mu B$ when $\boldsymbol{\mu}$ points in the direction opposite **B**.

Quick Quiz 29.5

Rank the magnitude of the torques acting on the rectangular loops shown in Figure 29.15, from highest to lowest. All loops are identical and carry the same current.

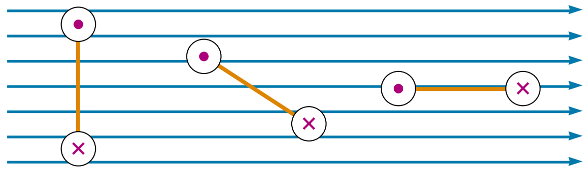


Figure 29.15 Which current loop (seen edge-on) experiences the greatest torque?

EXAMPLE 29.3 The Magnetic Dipole Moment of a Coil

A rectangular coil of dimensions 5.40 cm × 8.50 cm consists of 25 turns of wire and carries a current of 15.0 mA. A 0.350-T magnetic field is applied parallel to the plane of the loop. (a) Calculate the magnitude of its magnetic dipole moment.

Solution Because the coil has 25 turns, we modify Equation 29.10 to obtain

$$\mu_{\text{coil}} = NIA = (25)(15.0 \times 10^{-3}\text{ A})(0.054\,0\text{ m})(0.085\,0\text{ m})$$
$$= \boxed{1.72 \times 10^{-3}\text{ A}\cdot\text{m}^2}$$

(b) What is the magnitude of the torque acting on the loop?

Solution Because **B** is perpendicular to $\boldsymbol{\mu}_{\text{coil}}$, Equation 29.11 gives

$$\tau = \mu_{\text{coil}} B = (1.72 \times 10^{-3}\text{ A}\cdot\text{m}^2)(0.350\text{ T})$$
$$= \boxed{6.02 \times 10^{-4}\text{ N}\cdot\text{m}}$$

Exercise Show that the units $\text{A}\cdot\text{m}^2\cdot\text{T}$ reduce to the torque units $\text{N}\cdot\text{m}$.

Exercise Calculate the magnitude of the torque on the coil when the field makes an angle of (a) 60° and (b) 0° with $\boldsymbol{\mu}$.

Answer (a) $5.21 \times 10^{-4}\text{ N}\cdot\text{m}$; (b) zero.

web
For more information on torquers, visit the Web site of a company that supplies these devices to NASA:
http://www.smad.com

EXAMPLE 29.4 Satellite Attitude Control

Many satellites use coils called *torquers* to adjust their orientation. These devices interact with the Earth's magnetic field to create a torque on the spacecraft in the *x*, *y*, or *z* direction. The major advantage of this type of attitude-control system is that it uses solar-generated electricity and so does not consume any thruster fuel.

If a typical device has a magnetic dipole moment of $250\text{ A}\cdot\text{m}^2$, what is the maximum torque applied to a satellite when its torquer is turned on at an altitude where the magnitude of the Earth's magnetic field is 3.0×10^{-5} T?

Solution We once again apply Equation 29.11, recognizing that the maximum torque is obtained when the magnetic dipole moment of the torquer is perpendicular to the Earth's magnetic field:

$$\tau_{\max} = \mu B = (250\text{ A}\cdot\text{m}^2)(3.0 \times 10^{-5}\text{ T})$$
$$= \boxed{7.5 \times 10^{-3}\text{ N}\cdot\text{m}}$$

Exercise If the torquer requires 1.3 W of power at a potential difference of 28 V, how much current does it draw when it operates?

Answer 46 mA.

EXAMPLE 29.5 The D'Arsonval Galvanometer

An end view of a D'Arsonval galvanometer (see Section 28.5) is shown in Figure 29.16. When the turns of wire making up the coil carry a current, the magnetic field created by the magnet exerts on the coil a torque that turns it (along with its attached pointer) against the spring. Let us show that the angle of deflection of the pointer is directly proportional to the current in the coil.

Solution We can use Equation 29.11 to find the torque τ_m the magnetic field exerts on the coil. If we assume that the magnetic field through the coil is perpendicular to the normal to the plane of the coil, Equation 29.11 becomes

$$\tau_m = \mu B$$

(This is a reasonable assumption because the circular cross section of the magnet ensures radial magnetic field lines.) This magnetic torque is opposed by the torque due to the spring, which is given by the rotational version of Hooke's law, $\tau_s = -\kappa\varphi$, where κ is the torsional spring constant and φ is the angle through which the spring turns. Because the coil does not have an angular acceleration when the pointer is at rest, the sum of these torques must be zero:

$$(1) \qquad \tau_m + \tau_s = \mu B - \kappa\varphi = 0$$

Equation 29.10 allows us to relate the magnetic moment of the N turns of wire to the current through them:

$$\mu = NIA$$

We can substitute this expression for μ in Equation (1) to obtain

$$(NIA)B - \kappa\varphi = 0$$

$$\varphi = \boxed{\frac{NAB}{\kappa} I}$$

Thus, the angle of deflection of the pointer is directly proportional to the current in the loop. The factor NAB/κ tells us that deflection also depends on the design of the meter.

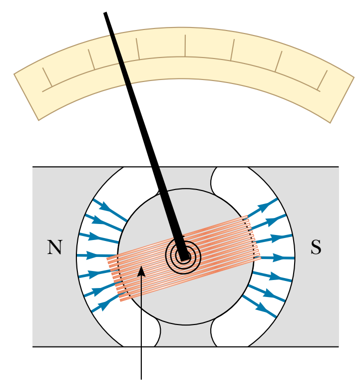


Figure 29.16 End view of a moving-coil galvanometer.

29.4 MOTION OF A CHARGED PARTICLE IN A UNIFORM MAGNETIC FIELD

12.3

In Section 29.1 we found that the magnetic force acting on a charged particle moving in a magnetic field is perpendicular to the velocity of the particle and that consequently the work done on the particle by the magnetic force is zero. Let us now consider the special case of a positively charged particle moving in a uniform magnetic field with the initial velocity vector of the particle perpendicular to the field. Let us assume that the direction of the magnetic field is into the page. Figure 29.17 shows that the particle moves in a circle in a plane perpendicular to the magnetic field.

The particle moves in this way because the magnetic force $\mathbf{F}_B$ is at right angles to $\mathbf{v}$ and $\mathbf{B}$ and has a constant magnitude qvB. As the force deflects the particle, the directions of $\mathbf{v}$ and $\mathbf{F}_B$ change continuously, as Figure 29.17 shows. Because $\mathbf{F}_B$ always points toward the center of the circle, **it changes only the direction of v and not its magnitude.** As Figure 29.17 illustrates, the rotation is counterclockwise for a positive charge. If q were negative, the rotation would be clockwise. We can use Equation 6.1 to equate this magnetic force to the radial force required to

QuickLab

Move a bar magnet across the screen of a black-and-white television and watch what happens to the picture. The electrons are deflected by the magnetic field as they approach the screen, causing distortion. (WARNING: Do not attempt to do this with a *color* television or computer monitor. These devices typically contain a metallic plate that can become magnetized by the bar magnet. If this happens, a repair shop will need to "degauss" the screen.)

keep the charge moving in a circle:

$$\sum F = ma_r$$

$$F_B = qvB = \frac{mv^2}{r}$$

$$r = \frac{mv}{qB} \tag{29.13}$$

That is, the radius of the path is proportional to the linear momentum mv of the particle and inversely proportional to the magnitude of the charge on the particle and to the magnitude of the magnetic field. The angular speed of the particle (from Eq. 10.10) is

$$\omega = \frac{v}{r} = \frac{qB}{m} \tag{29.14}$$

The period of the motion (the time that the particle takes to complete one revolution) is equal to the circumference of the circle divided by the linear speed of the particle:

$$T = \frac{2\pi r}{v} = \frac{2\pi}{\omega} = \frac{2\pi m}{qB} \tag{29.15}$$

These results show that the angular speed of the particle and the period of the circular motion do not depend on the linear speed of the particle or on the radius of the orbit. The angular speed ω is often referred to as the **cyclotron frequency** because charged particles circulate at this angular speed in the type of accelerator called a *cyclotron,* which is discussed in Section 29.5.

If a charged particle moves in a uniform magnetic field with its velocity at some arbitrary angle with respect to **B**, its path is a helix. For example, if the field is directed in the x direction, as shown in Figure 29.18, there is no component of force in the x direction. As a result, $a_x = 0$, and the x component of velocity remains constant. However, the magnetic force $q\mathbf{v} \times \mathbf{B}$ causes the components v_y and v_z to change in time, and the resulting motion is a helix whose axis is parallel to the magnetic field. The projection of the path onto the yz plane (viewed along the x axis) is a circle. (The projections of the path onto the xy and xz planes are sinusoids!) Equations 29.13 to 29.15 still apply provided that v is replaced by $v_\perp = \sqrt{v_y^2 + v_z^2}$.

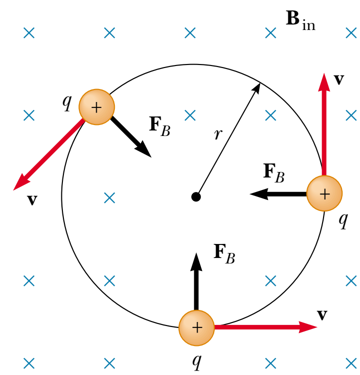


Figure 29.17 When the velocity of a charged particle is perpendicular to a uniform magnetic field, the particle moves in a circular path in a plane perpendicular to **B**. The magnetic force $\mathbf{F}_B$ acting on the charge is always directed toward the center of the circle.

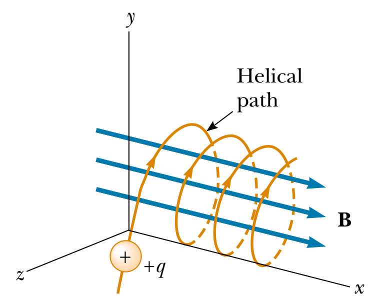


Figure 29.18 A charged particle having a velocity vector that has a component parallel to a uniform magnetic field moves in a helical path.

EXAMPLE 29.6 A Proton Moving Perpendicular to a Uniform Magnetic Field

A proton is moving in a circular orbit of radius 14 cm in a uniform 0.35-T magnetic field perpendicular to the velocity of the proton. Find the linear speed of the proton.

Solution From Equation 29.13, we have

$$v = \frac{qBr}{m_p} = \frac{(1.60 \times 10^{-19}\ \text{C})(0.35\ \text{T})(14 \times 10^{-2}\ \text{m})}{1.67 \times 10^{-27}\ \text{kg}}$$

$$= \boxed{4.7 \times 10^{6}\ \text{m/s}}$$

Exercise If an electron moves in a direction perpendicular to the same magnetic field with this same linear speed, what is the radius of its circular orbit?

Answer 7.6×10^{-5} m.

EXAMPLE 29.7 Bending an Electron Beam

In an experiment designed to measure the magnitude of a uniform magnetic field, electrons are accelerated from rest through a potential difference of 350 V. The electrons travel along a curved path because of the magnetic force exerted on them, and the radius of the path is measured to be 7.5 cm. (Fig. 29.19 shows such a curved beam of electrons.) If the magnetic field is perpendicular to the beam, (a) what is the magnitude of the field?

Solution First we must calculate the speed of the electrons. We can use the fact that the increase in their kinetic energy must equal the decrease in their potential energy $|e|\Delta V$ (because of conservation of energy). Then we can use Equation 29.13 to find the magnitude of the magnetic field. Because $K_i = 0$ and $K_f = m_e v^2/2$, we have

$$\tfrac{1}{2}m_e v^2 = |e|\Delta V$$

$$v = \sqrt{\frac{2|e|\Delta V}{m_e}} = \sqrt{\frac{2(1.60 \times 10^{-19}\ \text{C})(350\ \text{V})}{9.11 \times 10^{-31}\ \text{kg}}}$$

$$= 1.11 \times 10^{7}\ \text{m/s}$$

$$B = \frac{m_e v}{|e|r} = \frac{(9.11 \times 10^{-31}\ \text{kg})(1.11 \times 10^{7}\ \text{m/s})}{(1.60 \times 10^{-19}\ \text{C})(0.075\ \text{m})}$$

$$= \boxed{8.4 \times 10^{-4}\ \text{T}}$$

(b) What is the angular speed of the electrons?

Solution Using Equation 29.14, we find that

$$\omega = \frac{v}{r} = \frac{1.11 \times 10^{7}\ \text{m/s}}{0.075\ \text{m}} = \boxed{1.5 \times 10^{8}\ \text{rad/s}}$$

Exercise What is the period of revolution of the electrons?

Answer 43 ns.

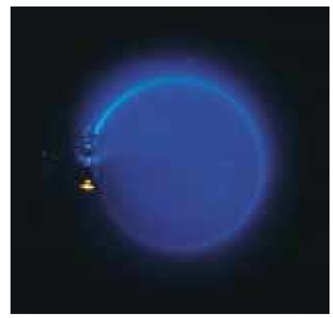


Figure 29.19 The bending of an electron beam in a magnetic field.

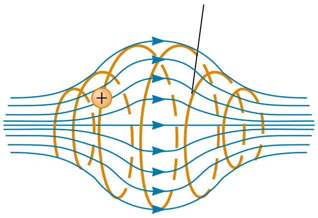


Figure 29.20 A charged particle moving in a nonuniform magnetic field (a magnetic bottle) spirals about the field (red path) and oscillates between the end points. The magnetic force exerted on the particle near either end of the bottle has a component that causes the particle to spiral back toward the center.

When charged particles move in a nonuniform magnetic field, the motion is complex. For example, in a magnetic field that is strong at the ends and weak in the middle, such as that shown in Figure 29.20, the particles can oscillate back and forth between the end points. A charged particle starting at one end spirals along the field lines until it reaches the other end, where it reverses its path and spirals back. This configuration is known as a *magnetic bottle* because charged particles can be trapped within it. The magnetic bottle has been used to confine a *plasma,* a gas consisting of ions and electrons. Such a plasma-confinement scheme could fulfill a crucial role in the control of nuclear fusion, a process that could supply us with an almost endless source of energy. Unfortunately, the magnetic bottle has its problems. If a large number of particles are trapped, collisions between them cause the particles to eventually leak from the system.

The Van Allen radiation belts consist of charged particles (mostly electrons and protons) surrounding the Earth in doughnut-shaped regions (Fig. 29.21). The particles, trapped by the Earth's nonuniform magnetic field, spiral around the field lines from pole to pole, covering the distance in just a few seconds. These particles originate mainly from the Sun, but some come from stars and other heavenly objects. For this reason, the particles are called *cosmic rays.* Most cosmic rays are deflected by the Earth's magnetic field and never reach the atmosphere. However, some of the particles become trapped; it is these particles that make up the Van Allen belts. When the particles are located over the poles, they sometimes collide with atoms in the atmosphere, causing the atoms to emit visible light. Such collisions are the origin of the beautiful Aurora Borealis, or Northern Lights, in the northern hemisphere and the Aurora Australis in the southern hemisphere.

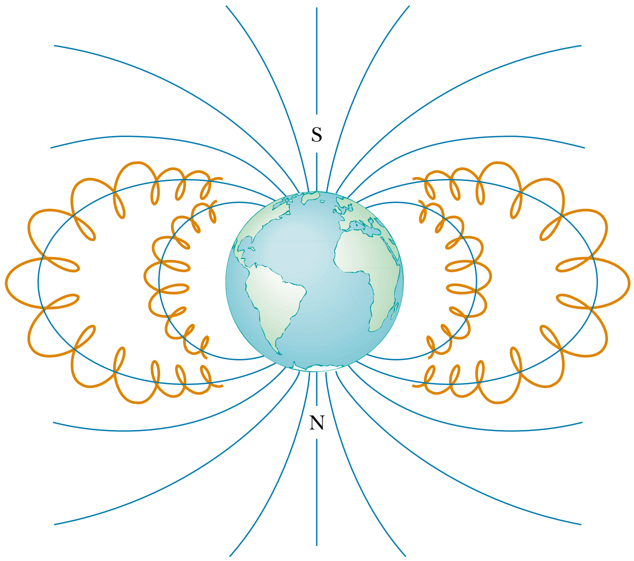
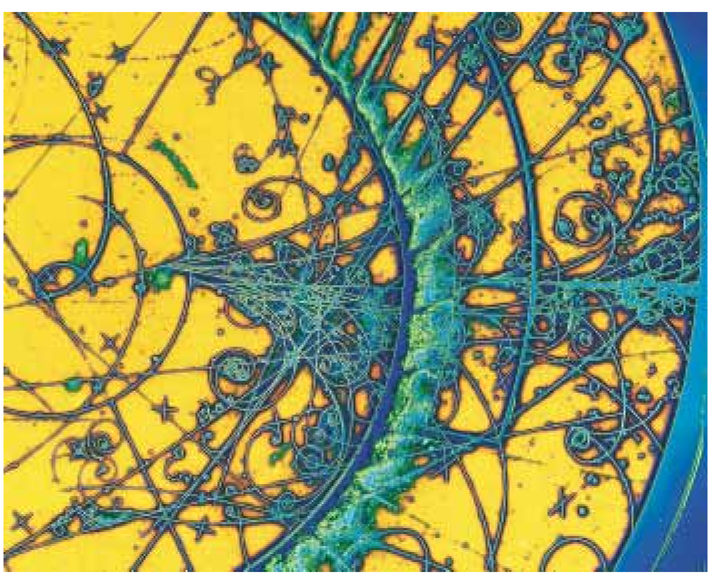


Figure 29.21 The Van Allen belts are made up of charged particles trapped by the Earth's nonuniform magnetic field. The magnetic field lines are in blue and the particle paths in red.

Auroras are usually confined to the polar regions because it is here that the Van Allen belts are nearest the Earth's surface. Occasionally, though, solar activity causes larger numbers of charged particles to enter the belts and significantly distort the normal magnetic field lines associated with the Earth. In these situations an aurora can sometimes be seen at lower latitudes. 12.1 & 12.11



This color-enhanced photograph, taken at CERN, the particle physics laboratory outside Geneva, Switzerland, shows a collection of tracks left by subatomic particles in a bubble chamber. A bubble chamber is a container filled with liquid hydrogen that is superheated, that is, momentarily raised above its normal boiling point by a sudden drop in pressure in the container. Any charged particle passing through the liquid in this state leaves behind a trail of tiny bubbles as the liquid boils in its wake. These bubbles are seen as fine tracks, showing the characteristic paths of different types of particles. The paths are curved because there is an intense applied magnetic field. The tightly wound spiral tracks are due to electrons and positrons.

Optional Section

29.5 APPLICATIONS INVOLVING CHARGED PARTICLES MOVING IN A MAGNETIC FIELD

A charge moving with a velocity $\mathbf{v}$ in the presence of both an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ experiences both an electric force $q\mathbf{E}$ and a magnetic force $q\mathbf{v} \times \mathbf{B}$. The total force (called the Lorentz force) acting on the charge is

Lorentz force

$$\sum \mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{29.16}$$

Velocity Selector

In many experiments involving moving charged particles, it is important that the particles all move with essentially the same velocity. This can be achieved by applying a combination of an electric field and a magnetic field oriented as shown in Figure 29.22. A uniform electric field is directed vertically downward (in the plane of the page in Fig. 29.22a), and a uniform magnetic field is applied in the direction perpendicular to the electric field (into the page in Fig. 29.22a). For q positive, the magnetic force $q\mathbf{v} \times \mathbf{B}$ is upward and the electric force $q\mathbf{E}$ is downward. When the magnitudes of the two fields are chosen so that $qE = qvB$, the particle moves in a straight horizontal line through the region of the fields. From the expression $qE = qvB$, we find that

$$v = \frac{E}{B} \tag{29.17}$$

Only those particles having speed v pass undeflected through the mutually perpendicular electric and magnetic fields. The magnetic force exerted on particles moving at speeds greater than this is stronger than the electric force, and the particles are deflected upward. Those moving at speeds less than this are deflected downward.

The Mass Spectrometer

A **mass spectrometer** separates ions according to their mass-to-charge ratio. In one version of this device, known as the *Bainbridge mass spectrometer,* a beam of ions first passes through a velocity selector and then enters a second uniform magnetic field $\mathbf{B}_0$ that has the same direction as the magnetic field in the selector (Fig. 29.23). Upon entering the second magnetic field, the ions move in a semicircle of

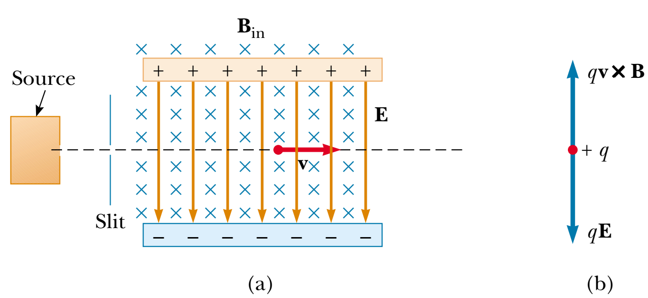


Figure 29.22 (a) A velocity selector. When a positively charged particle is in the presence of a magnetic field directed into the page and an electric field directed downward, it experiences a downward electric force $q\mathbf{E}$ and an upward magnetic force $q\mathbf{v} \times \mathbf{B}$. (b) When these forces balance, the particle moves in a horizontal line through the fields.

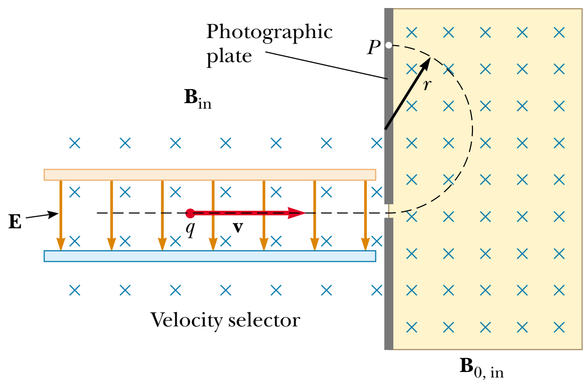


Figure 29.23 A mass spectrometer. Positively charged particles are sent first through a velocity selector and then into a region where the magnetic field $\mathbf{B}_0$ causes the particles to move in a semicircular path and strike a photographic film at *P*.

radius r before striking a photographic plate at P. If the ions are positively charged, the beam deflects upward, as Figure 29.23 shows. If the ions are negatively charged, the beam would deflect downward. From Equation 29.13, we can express the ratio m/q as

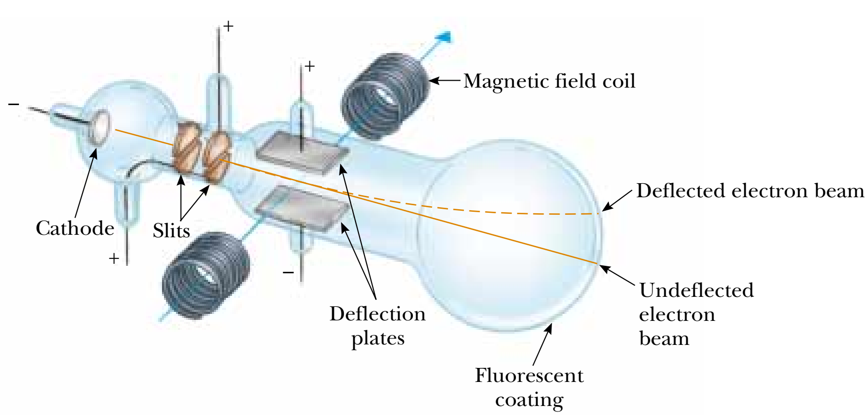
$$\frac{m}{q} = \frac{rB_0}{v}$$

Using Equation 29.17, we find that

$$\frac{m}{q} = \frac{rB_0B}{E} \tag{29.18}$$

Therefore, we can determine m/q by measuring the radius of curvature and knowing the field magnitudes B, B_0, and E. In practice, one usually measures the masses of various isotopes of a given ion, with the ions all carrying the same charge q. In this way, the mass ratios can be determined even if q is unknown.

A variation of this technique was used by J. J. Thomson (1856–1940) in 1897 to measure the ratio e/m_e for electrons. Figure 29.24a shows the basic apparatus he



(a)

(b)

Figure 29.24 (a) Thomson's apparatus for measuring e/m_e. Electrons are accelerated from the cathode, pass through two slits, and are deflected by both an electric field and a magnetic field (directed perpendicular to the electric field). The beam of electrons then strikes a fluorescent screen. (b) J. J. Thomson *(left)* in the Cavendish Laboratory, University of Cambridge. It is interesting to note that the man on the right, Frank Baldwin Jewett, is a distant relative of John W. Jewett, Jr., contributing author of this text.

used. Electrons are accelerated from the cathode and pass through two slits. They then drift into a region of perpendicular electric and magnetic fields. The magnitudes of the two fields are first adjusted to produce an undeflected beam. When the magnetic field is turned off, the electric field produces a measurable beam deflection that is recorded on the fluorescent screen. From the size of the deflection and the measured values of E and B, the charge-to-mass ratio can be determined. The results of this crucial experiment represent the discovery of the electron as a fundamental particle of nature.

Quick Quiz 29.6

When a photographic plate from a mass spectrometer like the one shown in Figure 29.23 is developed, the three patterns shown in Figure 29.25 are observed. Rank the particles that caused the patterns by speed and m/q ratio.

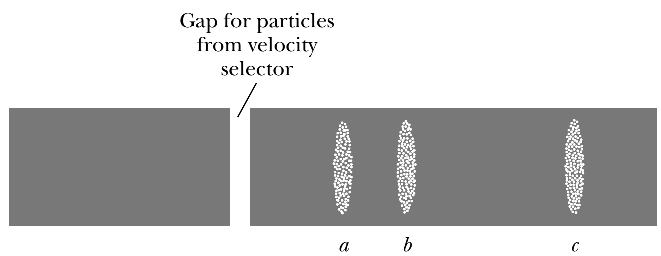
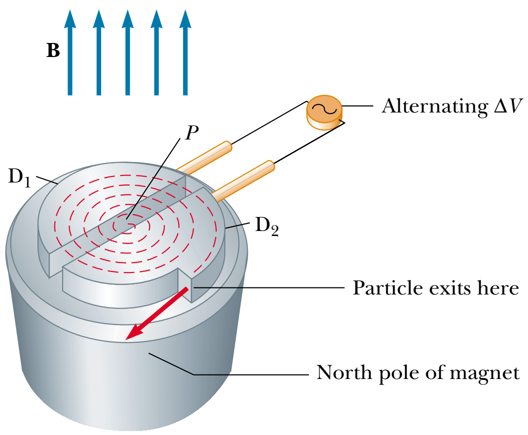


Figure 29.25

The Cyclotron

A **cyclotron** can accelerate charged particles to very high speeds. Both electric and magnetic forces have a key role. The energetic particles produced are used to bombard atomic nuclei and thereby produce nuclear reactions of interest to researchers. A number of hospitals use cyclotron facilities to produce radioactive substances for diagnosis and treatment.

A schematic drawing of a cyclotron is shown in Figure 29.26. The charges move inside two semicircular containers D_1 and D_2, referred to as *dees*. A high-frequency alternating potential difference is applied to the dees, and a uniform magnetic field is directed perpendicular to them. A positive ion released at P near the center of the magnet in one dee moves in a semicircular path (indicated by the dashed red line in the drawing) and arrives back at the gap in a time $T/2$, where T is the time needed to make one complete trip around the two dees, given by Equation 29.15. The frequency of the applied potential difference is adjusted so that the polarity of the dees is reversed in the same time it takes the ion to travel around one dee. If the applied potential difference is adjusted such that D_2 is at a lower electric potential than D_1 by an amount ΔV, the ion accelerates across the gap to D_2 and its kinetic energy increases by an amount $q\Delta V$. It then moves around D_2 in a semicircular path of greater radius (because its speed has increased). After a time $T/2$, it again arrives at the gap between the dees. By this time, the polarity across the dees is again reversed, and the ion is given another "kick" across the gap. The motion continues so that for each half-circle trip around one dee, the ion gains additional kinetic energy equal to $q\,\Delta V$. When the radius of its path is nearly that of the dees, the energetic ion leaves the system through the exit slit. It is important to note that the operation of the cyclotron is



(a)

(b)

Figure 29.26 (a) A cyclotron consists of an ion source at P, two dees D_1 and D_2 across which an alternating potential difference is applied, and a uniform magnetic field. (The south pole of the magnet is not shown.) The red dashed curved lines represent the path of the particles. (b) The first cyclotron, invented by E.O. Lawrence and M.S. Livingston in 1934.

based on the fact that T is independent of the speed of the ion and of the radius of the circular path.

We can obtain an expression for the kinetic energy of the ion when it exits the cyclotron in terms of the radius R of the dees. From Equation 29.13 we know that $v = qBR/m$. Hence, the kinetic energy is

$$K = \tfrac{1}{2}mv^2 = \frac{q^2B^2R^2}{2m} \tag{29.19}$$

When the energy of the ions in a cyclotron exceeds about 20 MeV, relativistic effects come into play. (Such effects are discussed in Chapter 39.) We observe that T increases and that the moving ions do not remain in phase with the applied potential difference. Some accelerators overcome this problem by modifying the period of the applied potential difference so that it remains in phase with the moving ions.

web

More information on these accelerators is available at
http://www.fnal.gov or
http://www.CERN.ch
The CERN site also discusses the creation of the World Wide Web there by physicists in the mid-1990s.

Optional Section

29.6 THE HALL EFFECT

When a current-carrying conductor is placed in a magnetic field, a potential difference is generated in a direction perpendicular to both the current and the magnetic field. This phenomenon, first observed by Edwin Hall (1855–1938) in 1879, is known as the *Hall effect.* It arises from the deflection of charge carriers to one side of the conductor as a result of the magnetic force they experience. The Hall effect gives information regarding the sign of the charge carriers and their density; it can also be used to measure the magnitude of magnetic fields.

The arrangement for observing the Hall effect consists of a flat conductor carrying a current I in the x direction, as shown in Figure 29.27. A uniform magnetic field $\mathbf{B}$ is applied in the y direction. If the charge carriers are electrons moving in the negative x direction with a drift velocity $\mathbf{v}_d$, they experience an upward mag-

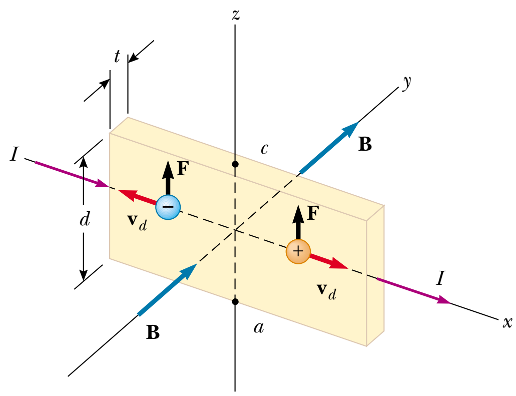


Figure 29.27 To observe the Hall effect, a magnetic field is applied to a current-carrying conductor. When I is in the x direction and $\mathbf{B}$ in the y direction, both positive and negative charge carriers are deflected upward in the magnetic field. The Hall voltage is measured between points a and c.

netic force $\mathbf{F}_B = q\mathbf{v}_d \times \mathbf{B}$, are deflected upward, and accumulate at the upper edge of the flat conductor, leaving an excess of positive charge at the lower edge (Fig. 29.28a). This accumulation of charge at the edges increases until the electric force resulting from the charge separation balances the magnetic force acting on the carriers. When this equilibrium condition is reached, the electrons are no longer deflected upward. A sensitive voltmeter or potentiometer connected across the sample, as shown in Figure 29.28, can measure the potential difference—known as the **Hall voltage** ΔV_H—generated across the conductor.

If the charge carriers are positive and hence move in the positive x direction, as shown in Figures 29.27 and 29.28b, they also experience an upward magnetic force $q\mathbf{v}_d \times \mathbf{B}$. This produces a buildup of positive charge on the upper edge and leaves an excess of negative charge on the lower edge. Hence, the sign of the Hall voltage generated in the sample is opposite the sign of the Hall voltage resulting from the deflection of electrons. The sign of the charge carriers can therefore be determined from a measurement of the polarity of the Hall voltage.

In deriving an expression for the Hall voltage, we first note that the magnetic force exerted on the carriers has magnitude qv_dB. In equilibrium, this force is balanced by the electric force qE_H, where E_H is the magnitude of the electric field due to the charge separation (sometimes referred to as the *Hall field*). Therefore,

$$qv_dB = qE_H$$
$$E_H = v_dB$$

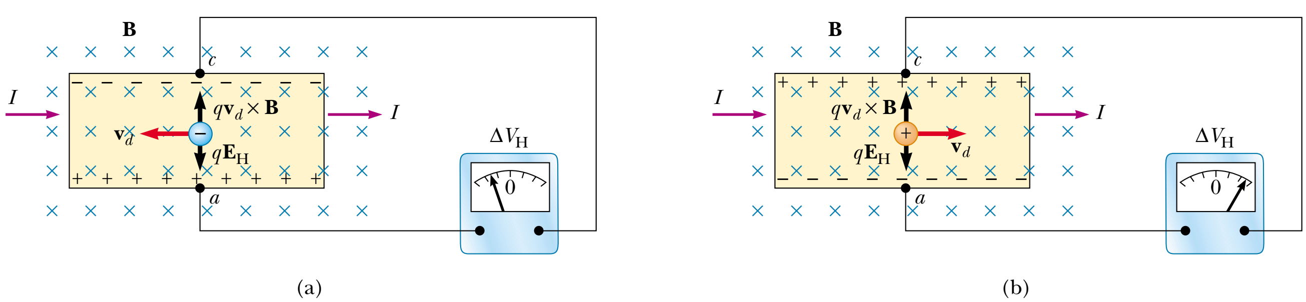


Figure 29.28 (a) When the charge carriers in a Hall effect apparatus are negative, the upper edge of the conductor becomes negatively charged, and c is at a lower electric potential than a. (b) When the charge carriers are positive, the upper edge becomes positively charged, and c is at a higher potential than a. In either case, the charge carriers are no longer deflected when the edges become fully charged, that is, when there is a balance between the electrostatic force qE_H and the magnetic deflection force qvB.

If d is the width of the conductor, the Hall voltage is

$$\Delta V_H = E_H d = v_d B d \quad \textbf{(29.20)}$$

Thus, the measured Hall voltage gives a value for the drift speed of the charge carriers if d and B are known.

We can obtain the charge carrier density n by measuring the current in the sample. From Equation 27.4, we can express the drift speed as

$$v_d = \frac{I}{nqA} \quad \textbf{(29.21)}$$

where A is the cross-sectional area of the conductor. Substituting Equation 29.21 into Equation 29.20, we obtain

$$\Delta V_H = \frac{IBd}{nqA} \quad \textbf{(29.22)}$$

Because $A = td$, where t is the thickness of the conductor, we can also express Equation 29.22 as

$$\Delta V_H = \frac{IB}{nqt} = \frac{R_H IB}{t} \quad \textbf{(29.23)}$$

The Hall voltage

where $R_H = 1/nq$ is the **Hall coefficient.** This relationship shows that a properly calibrated conductor can be used to measure the magnitude of an unknown magnetic field.

Because all quantities in Equation 29.23 other than nq can be measured, a value for the Hall coefficient is readily obtainable. The sign and magnitude of R_H give the sign of the charge carriers and their number density. In most metals, the charge carriers are electrons, and the charge carrier density determined from Hall-effect measurements is in good agreement with calculated values for such metals as lithium (Li), sodium (Na), copper (Cu), and silver (Ag), whose atoms each give up one electron to act as a current carrier. In this case, n is approximately equal to the number of conducting electrons per unit volume. However, this classical model is not valid for metals such as iron (Fe), bismuth (Bi), and cadmium (Cd) or for semiconductors. These discrepancies can be explained only by using a model based on the quantum nature of solids.

web

In 1980, Klaus von Klitzing discovered that the Hall voltage is quantized. He won the Nobel Prize for this discovery in 1985. For a discussion of the quantum Hall effect and some of its consequences, visit our Web site at **http://www.saunderscollege.com/physics/**

EXAMPLE 29.8 The Hall Effect for Copper

A rectangular copper strip 1.5 cm wide and 0.10 cm thick carries a current of 5.0 A. Find the Hall voltage for a 1.2-T magnetic field applied in a direction perpendicular to the strip.

Solution If we assume that one electron per atom is available for conduction, we can take the charge carrier density to be $n = 8.49 \times 10^{28}$ electrons/m^3 (see Example 27.1). Substituting this value and the given data into Equation 29.23 gives

$$\Delta V_H = \frac{IB}{nqt}$$

$$= \frac{(5.0\ \text{A})(1.2\ \text{T})}{(8.49 \times 10^{28}\ \text{m}^{-3})(1.6 \times 10^{-19}\ \text{C})(0.001\,0\ \text{m})}$$

$$\Delta V_H = \boxed{0.44\ \mu\text{V}}$$

Such an extremely small Hall voltage is expected in good conductors. (Note that the width of the conductor is not needed in this calculation.)

In semiconductors, n is much smaller than it is in metals that contribute one electron per atom to the current; hence, the Hall voltage is usually greater because it varies as the inverse of n. Currents of the order of 0.1 mA are generally used for such materials. Consider a piece of silicon that has the same dimensions as the copper strip in this example and whose value for $n = 1.0 \times 10^{20}$ electrons/m^3. Taking $B = 1.2$ T and $I = 0.10$ mA, we find that $\Delta V_H = 7.5$ mV. A potential difference of this magnitude is readily measured.

SUMMARY

The magnetic force that acts on a charge q moving with a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F}_B = q\mathbf{v} \times \mathbf{B} \tag{29.1}$$

The direction of this magnetic force is perpendicular both to the velocity of the particle and to the magnetic field. The magnitude of this force is

$$F_B = |q|vB \sin\theta \tag{29.2}$$

where θ is the smaller angle between $\mathbf{v}$ and $\mathbf{B}$. The SI unit of $\mathbf{B}$ is the **tesla** (T), where $1\ \text{T} = 1\ \text{N/A}\cdot\text{m}$.

When a charged particle moves in a magnetic field, the work done by the magnetic force on the particle is zero because the displacement is always perpendicular to the direction of the force. The magnetic field can alter the direction of the particle's velocity vector, but it cannot change its speed.

If a straight conductor of length L carries a current I, the force exerted on that conductor when it is placed in a uniform magnetic field $\mathbf{B}$ is

$$\mathbf{F}_B = I\mathbf{L} \times \mathbf{B} \tag{29.3}$$

where the direction of $\mathbf{L}$ is in the direction of the current and $|\mathbf{L}| = L$.

If an arbitrarily shaped wire carrying a current I is placed in a magnetic field, the magnetic force exerted on a very small segment $d\mathbf{s}$ is

$$d\mathbf{F}_B = I\,d\mathbf{s} \times \mathbf{B} \tag{29.4}$$

To determine the total magnetic force on the wire, one must integrate Equation 29.4, keeping in mind that both $\mathbf{B}$ and $d\mathbf{s}$ may vary at each point. Integration gives for the force exerted on a current-carrying conductor of arbitrary shape in a uniform magnetic field

$$\mathbf{F}_B = I\mathbf{L}' \times \mathbf{B} \tag{29.7}$$

where $\mathbf{L}'$ is a vector directed from one end of the conductor to the opposite end. Because integration of Equation 29.4 for a closed loop yields a zero result, the net magnetic force on any closed loop carrying a current in a uniform magnetic field is zero.

The **magnetic dipole moment** $\boldsymbol{\mu}$ of a loop carrying a current I is

$$\boldsymbol{\mu} = I\mathbf{A} \tag{29.10}$$

where the area vector $\mathbf{A}$ is perpendicular to the plane of the loop and $|\mathbf{A}|$ is equal to the area of the loop. The SI unit of $\boldsymbol{\mu}$ is $\text{A}\cdot\text{m}^2$.

The torque $\boldsymbol{\tau}$ on a current loop placed in a uniform magnetic field $\mathbf{B}$ is

$$\boldsymbol{\tau} = \boldsymbol{\mu} \times \mathbf{B} \tag{29.11}$$

and the potential energy of a magnetic dipole in a magnetic field is

$$U = -\boldsymbol{\mu}\cdot\mathbf{B} \tag{29.12}$$

If a charged particle moves in a uniform magnetic field so that its initial velocity is perpendicular to the field, the particle moves in a circle, the plane of which is perpendicular to the magnetic field. The radius of the circular path is

$$r = \frac{mv}{qB} \tag{29.13}$$

where m is the mass of the particle and q is its charge. The angular speed of the charged particle is

$$\omega = \frac{qB}{m} \tag{29.14}$$

QUESTIONS

1. At a given instant, a proton moves in the positive x direction in a region where a magnetic field is directed in the negative z direction. What is the direction of the magnetic force? Does the proton continue to move in the positive x direction? Explain.
2. Two charged particles are projected into a region where a magnetic field is directed perpendicular to their velocities. If the charges are deflected in opposite directions, what can be said about them?
3. If a charged particle moves in a straight line through some region of space, can one say that the magnetic field in that region is zero?
4. Suppose an electron is chasing a proton up this page when suddenly a magnetic field directed perpendicular into the page is turned on. What happens to the particles?
5. How can the motion of a moving charged particle be used to distinguish between a magnetic field and an electric field? Give a specific example to justify your argument.
6. List several similarities and differences between electric and magnetic forces.
7. Justify the following statement: "It is impossible for a constant (in other words, a time-independent) magnetic field to alter the speed of a charged particle."
8. In view of the preceding statement, what is the role of a magnetic field in a cyclotron?
9. A current-carrying conductor experiences no magnetic force when placed in a certain manner in a uniform magnetic field. Explain.
10. Is it possible to orient a current loop in a uniform magnetic field such that the loop does not tend to rotate? Explain.
11. How can a current loop be used to determine the presence of a magnetic field in a given region of space?
12. What is the net force acting on a compass needle in a uniform magnetic field?
13. What type of magnetic field is required to exert a resultant force on a magnetic dipole? What is the direction of the resultant force?
14. A proton moving horizontally enters a region where a uniform magnetic field is directed perpendicular to the proton's velocity, as shown in Figure Q29.14. Describe the subsequent motion of the proton. How would an electron behave under the same circumstances?
15. In a magnetic bottle, what causes the direction of the velocity of the confined charged particles to reverse? (*Hint:* Find the direction of the magnetic force acting on the particles in a region where the field lines converge.)
16. In the cyclotron, why do particles of different velocities take the same amount of time to complete one half-circle trip around one dee?

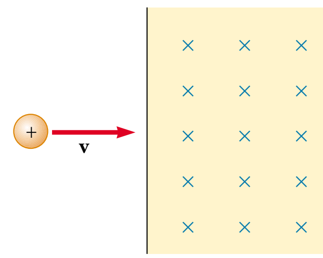


Figure Q29.14

17. The *bubble chamber* is a device used for observing tracks of particles that pass through the chamber, which is immersed in a magnetic field. If some of the tracks are spirals and others are straight lines, what can you say about the particles?
18. Can a constant magnetic field set into motion an electron initially at rest? Explain your answer.
19. You are designing a magnetic probe that uses the Hall effect to measure magnetic fields. Assume that you are restricted to using a given material and that you have already made the probe as thin as possible. What, if anything, can be done to increase the Hall voltage produced for a given magnetic field?
20. The electron beam shown in Figure Q29.20 is projected to the right. The beam deflects downward in the presence of a magnetic field produced by a pair of current-carrying coils. (a) What is the direction of the magnetic field? (b) What would happen to the beam if the current in the coils were reversed?

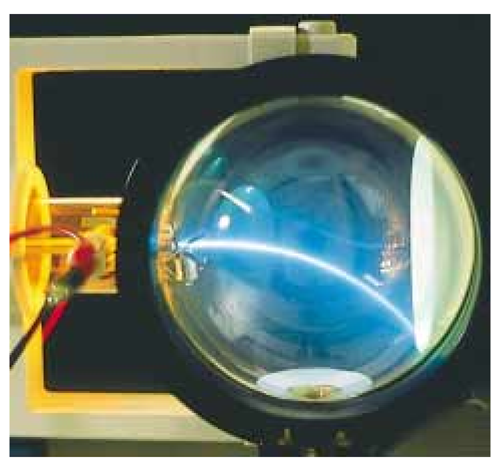


Figure Q29.20

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 29.1 The Magnetic Field

WEB **1.** Determine the initial direction of the deflection of charged particles as they enter the magnetic fields, as shown in Figure P29.1.

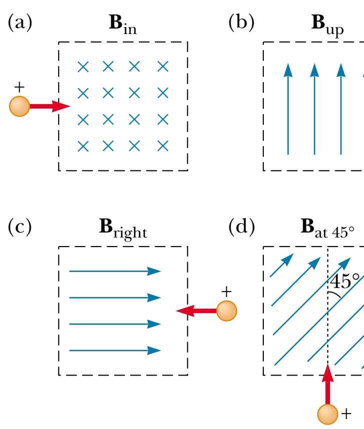


Figure P29.1

2. Consider an electron near the Earth's equator. In which direction does it tend to deflect if its velocity is directed (a) downward, (b) northward, (c) westward, or (d) southeastward?

3. An electron moving along the positive x axis perpendicular to a magnetic field experiences a magnetic deflection in the negative y direction. What is the direction of the magnetic field?

4. A proton travels with a speed of 3.00×10^6 m/s at an angle of 37.0° with the direction of a magnetic field of 0.300 T in the $+y$ direction. What are (a) the magnitude of the magnetic force on the proton and (b) its acceleration?

5. A proton moves in a direction perpendicular to a uniform magnetic field **B** at 1.00×10^7 m/s and experiences an acceleration of 2.00×10^{13} m/s^2 in the $+x$ direction when its velocity is in the $+z$ direction. Determine the magnitude and direction of the field.

6. An electron is accelerated through 2 400 V from rest and then enters a region where there is a uniform 1.70-T magnetic field. What are (a) the maximum and (b) the minimum values of the magnetic force this charge can experience?

7. At the equator, near the surface of the Earth, the magnetic field is approximately 50.0 μT northward, and the electric field is about 100 N/C downward in fair weather. Find the gravitational, electric, and magnetic forces on an electron with an instantaneous velocity of 6.00×10^6 m/s directed to the east in this environment.

8. A 30.0-g metal ball having net charge $Q = 5.00$ μC is thrown out of a window horizontally at a speed $v = 20.0$ m/s. The window is at a height $h = 20.0$ m above the ground. A uniform horizontal magnetic field of magnitude $B = 0.010\,0$ T is perpendicular to the plane of the ball's trajectory. Find the magnetic force acting on the ball just before it hits the ground.

9. A proton moving at 4.00×10^6 m/s through a magnetic field of 1.70 T experiences a magnetic force of magnitude 8.20×10^{-13} N. What is the angle between the proton's velocity and the field?

10. An electron has a velocity of 1.20 km/s (in the positive x direction) and an acceleration of 2.00×10^{12} m/s^2 (in the positive z direction) in uniform electric and magnetic fields. If the electric field has a magnitude of 20.0 N/C (in the positive z direction), what can you determine about the magnetic field in the region? What can you not determine?

11. A proton moves with a velocity of $\mathbf{v} = (2\mathbf{i} - 4\mathbf{j} + \mathbf{k})$ m/s in a region in which the magnetic field is $\mathbf{B} = (\mathbf{i} + 2\mathbf{j} - 3\mathbf{k})$ T. What is the magnitude of the magnetic force this charge experiences?

12. An electron is projected into a uniform magnetic field $\mathbf{B} = (1.40\mathbf{i} + 2.10\mathbf{j})$ T. Find the vector expression for the force on the electron when its velocity is $\mathbf{v} = 3.70 \times 10^5\,\mathbf{j}$ m/s.

Section 29.2 Magnetic Force Acting on a Current-Carrying Conductor

WEB **13.** A wire having a mass per unit length of 0.500 g/cm carries a 2.00-A current horizontally to the south. What are the direction and magnitude of the minimum magnetic field needed to lift this wire vertically upward?

14. A wire carries a steady current of 2.40 A. A straight section of the wire is 0.750 m long and lies along the x axis within a uniform magnetic field of magnitude $B = 1.60$ T in the positive z direction. If the current is in the $+x$ direction, what is the magnetic force on the section of wire?

15. A wire 2.80 m in length carries a current of 5.00 A in a region where a uniform magnetic field has a magnitude of 0.390 T. Calculate the magnitude of the magnetic force on the wire if the angle between the magnetic field and the current is (a) 60.0°, (b) 90.0°, (c) 120°.

16. A conductor suspended by two flexible wires as shown in Figure P29.16 has a mass per unit length of 0.040 0 kg/m. What current must exist in the conductor for the tension in the supporting wires to be zero when the magnetic

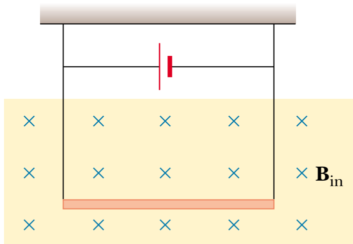


Figure P29.16

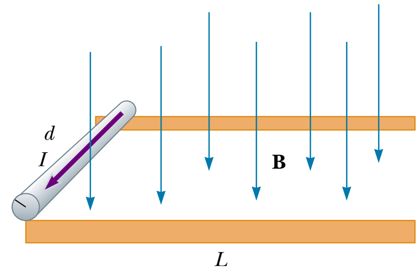


Figure P29.19 Problems 19 and 20.

field is 3.60 T into the page? What is the required direction for the current?

17. Imagine a very long, uniform wire with a linear mass density of 1.00 g/m that encircles the Earth at its magnetic equator. Suppose that the planet's magnetic field is 50.0 μT horizontally north throughout this region. What are the magnitude and direction of the current in the wire that keep it levitated just above the ground?

18. In Figure P29.18, the cube is 40.0 cm on each edge. Four straight segments of wire—*ab, bc, cd,* and *da*—form a closed loop that carries a current $I = 5.00$ A, in the direction shown. A uniform magnetic field of magnitude $B = 0.020\ 0$ T is in the positive y direction. Determine the magnitude and direction of the magnetic force on each segment.

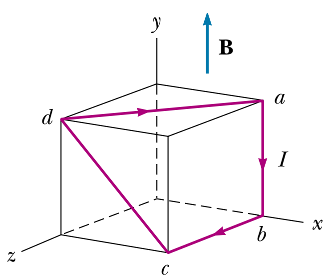


Figure P29.18

19. Review Problem. A rod with a mass of 0.720 kg and a radius of 6.00 cm rests on two parallel rails (Fig. P29.19) that are $d = 12.0$ cm apart and $L = 45.0$ cm long. The rod carries a current of $I = 48.0$ A (in the direction shown) and rolls along the rails without slipping. If it starts from rest, what is the speed of the rod as it leaves the rails if a uniform magnetic field of magnitude 0.240 T is directed perpendicular to the rod and the rails?

20. Review Problem. A rod of mass m and radius R rests on two parallel rails (Fig. P29.19) that are a distance d apart and have a length L. The rod carries a current I (in the direction shown) and rolls along the rails without slipping. If it starts from rest, what is the speed of the rod as it leaves the rails if a uniform magnetic field **B** is directed perpendicular to the rod and the rails?

WEB **21.** *A nonuniform magnetic field exerts a net force on a magnetic dipole.* A strong magnet is placed under a horizontal conducting ring of radius r that carries current I, as shown in Figure P29.21. If the magnetic field **B** makes an angle θ with the vertical at the ring's location, what are the magnitude and direction of the resultant force on the ring?

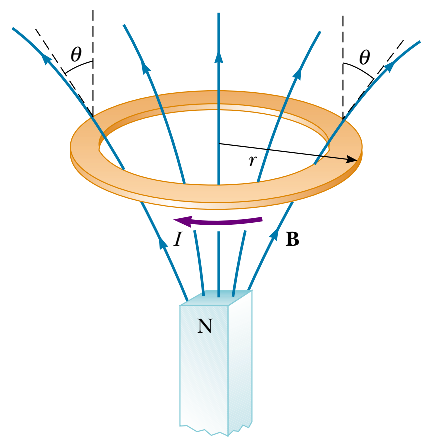


Figure P29.21

22. Assume that in Atlanta, Georgia, the Earth's magnetic field is 52.0 μT northward at 60.0° below the horizontal. A tube in a neon sign carries a current of 35.0 mA between two diagonally opposite corners of a shop window, which lies in a north–south vertical plane. The current enters the tube at the bottom south corner of the window. It exits at the opposite corner, which is 1.40 m farther north and 0.850 m higher up. Between these two points, the glowing tube spells out DONUTS. Use the theorem proved as "Case 1" in the text to determine the total vector magnetic force on the tube.

Section 29.3 Torque on a Current Loop in a Uniform Magnetic Field

23. A current of 17.0 mA is maintained in a single circular loop with a circumference of 2.00 m. A magnetic field

of 0.800 T is directed parallel to the plane of the loop. (a) Calculate the magnetic moment of the loop. (b) What is the magnitude of the torque exerted on the loop by the magnetic field?

24. A small bar magnet is suspended in a uniform 0.250-T magnetic field. The maximum torque experienced by the bar magnet is 4.60×10^{-3} N·m. Calculate the magnetic moment of the bar magnet.

WEB **25.** A rectangular loop consists of $N = 100$ closely wrapped turns and has dimensions $a = 0.400$ m and $b = 0.300$ m. The loop is hinged along the y axis, and its plane makes an angle $\theta = 30.0°$ with the x axis (Fig. P29.25). What is the magnitude of the torque exerted on the loop by a uniform magnetic field $B = 0.800$ T directed along the x axis when the current is $I = 1.20$ A in the direction shown? What is the expected direction of rotation of the loop?

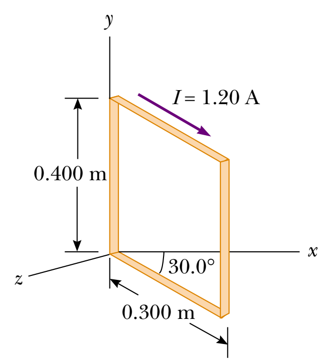


Figure P29.25

26. A long piece of wire of mass 0.100 kg and total length of 4.00 m is used to make a square coil with a side of 0.100 m. The coil is hinged along a horizontal side, carries a 3.40-A current, and is placed in a vertical magnetic field with a magnitude of 0.010 0 T. (a) Determine the angle that the plane of the coil makes with the vertical when the coil is in equilibrium. (b) Find the torque acting on the coil due to the magnetic force at equilibrium.

27. A 40.0-cm length of wire carries a current of 20.0 A. It is bent into a loop and placed with its normal perpendicular to a magnetic field with a strength of 0.520 T. What is the torque on the loop if it is bent into (a) an equilateral triangle, (b) a square, (c) a circle? (d) Which torque is greatest?

28. A current loop with dipole moment $\boldsymbol{\mu}$ is placed in a uniform magnetic field $\mathbf{B}$. Prove that its potential energy is $U = -\boldsymbol{\mu} \cdot \mathbf{B}$. You may imitate the discussion of the potential energy of an electric dipole in an electric field given in Chapter 26.

29. The needle of a magnetic compass has a magnetic moment of 9.70 mA·m^2. At its location, the Earth's magnetic field is 55.0 μT north at 48.0° below the horizontal. (a) Identify the orientations at which the compass needle has minimum potential energy and maximum potential energy. (b) How much work must be done on the needle for it to move from the former to the latter orientation?

30. A wire is formed into a circle having a diameter of 10.0 cm and is placed in a uniform magnetic field of 3.00 mT. A current of 5.00 A passes through the wire. Find (a) the maximum torque on the wire and (b) the range of potential energy of the wire in the field for different orientations of the circle.

Section 29.4 Motion of a Charged Particle in a Uniform Magnetic Field

31. The magnetic field of the Earth at a certain location is directed vertically downward and has a magnitude of 50.0 μT. A proton is moving horizontally toward the west in this field with a speed of 6.20×10^6 m/s. (a) What are the direction and magnitude of the magnetic force that the field exerts on this charge? (b) What is the radius of the circular arc followed by this proton?

32. A singly charged positive ion has a mass of 3.20×10^{-26} kg. After being accelerated from rest through a potential difference of 833 V, the ion enters a magnetic field of 0.920 T along a direction perpendicular to the direction of the field. Calculate the radius of the path of the ion in the field.

33. Review Problem. One electron collides elastically with a second electron initially at rest. After the collision, the radii of their trajectories are 1.00 cm and 2.40 cm. The trajectories are perpendicular to a uniform magnetic field of magnitude 0.044 0 T. Determine the energy (in keV) of the incident electron.

34. A proton moving in a circular path perpendicular to a constant magnetic field takes 1.00 μs to complete one revolution. Determine the magnitude of the magnetic field.

35. A proton (charge $+e$, mass m_p), a deuteron (charge $+e$, mass $2m_p$), and an alpha particle (charge $+2e$, mass $4m_p$) are accelerated through a common potential difference ΔV. The particles enter a uniform magnetic field $\mathbf{B}$ with a velocity in a direction perpendicular to $\mathbf{B}$. The proton moves in a circular path of radius r_p. Determine the values of the radii of the circular orbits for the deuteron r_d and the alpha particle r_α in terms of r_p.

36. Review Problem. An electron moves in a circular path perpendicular to a constant magnetic field with a magnitude of 1.00 mT. If the angular momentum of the electron about the center of the circle is 4.00×10^{-25} J·s, determine (a) the radius of the circular path and (b) the speed of the electron.

37. Calculate the cyclotron frequency of a proton in a magnetic field with a magnitude of 5.20 T.

38. A singly charged ion of mass m is accelerated from rest by a potential difference ΔV. It is then deflected by a uniform magnetic field (perpendicular to the ion's velocity) into a semicircle of radius R. Now a doubly

charged ion of mass m' is accelerated through the same potential difference and deflected by the same magnetic field into a semicircle of radius $R' = 2R$. What is the ratio of the ions' masses?

39. A cosmic-ray proton in interstellar space has an energy of 10.0 MeV and executes a circular orbit having a radius equal to that of Mercury's orbit around the Sun (5.80×10^{10} m). What is the magnetic field in that region of space?

40. A singly charged positive ion moving at 4.60×10^5 m/s leaves a circular track of radius 7.94 mm along a direction perpendicular to the 1.80-T magnetic field of a bubble chamber. Compute the mass (in atomic mass units) of this ion, and identify it from that value.

(Optional)

Section 29.5 Applications Involving Charged Particles Moving in a Magnetic Field

41. A velocity selector consists of magnetic and electric fields described by the expressions $\mathbf{E} = E\mathbf{k}$ and $\mathbf{B} = B\mathbf{j}$. If $B = 0.015\,0$ T, find the value of E such that a 750-eV electron moving along the positive x axis is undeflected.

42. (a) Singly charged uranium-238 ions are accelerated through a potential difference of 2.00 kV and enter a uniform magnetic field of 1.20 T directed perpendicular to their velocities. Determine the radius of their circular path. (b) Repeat for uranium-235 ions. How does the ratio of these path radii depend on the accelerating voltage and the magnetic field strength?

43. Consider the mass spectrometer shown schematically in Figure 29.23. The electric field between the plates of the velocity selector is 2 500 V/m, and the magnetic field in both the velocity selector and the deflection chamber has a magnitude of 0.035 0 T. Calculate the radius of the path for a singly charged ion having a mass $m = 2.18 \times 10^{-26}$ kg.

44. What is the required radius of a cyclotron designed to accelerate protons to energies of 34.0 MeV using a magnetic field of 5.20 T?

45. A cyclotron designed to accelerate protons has a magnetic field with a magnitude of 0.450 T over a region of radius 1.20 m. What are (a) the cyclotron frequency and (b) the maximum speed acquired by the protons?

46. At the Fermilab accelerator in Batavia, Illinois, protons having momentum 4.80×10^{-16} kg · m/s are held in a circular orbit of radius 1.00 km by an upward magnetic field. What is the magnitude of this field?

WEB 47. The picture tube in a television uses magnetic deflection coils rather than electric deflection plates. Suppose an electron beam is accelerated through a 50.0-kV potential difference and then travels through a region of uniform magnetic field 1.00 cm wide. The screen is located 10.0 cm from the center of the coils and is 50.0 cm wide. When the field is turned off, the electron beam hits the center of the screen. What field magnitude is necessary to deflect the beam to the side of the screen? Ignore relativistic corrections.

(Optional)

Section 29.6 The Hall Effect

48. A flat ribbon of silver having a thickness $t = 0.200$ mm is used in a Hall-effect measurement of a uniform magnetic field perpendicular to the ribbon, as shown in Figure P29.48. The Hall coefficient for silver is $R_H = 0.840 \times 10^{-10}$ m^3/C. (a) What is the density of charge carriers in silver? (b) If a current $I = 20.0$ A produces a Hall voltage $\Delta V_H = 15.0$ μV, what is the magnitude of the applied magnetic field?

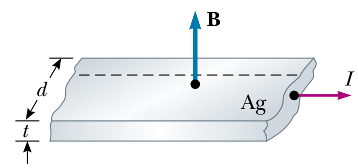


Figure P29.48

49. A section of conductor 0.400 cm thick is used in a Hall-effect measurement. A Hall voltage of 35.0 μV is measured for a current of 21.0 A in a magnetic field of 1.80 T. Calculate the Hall coefficient for the conductor.

50. A flat copper ribbon 0.330 mm thick carries a steady current of 50.0 A and is located in a uniform 1.30-T magnetic field directed perpendicular to the plane of the ribbon. If a Hall voltage of 9.60 μV is measured across the ribbon, what is the charge density of the free electrons? What effective number of free electrons per atom does this result indicate?

51. In an experiment designed to measure the Earth's magnetic field using the Hall effect, a copper bar 0.500 cm thick is positioned along an east–west direction. If a current of 8.00 A in the conductor results in a Hall voltage of 5.10 pV, what is the magnitude of the Earth's magnetic field? (Assume that $n = 8.48 \times 10^{28}$ electrons/m^3 and that the plane of the bar is rotated to be perpendicular to the direction of $\mathbf{B}$.)

52. A Hall-effect probe operates with a 120-mA current. When the probe is placed in a uniform magnetic field with a magnitude of 0.080 0 T, it produces a Hall voltage of 0.700 μV. (a) When it is measuring an unknown magnetic field, the Hall voltage is 0.330 μV. What is the unknown magnitude of the field? (b) If the thickness of the probe in the direction of $\mathbf{B}$ is 2.00 mm, find the charge-carrier density (each of charge e).

ADDITIONAL PROBLEMS

53. An electron enters a region of magnetic field of magnitude 0.100 T, traveling perpendicular to the linear boundary of the region. The direction of the field is perpendicular to the velocity of the electron. (a) Determine the time it takes for the electron to leave the "field-filled" region, noting that its path is a semicircle. (b) Find the kinetic energy of the electron if the maximum depth of penetration in the field is 2.00 cm.

54. A 0.200-kg metal rod carrying a current of 10.0 A glides on two horizontal rails 0.500 m apart. What vertical magnetic field is required to keep the rod moving at a constant speed if the coefficient of kinetic friction between the rod and rails is 0.100?

55. Sodium melts at 99°C. Liquid sodium, an excellent thermal conductor, is used in some nuclear reactors to cool the reactor core. The liquid sodium is moved through pipes by pumps that exploit the force on a moving charge in a magnetic field. The principle is as follows: Assume that the liquid metal is in an electrically insulating pipe having a rectangular cross-section of width w and height h. A uniform magnetic field perpendicular to the pipe affects a section of length L (Fig. P29.55). An electric current directed perpendicular to the pipe and to the magnetic field produces a current density J in the liquid sodium. (a) Explain why this arrangement produces on the liquid a force that is directed along the length of the pipe. (b) Show that the section of liquid in the magnetic field experiences a pressure increase JLB.

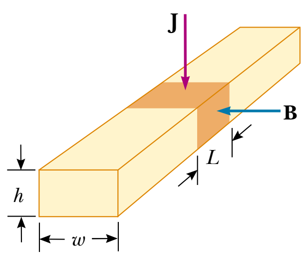


Figure P29.55

56. Protons having a kinetic energy of 5.00 MeV are moving in the positive x direction and enter a magnetic field $\mathbf{B} = (0.050\,0\ \mathbf{k})$ T directed out of the plane of the page and extending from $x = 0$ to $x = 1.00$ m, as shown in Figure P29.56. (a) Calculate the y component of the protons' momentum as they leave the magnetic field. (b) Find the angle α between the initial velocity vector of the proton beam and the velocity vector after the beam emerges from the field. (*Hint:* Neglect relativistic effects and note that 1 eV = 1.60×10^{-19} J.)

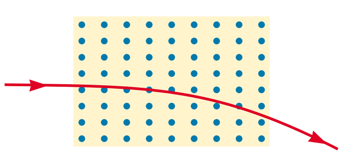


Figure P29.56

57. (a) A proton moving in the $+x$ direction with velocity $\mathbf{v} = v_i\mathbf{i}$ experiences a magnetic force $\mathbf{F} = F_i\mathbf{j}$. Explain what you can and cannot infer about $\mathbf{B}$ from this information. (b) In terms of F_i, what would be the force on a proton in the same field moving with velocity $\mathbf{v} = -v_i\mathbf{i}$? (c) What would be the force on an electron in the same field moving with velocity $\mathbf{v} = v_i\mathbf{i}$?

58. **Review Problem.** A wire having a linear mass density of 1.00 g/cm is placed on a horizontal surface that has a coefficient of friction of 0.200. The wire carries a current of 1.50 A toward the east and slides horizontally to the north. What are the magnitude and direction of the smallest magnetic field that enables the wire to move in this fashion?

59. A positive charge $q = 3.20 \times 10^{-19}$ C moves with a velocity $\mathbf{v} = (2\mathbf{i} + 3\mathbf{j} - 1\mathbf{k})$ m/s through a region where both a uniform magnetic field and a uniform electric field exist. (a) What is the total force on the moving charge (in unit–vector notation) if $\mathbf{B} = (2\mathbf{i} + 4\mathbf{j} + 1\mathbf{k})$ T and $\mathbf{E} = (4\mathbf{i} - 1\mathbf{j} - 2\mathbf{k})$ V/m? (b) What angle does the force vector make with the positive x axis?

60. A cosmic-ray proton traveling at half the speed of light is heading directly toward the center of the Earth in the plane of the Earth's equator. Will it hit the Earth? Assume that the Earth's magnetic field is uniform over the planet's equatorial plane with a magnitude of 50.0 μT, extending out 1.30×10^7 m from the surface of the Earth. Assume that the field is zero at greater distances. Calculate the radius of curvature of the proton's path in the magnetic field. Ignore relativistic effects.

61. The circuit in Figure P29.61 consists of wires at the top and bottom and identical metal springs as the left and right sides. The wire at the bottom has a mass of 10.0 g and is 5.00 cm long. The springs stretch 0.500 cm under the weight of the wire, and the circuit has a total resistance of 12.0 Ω. When a magnetic field is turned on, directed out of the page, the springs stretch an additional 0.300 cm. What is the magnitude of the magnetic field? (The upper portion of the circuit is fixed.)

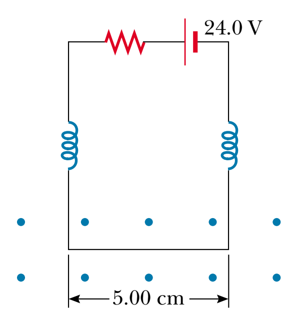


Figure P29.61

62. A hand-held electric mixer contains an electric motor. Model the motor as a single flat compact circular coil carrying electric current in a region where a magnetic field is produced by an external permanent magnet. You need consider only one instant in the operation of the motor. (We will consider motors again in Chapter 31.) The coil moves because the magnetic field exerts torque on the coil, as described in Section 29.3. Make

order-of-magnitude estimates of the magnetic field, the torque on the coil, the current in it, its area, and the number of turns in the coil, so that they are related according to Equation 29.11. Note that the input power to the motor is electric, given by $\mathcal{P} = I\,\Delta V$, and the useful output power is mechanical, given by $\mathcal{P} = \tau\omega$.

63. A metal rod having a mass per unit length of 0.010 0 kg/m carries a current of $I = 5.00$ A. The rod hangs from two wires in a uniform vertical magnetic field, as shown in Figure P29.63. If the wires make an angle $\theta = 45.0°$ with the vertical when in equilibrium, determine the magnitude of the magnetic field.

64. A metal rod having a mass per unit length μ carries a current I. The rod hangs from two wires in a uniform vertical magnetic field, as shown in Figure P29.63. If the wires make an angle θ with the vertical when in equilibrium, determine the magnitude of the magnetic field.

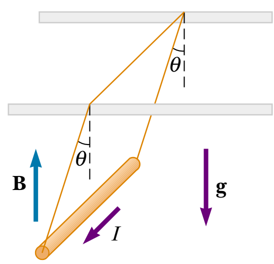


Figure P29.63 Problems 63 and 64.

65. A cyclotron is sometimes used for carbon dating, which we consider in Section 44.6. Carbon-14 and carbon-12 ions are obtained from a sample of the material to be dated and accelerated in the cyclotron. If the cyclotron has a magnetic field of magnitude 2.40 T, what is the difference in cyclotron frequencies for the two ions?

66. A uniform magnetic field of magnitude 0.150 T is directed along the positive x axis. A positron moving at 5.00×10^6 m/s enters the field along a direction that makes an angle of 85.0° with the x axis (Fig. P29.66). The motion of the particle is expected to be a helix, as described in Section 29.4. Calculate (a) the pitch p and (b) the radius r of the trajectory.

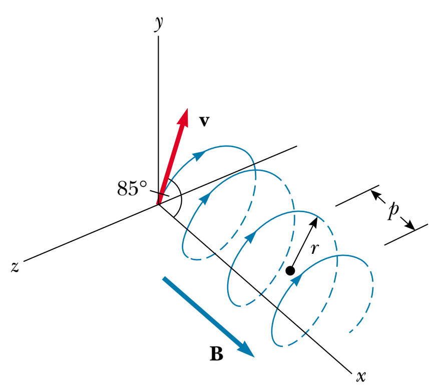


Figure P29.66

67. Consider an electron orbiting a proton and maintained in a fixed circular path of radius $R = 5.29 \times 10^{-11}$ m by the Coulomb force. Treating the orbiting charge as a current loop, calculate the resulting torque when the system is in a magnetic field of 0.400 T directed perpendicular to the magnetic moment of the electron.

68. A singly charged ion completes five revolutions in a uniform magnetic field of magnitude 5.00×10^{-2} T in 1.50 ms. Calculate the mass of the ion in kilograms.

69. A proton moving in the plane of the page has a kinetic energy of 6.00 MeV. It enters a magnetic field of magnitude $B = 1.00$ T directed into the page, moving at an angle of $\theta = 45.0°$ with the straight linear boundary of the field, as shown in Figure P29.69. (a) Find the distance x from the point of entry to where the proton leaves the field. (b) Determine the angle θ' between the boundary and the proton's velocity vector as it leaves the field.

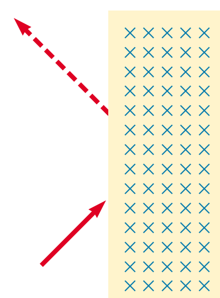


Figure P29.69

70. Table P29.70 shows measurements of a Hall voltage and corresponding magnetic field for a probe used to measure magnetic fields. (a) Plot these data, and deduce a relationship between the two variables. (b) If the mea-

TABLE P29.70

ΔV_H (μV)	B (T)
0	0.00
11	0.10
19	0.20
28	0.30
42	0.40
50	0.50
61	0.60
68	0.70
79	0.80
90	0.90
102	1.00

surements were taken with a current of 0.200 A and the sample is made from a material having a charge-carrier density of $1.00 \times 10^{26}/m^3$, what is the thickness of the sample?

71. A heart surgeon monitors the flow rate of blood through an artery using an electromagnetic flowmeter (Fig. P29.71). Electrodes A and B make contact with the outer surface of the blood vessel, which has interior diameter 3.00 mm. (a) For a magnetic field magnitude of 0.040 0 T, an emf of 160 μV appears between the electrodes. Calculate the speed of the blood. (b) Verify that electrode A is positive, as shown. Does the sign of the emf depend on whether the mobile ions in the blood are predominantly positively or negatively charged? Explain.

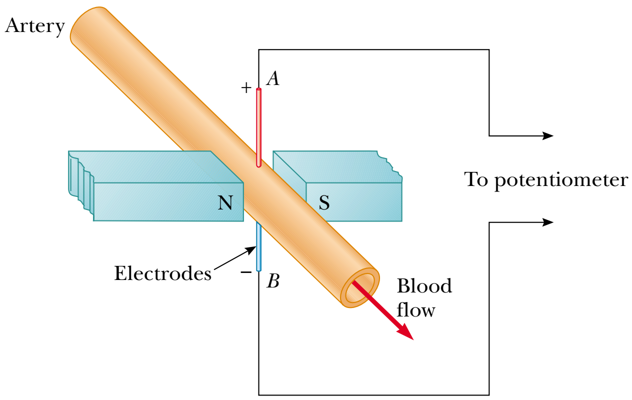


Figure P29.71

72. As illustrated in Figure P29.72, a particle of mass m having positive charge q is initially traveling upward with velocity $v\mathbf{j}$. At the origin of coordinates it enters a region between $y = 0$ and $y = h$ containing a uniform magnetic field $B\mathbf{k}$ directed perpendicular out of the page. (a) What is the critical value of v such that the particle just reaches $y = h$? Describe the path of the particle under this condition, and predict its final velocity. (b) Specify the path of the particle and its final velocity if v is less than the critical value. (c) Specify the path of the particle and its final velocity if v is greater than the critical value.

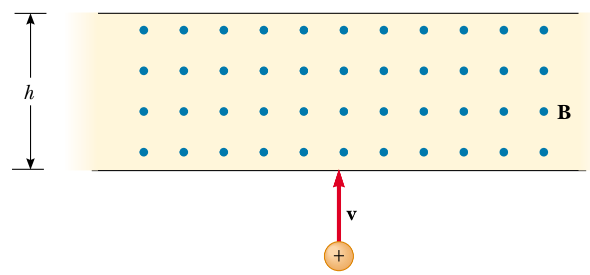


Figure P29.72

Answers to Quick Quizzes

29.1 Zero. Because the magnetic force exerted by the field on the charge is always perpendicular to the velocity of the charge, the field can never do any work on the charge: $W = \mathbf{F}_B \cdot d\mathbf{s} = (\mathbf{F}_B \cdot \mathbf{v})\,dt = 0$. Work requires a component of force along the direction of motion.

29.2 Unaffected. The magnetic force exerted by a magnetic field on a charge is proportional to the charge's velocity relative to the field. If the charge is stationary, as in this situation, there is no magnetic force.

29.3 (c), (b), (a), (d). As Example 29.2 shows, we need to be concerned only with the "effective length" of wire perpendicular to the magnetic field or, stated another way, the length of the "magnetic field shadow" cast by the wire. For (c), 4 m of wire is perpendicular to the field. The short vertical pieces experience no magnetic force because their currents are parallel to the field. When the wire in (b) is broken into many short vertical and horizontal segments alternately parallel and perpendicular to the field, we find a total of 3.5 m of horizontal segments perpendicular to the field and therefore experiencing a force. Next comes (a), with 3 m of wire effectively perpendicular to the field. Only 2 m of the wire in (d) experiences a force. The portion carrying current from 2 m to 4 m does experience a force directed out of the page, but this force is canceled by an oppositely directed force acting on the current as it moves from 4 m to 2 m.

29.4 Because it is in the region of the stronger magnetic field, side ③ experiences a greater force than side ①: $F_3 > F_1$. Therefore, in addition to the torque resulting from the two forces, a net force is exerted downward on the loop.

29.5 (c), (b), (a). Because all loops enclose the same area and carry the same current, the magnitude of $\boldsymbol{\mu}$ is the same for all. For (c), $\boldsymbol{\mu}$ points upward and is perpendicular to the magnetic field and $\tau = \mu B$. This is the maximum torque possible. The next largest cross product of $\boldsymbol{\mu}$ and $\mathbf{B}$ is for (b), in which $\boldsymbol{\mu}$ points toward the upper right (as illustrated in Fig. 29.13b). Finally, $\boldsymbol{\mu}$ for the loop in (a) points along the direction of $\mathbf{B}$; thus, the torque is zero.

29.6 The velocity selector ensures that all three types of particles have the same speed. We cannot determine individual masses or charges, but we can rank the particles by m/q ratio. Equation 29.18 indicates that those particles traveling through the circle of greatest radius have the greatest m/q ratio. Thus, the m/q ranking, from greatest to least value, is c, b, a.

PUZZLER

All three of these commonplace items use magnetism to store information. The cassette can store more than an hour of music, the floppy disk can hold the equivalent of hundreds of pages of information, and many hours of television programming can be recorded on the videotape. How do these devices work? *(George Semple)*

chapter

30

Sources of the Magnetic Field

Chapter Outline

In the preceding chapter, we discussed the magnetic force exerted on a charged particle moving in a magnetic field. To complete the description of the magnetic interaction, this chapter deals with the origin of the magnetic field—moving charges. We begin by showing how to use the law of Biot and Savart to calculate the magnetic field produced at some point in space by a small current element. Using this formalism and the principle of superposition, we then calculate the total magnetic field due to various current distributions. Next, we show how to determine the force between two current-carrying conductors, which leads to the definition of the ampere. We also introduce Ampère's law, which is useful in calculating the magnetic field of a highly symmetric configuration carrying a steady current.

This chapter is also concerned with the complex processes that occur in magnetic materials. All magnetic effects in matter can be explained on the basis of atomic magnetic moments, which arise both from the orbital motion of the electrons and from an intrinsic property of the electrons known as spin.

30.1 THE BIOT–SAVART LAW

Shortly after Oersted's discovery in 1819 that a compass needle is deflected by a current-carrying conductor, Jean-Baptiste Biot (1774–1862) and Félix Savart (1791–1841) performed quantitative experiments on the force exerted by an electric current on a nearby magnet. From their experimental results, Biot and Savart arrived at a mathematical expression that gives the magnetic field at some point in space in terms of the current that produces the field. That expression is based on the following experimental observations for the magnetic field $d\mathbf{B}$ at a point P associated with a length element $d\mathbf{s}$ of a wire carrying a steady current I (Fig. 30.1):

Properties of the magnetic field created by an electric current

- The vector $d\mathbf{B}$ is perpendicular both to $d\mathbf{s}$ (which points in the direction of the current) and to the unit vector $\hat{\mathbf{r}}$ directed from $d\mathbf{s}$ to P.
- The magnitude of $d\mathbf{B}$ is inversely proportional to r^2, where r is the distance from $d\mathbf{s}$ to P.
- The magnitude of $d\mathbf{B}$ is proportional to the current and to the magnitude ds of the length element $d\mathbf{s}$.
- The magnitude of $d\mathbf{B}$ is proportional to $\sin\theta$, where θ is the angle between the vectors $d\mathbf{s}$ and $\hat{\mathbf{r}}$.

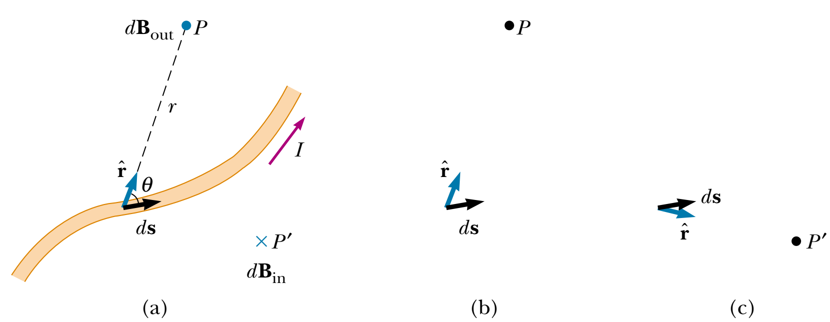


Figure 30.1 (a) The magnetic field $d\mathbf{B}$ at point P due to the current I through a length element $d\mathbf{s}$ is given by the Biot–Savart law. The direction of the field is out of the page at P and into the page at P'. (b) The cross product $d\mathbf{s} \times \hat{\mathbf{r}}$ points out of the page when $\hat{\mathbf{r}}$ points toward P. (c) The cross product $d\mathbf{s} \times \hat{\mathbf{r}}$ points into the page when $\hat{\mathbf{r}}$ points toward P'.

These observations are summarized in the mathematical formula known today as the **Biot–Savart law:**

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I\, d\mathbf{s} \times \hat{\mathbf{r}}}{r^2} \qquad \textbf{(30.1)}$$

Biot–Savart law

where μ_0 is a constant called the **permeability of free space:**

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A} \qquad \textbf{(30.2)}$$

Permeability of free space

It is important to note that the field $d\mathbf{B}$ in Equation 30.1 is the field created by the current in only a small length element $d\mathbf{s}$ of the conductor. To find the total magnetic field $\mathbf{B}$ created at some point by a current of finite size, we must sum up contributions from all current elements $I d\mathbf{s}$ that make up the current. That is, we must evaluate $\mathbf{B}$ by integrating Equation 30.1:

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\mathbf{s} \times \hat{\mathbf{r}}}{r^2} \qquad \textbf{(30.3)}$$

where the integral is taken over the entire current distribution. This expression must be handled with special care because the integrand is a cross product and therefore a vector quantity. We shall see one case of such an integration in Example 30.1.

Although we developed the Biot–Savart law for a current-carrying wire, it is also valid for a current consisting of charges flowing through space, such as the electron beam in a television set. In that case, $d\mathbf{s}$ represents the length of a small segment of space in which the charges flow.

Interesting similarities exist between the Biot–Savart law for magnetism and Coulomb's law for electrostatics. The current element produces a magnetic field, whereas a point charge produces an electric field. Furthermore, the magnitude of the magnetic field varies as the inverse square of the distance from the current element, as does the electric field due to a point charge. However, the directions of the two fields are quite different. The electric field created by a point charge is radial, but the magnetic field created by a current element is perpendicular to both the length element $d\mathbf{s}$ and the unit vector $\hat{\mathbf{r}}$, as described by the cross product in Equation 30.1. Hence, if the conductor lies in the plane of the page, as shown in Figure 30.1, $d\mathbf{B}$ points out of the page at P and into the page at P'.

Another difference between electric and magnetic fields is related to the source of the field. An electric field is established by an isolated electric charge. The Biot–Savart law gives the magnetic field of an isolated current element at some point, but such an isolated current element cannot exist the way an isolated electric charge can. A current element *must* be part of an extended current distribution because we must have a complete circuit in order for charges to flow. Thus, the Biot–Savart law is only the first step in a calculation of a magnetic field; it must be followed by an integration over the current distribution.

In the examples that follow, it is important to recognize that **the magnetic field determined in these calculations is the field created by a current-carrying conductor.** This field is not to be confused with any additional fields that may be present outside the conductor due to other sources, such as a bar magnet placed nearby.

EXAMPLE 30.1 Magnetic Field Surrounding a Thin, Straight Conductor

Consider a thin, straight wire carrying a constant current I and placed along the x axis as shown in Figure 30.2. Determine the magnitude and direction of the magnetic field at point P due to this current.

Solution From the Biot–Savart law, we expect that the magnitude of the field is proportional to the current in the wire and decreases as the distance a from the wire to point P increases. We start by considering a length element $d\mathbf{s}$ located a distance r from P. The direction of the magnetic field at point P due to the current in this element is out of the page because $d\mathbf{s} \times \hat{\mathbf{r}}$ is out of the page. In fact, since *all* of the current elements $I\,d\mathbf{s}$ lie in the plane of the page, they all produce a magnetic field directed out of the page at point P. Thus, we have the direction of the magnetic field at point P, and we need only find the magnitude.

Taking the origin at O and letting point P be along the positive y axis, with $\mathbf{k}$ being a unit vector pointing out of the page, we see that

$$d\mathbf{s} \times \hat{\mathbf{r}} = \mathbf{k}\,|d\mathbf{s} \times \hat{\mathbf{r}}| = \mathbf{k}(dx \sin\theta)$$

where, from Chapter 3, $|d\mathbf{s} \times \hat{\mathbf{r}}|$ represents the magnitude of $d\mathbf{s} \times \hat{\mathbf{r}}$. Because $\hat{\mathbf{r}}$ is a unit vector, the unit of the cross product is simply the unit of $d\mathbf{s}$, which is length. Substitution into Equation 30.1 gives

$$d\mathbf{B} = (dB)\,\mathbf{k} = \frac{\mu_0 I}{4\pi}\frac{dx \sin\theta}{r^2}\,\mathbf{k}$$

Because all current elements produce a magnetic field in the $\mathbf{k}$ direction, let us restrict our attention to the magnitude of the field due to one current element, which is

$$(1) \qquad dB = \frac{\mu_0 I}{4\pi}\frac{dx \sin\theta}{r^2}$$

To integrate this expression, we must relate the variables θ, x, and r. One approach is to express x and r in terms of θ. From the geometry in Figure 30.2a, we have

$$(2) \qquad r = \frac{a}{\sin\theta} = a \csc\theta$$

Because $\tan\theta = a/(-x)$ from the right triangle in Figure 30.2a (the negative sign is necessary because $d\mathbf{s}$ is located at a negative value of x), we have

$$x = -a \cot\theta$$

Taking the derivative of this expression gives

$$(3) \qquad dx = a \csc^2\theta\, d\theta$$

Substitution of Equations (2) and (3) into Equation (1) gives

$$(4) \qquad dB = \frac{\mu_0 I}{4\pi}\frac{a \csc^2\theta \sin\theta\, d\theta}{a^2 \csc^2\theta} = \frac{\mu_0 I}{4\pi a}\sin\theta\, d\theta$$

an expression in which the only variable is θ. We can now obtain the magnitude of the magnetic field at point P by integrating Equation (4) over all elements, subtending angles ranging from θ_1 to θ_2 as defined in Figure 30.2b:

$$B = \frac{\mu_0 I}{4\pi a}\int_{\theta_1}^{\theta_2} \sin\theta\, d\theta = \frac{\mu_0 I}{4\pi a}(\cos\theta_1 - \cos\theta_2) \qquad \textbf{(30.4)}$$

We can use this result to find the magnetic field of any straight current-carrying wire if we know the geometry and hence the angles θ_1 and θ_2. Consider the special case of an infinitely long, straight wire. If we let the wire in Figure 30.2b become infinitely long, we see that $\theta_1 = 0$ and $\theta_2 = \pi$ for length elements ranging between positions $x = -\infty$ and $x = +\infty$. Because $(\cos\theta_1 - \cos\theta_2) = (\cos 0 - \cos\pi) = 2$, Equation 30.4 becomes

$$B = \frac{\mu_0 I}{2\pi a} \qquad \textbf{(30.5)}$$

Equations 30.4 and 30.5 both show that the magnitude of

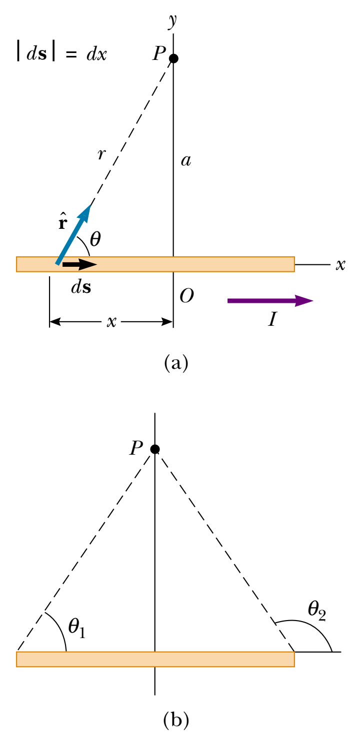


Figure 30.2 (a) A thin, straight wire carrying a current I. The magnetic field at point P due to the current in each element $d\mathbf{s}$ of the wire is out of the page, so the net field at point P is also out of the page. (b) The angles θ_1 and θ_2, used for determining the net field. When the wire is infinitely long, $\theta_1 = 0$ and $\theta_2 = 180°$.

the magnetic field is proportional to the current and decreases with increasing distance from the wire, as we expected. Notice that Equation 30.5 has the same mathematical form as the expression for the magnitude of the electric field due to a long charged wire (see Eq. 24.7).

Exercise Calculate the magnitude of the magnetic field 4.0 cm from an infinitely long, straight wire carrying a current of 5.0 A.

Answer 2.5×10^{-5} T.

The result of Example 30.1 is important because a current in the form of a long, straight wire occurs often. Figure 30.3 is a three-dimensional view of the magnetic field surrounding a long, straight current-carrying wire. Because of the symmetry of the wire, the magnetic field lines are circles concentric with the wire and lie in planes perpendicular to the wire. The magnitude of **B** is constant on any circle of radius a and is given by Equation 30.5. A convenient rule for determining the direction of **B** is to grasp the wire with the right hand, positioning the thumb along the direction of the current. The four fingers wrap in the direction of the magnetic field.

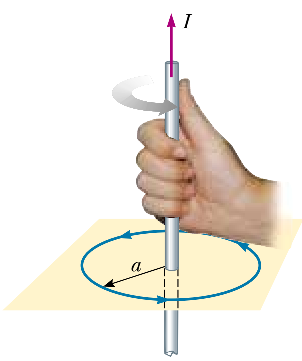


Figure 30.3 The right-hand rule for determining the direction of the magnetic field surrounding a long, straight wire carrying a current. Note that the magnetic field lines form circles around the wire.

EXAMPLE 30.2 Magnetic Field Due to a Curved Wire Segment

Calculate the magnetic field at point O for the current-carrying wire segment shown in Figure 30.4. The wire consists of two straight portions and a circular arc of radius R, which subtends an angle θ. The arrowheads on the wire indicate the direction of the current.

Solution The magnetic field at O due to the current in the straight segments AA' and CC' is zero because $d\mathbf{s}$ is parallel to $\hat{\mathbf{r}}$ along these paths; this means that $d\mathbf{s} \times \hat{\mathbf{r}} = 0$. Each length element $d\mathbf{s}$ along path AC is at the same distance R from O, and the current in each contributes a field element $d\mathbf{B}$ directed into the page at O. Furthermore, at every point on AC, $d\mathbf{s}$ is perpendicular to $\hat{\mathbf{r}}$; hence, $|d\mathbf{s} \times \hat{\mathbf{r}}| = ds$. Using this information and Equation 30.1, we can find the magnitude of the field at O due to the current in an element of length ds:

$$dB = \frac{\mu_0 I}{4\pi} \frac{ds}{R^2}$$

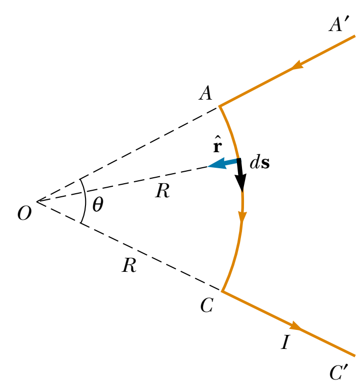


Figure 30.4 The magnetic field at O due to the current in the curved segment AC is into the page. The contribution to the field at O due to the current in the two straight segments is zero.

Because I and R are constants, we can easily integrate this expression over the curved path AC:

$$B = \frac{\mu_0 I}{4\pi R^2} \int ds = \frac{\mu_0 I}{4\pi R^2} s = \boxed{\frac{\mu_0 I}{4\pi R} \theta} \qquad \textbf{(30.6)}$$

where we have used the fact that $s = R\theta$ with θ measured in radians. The direction of $\mathbf{B}$ is into the page at O because $d\mathbf{s} \times \hat{\mathbf{r}}$ is into the page for every length element.

Exercise A circular wire loop of radius R carries a current I. What is the magnitude of the magnetic field at its center?

Answer $\mu_0 I/2R$.

EXAMPLE 30.3 Magnetic Field on the Axis of a Circular Current Loop

Consider a circular wire loop of radius R located in the yz plane and carrying a steady current I, as shown in Figure 30.5. Calculate the magnetic field at an axial point P a distance x from the center of the loop.

Solution In this situation, note that every length element $d\mathbf{s}$ is perpendicular to the vector $\hat{\mathbf{r}}$ at the location of the element. Thus, for any element, $|d\mathbf{s} \times \hat{\mathbf{r}}| = (ds)(1) \sin 90° = ds$. Furthermore, all length elements around the loop are at the same distance r from P, where $r^2 = x^2 + R^2$. Hence, the magnitude of $d\mathbf{B}$ due to the current in any length element $d\mathbf{s}$ is

$$dB = \frac{\mu_0 I}{4\pi} \frac{|d\mathbf{s} \times \hat{\mathbf{r}}|}{r^2} = \frac{\mu_0 I}{4\pi} \frac{ds}{(x^2 + R^2)}$$

The direction of $d\mathbf{B}$ is perpendicular to the plane formed by $\hat{\mathbf{r}}$ and $d\mathbf{s}$, as shown in Figure 30.5. We can resolve this vector into a component dB_x along the x axis and a component dB_y perpendicular to the x axis. When the components dB_y are summed over all elements around the loop, the resultant component is zero. That is, by symmetry the current in any element on one side of the loop sets up a perpendicular component of $d\mathbf{B}$ that cancels the perpendicular component set up by the current through the element diametrically opposite it. Therefore, *the resultant field at P must be along the x axis* and we can find it by integrating the components $dB_x = dB \cos \theta$. That is, $\mathbf{B} = B_x \mathbf{i}$, where

$$B_x = \oint dB \cos \theta = \frac{\mu_0 I}{4\pi} \oint \frac{ds \cos \theta}{x^2 + R^2}$$

and we must take the integral over the entire loop. Because θ, x, and R are constants for all elements of the loop and because $\cos \theta = R/(x^2 + R^2)^{1/2}$, we obtain

$$B_x = \frac{\mu_0 IR}{4\pi(x^2 + R^2)^{3/2}} \oint ds = \boxed{\frac{\mu_0 IR^2}{2(x^2 + R^2)^{3/2}}} \qquad \textbf{(30.7)}$$

where we have used the fact that $\oint ds = 2\pi R$ (the circumference of the loop).

To find the magnetic field at the center of the loop, we set $x = 0$ in Equation 30.7. At this special point, therefore,

$$B = \frac{\mu_0 I}{2R} \qquad (\text{at } x = 0) \qquad \textbf{(30.8)}$$

which is consistent with the result of the exercise in Example 30.2.

It is also interesting to determine the behavior of the magnetic field far from the loop—that is, when x is much greater than R. In this case, we can neglect the term R^2 in the denominator of Equation 30.7 and obtain

$$B \approx \frac{\mu_0 IR^2}{2x^3} \qquad (\text{for } x \gg R) \qquad \textbf{(30.9)}$$

Because the magnitude of the magnetic moment μ of the loop is defined as the product of current and loop area (see Eq. 29.10)—$\mu = I(\pi R^2)$ for our circular loop—we can express Equation 30.9 as

$$B \approx \frac{\mu_0}{2\pi} \frac{\mu}{x^3} \qquad \textbf{(30.10)}$$

This result is similar in form to the expression for the electric field due to an electric dipole, $E = k_e(2qa/y^3)$ (see Example

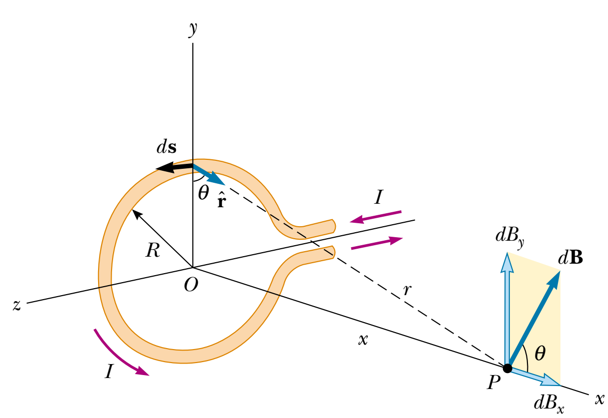


Figure 30.5 Geometry for calculating the magnetic field at a point P lying on the axis of a current loop. By symmetry, the total field $\mathbf{B}$ is along this axis.

23.6), where $2qa = p$ is the electric dipole moment as defined in Equation 26.16.

The pattern of the magnetic field lines for a circular current loop is shown in Figure 30.6a. For clarity, the lines are drawn for only one plane—one that contains the axis of the loop. Note that the field-line pattern is axially symmetric and looks like the pattern around a bar magnet, shown in Figure 30.6c.

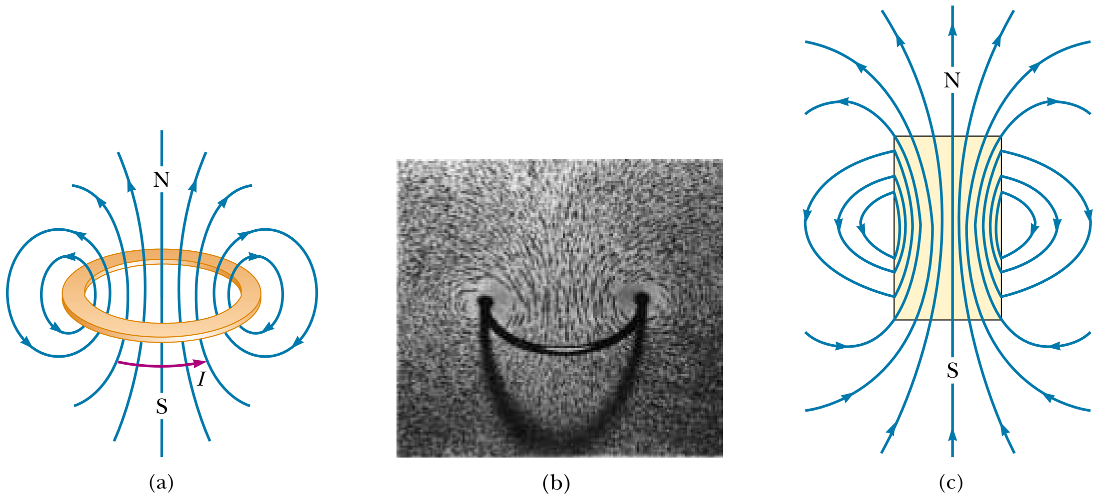


Figure 30.6 (a) Magnetic field lines surrounding a current loop. (b) Magnetic field lines surrounding a current loop, displayed with iron filings *(Education Development Center, Newton, MA).* (c) Magnetic field lines surrounding a bar magnet. Note the similarity between this line pattern and that of a current loop.

30.2 THE MAGNETIC FORCE BETWEEN TWO PARALLEL CONDUCTORS

In Chapter 29 we described the magnetic force that acts on a current-carrying conductor placed in an external magnetic field. Because a current in a conductor sets up its own magnetic field, it is easy to understand that two current-carrying conductors exert magnetic forces on each other. As we shall see, such forces can be used as the basis for defining the ampere and the coulomb.

Consider two long, straight, parallel wires separated by a distance a and carrying currents I_1 and I_2 in the same direction, as illustrated in Figure 30.7. We can determine the force exerted on one wire due to the magnetic field set up by the other wire. Wire 2, which carries a current I_2, creates a magnetic field $\mathbf{B}_2$ at the location of wire 1. The direction of $\mathbf{B}_2$ is perpendicular to wire 1, as shown in Figure 30.7. According to Equation 29.3, the magnetic force on a length ℓ of wire 1 is $\mathbf{F}_1 = I_1\boldsymbol{\ell} \times \mathbf{B}_2$. Because $\boldsymbol{\ell}$ is perpendicular to $\mathbf{B}_2$ in this situation, the magnitude of $\mathbf{F}_1$ is $F_1 = I_1 \ell B_2$. Because the magnitude of $\mathbf{B}_2$ is given by Equation 30.5, we see that

$$F_1 = I_1 \ell B_2 = I_1 \ell \left(\frac{\mu_0 I_2}{2\pi a} \right) = \frac{\mu_0 I_1 I_2}{2\pi a} \ell \tag{30.11}$$

The direction of $\mathbf{F}_1$ is toward wire 2 because $\boldsymbol{\ell} \times \mathbf{B}_2$ is in that direction. If the field set up at wire 2 by wire 1 is calculated, the force $\mathbf{F}_2$ acting on wire 2 is found to be equal in magnitude and opposite in direction to $\mathbf{F}_1$. This is what we expect be-

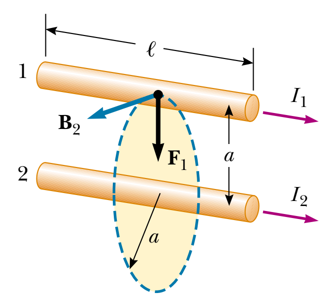


Figure 30.7 Two parallel wires that each carry a steady current exert a force on each other. The field $\mathbf{B}_2$ due to the current in wire 2 exerts a force of magnitude $F_1 = I_1 \ell B_2$ on wire 1. The force is attractive if the currents are parallel (as shown) and repulsive if the currents are antiparallel.

cause Newton's third law must be obeyed.[1] When the currents are in opposite directions (that is, when one of the currents is reversed in Fig. 30.7), the forces are reversed and the wires repel each other. Hence, we find that **parallel conductors carrying currents in the same direction attract each other, and parallel conductors carrying currents in opposite directions repel each other.**

Because the magnitudes of the forces are the same on both wires, we denote the magnitude of the magnetic force between the wires as simply F_B. We can rewrite this magnitude in terms of the force per unit length:

$$\frac{F_B}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi a} \tag{30.12}$$

The force between two parallel wires is used to define the **ampere** as follows:

Definition of the ampere

When the magnitude of the force per unit length between two long, parallel wires that carry identical currents and are separated by 1 m is 2×10^{-7} N/m, the current in each wire is defined to be 1 A.

web

Visit **http://physics.nist.gov/cuu/Units/ampere.html** for more information.

The value 2×10^{-7} N/m is obtained from Equation 30.12 with $I_1 = I_2 = 1$ A and $a = 1$ m. Because this definition is based on a force, a mechanical measurement can be used to standardize the ampere. For instance, the National Institute of Standards and Technology uses an instrument called a *current balance* for primary current measurements. The results are then used to standardize other, more conventional instruments, such as ammeters.

The SI unit of charge, the **coulomb,** is defined in terms of the ampere:

Definition of the coulomb

When a conductor carries a steady current of 1 A, the quantity of charge that flows through a cross-section of the conductor in 1 s is 1 C.

In deriving Equations 30.11 and 30.12, we assumed that both wires are long compared with their separation distance. In fact, only one wire needs to be long. The equations accurately describe the forces exerted on each other by a long wire and a straight parallel wire of limited length ℓ.

Quick Quiz 30.1

For $I_1 = 2$ A and $I_2 = 6$ A in Figure 30.7, which is true: (a) $F_1 = 3F_2$, (b) $F_1 = F_2/3$, or (c) $F_1 = F_2$?

Quick Quiz 30.2

A loose spiral spring is hung from the ceiling, and a large current is sent through it. Do the coils move closer together or farther apart?

[1] Although the total force exerted on wire 1 is equal in magnitude and opposite in direction to the total force exerted on wire 2, Newton's third law does not apply when one considers two small elements of the wires that are not exactly opposite each other. This apparent violation of Newton's third law and of the law of conservation of momentum is described in more advanced treatments on electricity and magnetism.

30.3 AMPÈRE'S LAW

12.4 Oersted's 1819 discovery about deflected compass needles demonstrates that a current-carrying conductor produces a magnetic field. Figure 30.8a shows how this effect can be demonstrated in the classroom. Several compass needles are placed in a horizontal plane near a long vertical wire. When no current is present in the wire, all the needles point in the same direction (that of the Earth's magnetic field), as expected. When the wire carries a strong, steady current, the needles all deflect in a direction tangent to the circle, as shown in Figure 30.8b. These observations demonstrate that the direction of the magnetic field produced by the current in the wire is consistent with the right-hand rule described in Figure 30.3. When the current is reversed, the needles in Figure 30.8b also reverse.

Because the compass needles point in the direction of **B**, we conclude that the lines of **B** form circles around the wire, as discussed in the preceding section. By symmetry, the magnitude of **B** is the same everywhere on a circular path centered on the wire and lying in a plane perpendicular to the wire. By varying the current and distance a from the wire, we find that B is proportional to the current and inversely proportional to the distance from the wire, as Equation 30.5 describes.

Now let us evaluate the product $\mathbf{B}\cdot d\mathbf{s}$ for a small length element $d\mathbf{s}$ on the circular path defined by the compass needles, and sum the products for all elements over the closed circular path. Along this path, the vectors $d\mathbf{s}$ and $\mathbf{B}$ are parallel at each point (see Fig. 30.8b), so $\mathbf{B}\cdot d\mathbf{s} = B\,ds$. Furthermore, the magnitude of $\mathbf{B}$ is constant on this circle and is given by Equation 30.5. Therefore, the sum of the products $B\,ds$ over the closed path, which is equivalent to the line integral of $\mathbf{B}\cdot d\mathbf{s}$, is

$$\oint \mathbf{B}\cdot d\mathbf{s} = B\oint ds = \frac{\mu_0 I}{2\pi r}(2\pi r) = \mu_0 I$$

where $\oint ds = 2\pi r$ is the circumference of the circular path. Although this result was calculated for the special case of a circular path surrounding a wire, it holds

Andre-Marie Ampère
(1775–1836) Ampère, a Frenchman, is credited with the discovery of electromagnetism—the relationship between electric currents and magnetic fields. Ampère's genius, particularly in mathematics, became evident by the time he was 12 years old; his personal life, however, was filled with tragedy. His father, a wealthy city official, was guillotined during the French Revolution, and his wife died young, in 1803. Ampère died at the age of 61 of pneumonia. His judgment of his life is clear from the epitaph he chose for his gravestone: *Tandem Felix* (Happy at Last). *(AIP Emilio Segre Visual Archive)*

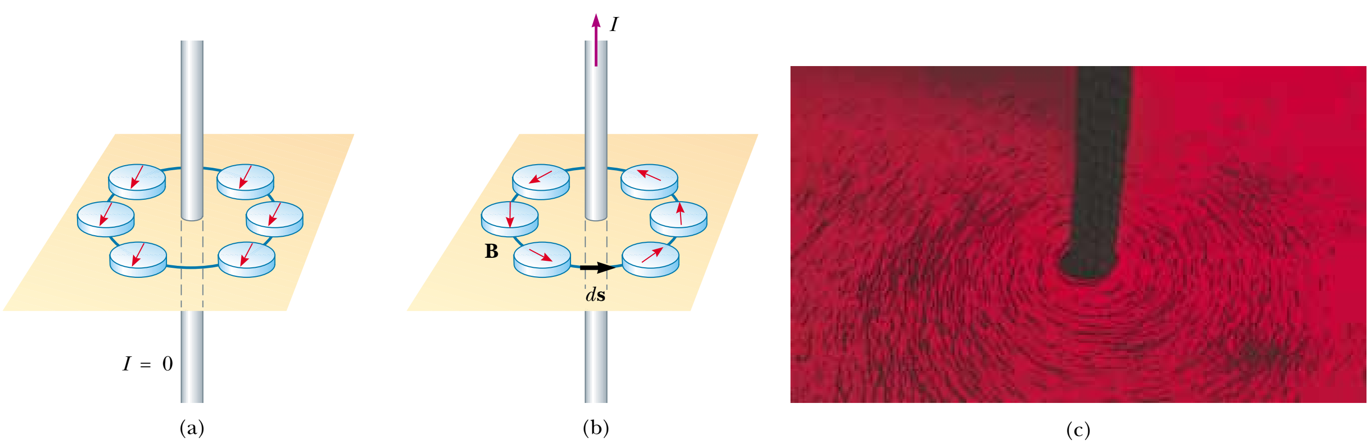


Figure 30.8 (a) When no current is present in the wire, all compass needles point in the same direction (toward the Earth's north pole). (b) When the wire carries a strong current, the compass needles deflect in a direction tangent to the circle, which is the direction of the magnetic field created by the current. (c) Circular magnetic field lines surrounding a current-carrying conductor, displayed with iron filings.

for a closed path of *any* shape surrounding a *current* that exists in an unbroken circuit. The general case, known as **Ampère's law,** can be stated as follows:

Ampère's law

The line integral of $\mathbf{B} \cdot d\mathbf{s}$ around any closed path equals $\mu_0 I$, where I is the total continuous current passing through any surface bounded by the closed path.

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I \quad \textbf{(30.13)}$$

Ampère's law describes the creation of magnetic fields by all continuous current configurations, but at our mathematical level it is useful only for calculating the magnetic field of current configurations having a high degree of symmetry. Its use is similar to that of Gauss's law in calculating electric fields for highly symmetric charge distributions.

Quick Quiz 30.3

Rank the magnitudes of $\oint \mathbf{B} \cdot d\mathbf{s}$ for the closed paths in Figure 30.9, from least to greatest.

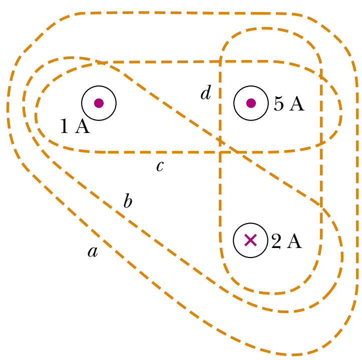


Figure 30.9 Four closed paths around three current-carrying wires.

Quick Quiz 30.4

Rank the magnitudes of $\oint \mathbf{B} \cdot d\mathbf{s}$ for the closed paths in Figure 30.10, from least to greatest.

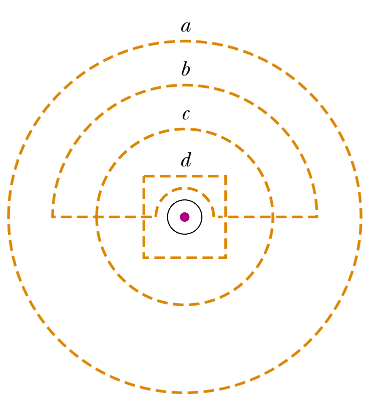


Figure 30.10 Several closed paths near a single current-carrying wire.

EXAMPLE 30.4 The Magnetic Field Created by a Long Current-Carrying Wire

A long, straight wire of radius R carries a steady current I_0 that is uniformly distributed through the cross-section of the wire (Fig. 30.11). Calculate the magnetic field a distance r from the center of the wire in the regions $r \geq R$ and $r < R$.

Solution For the $r \geq R$ case, we should get the same result we obtained in Example 30.1, in which we applied the Biot–Savart law to the same situation. Let us choose for our path of integration circle 1 in Figure 30.11. From symmetry, $\mathbf{B}$ must be constant in magnitude and parallel to $d\mathbf{s}$ at every point on this circle. Because the total current passing through the plane of the circle is I_0, Ampère's law gives

$$\oint \mathbf{B} \cdot d\mathbf{s} = B \oint ds = B(2\pi r) = \mu_0 I_0$$

$$\boxed{B = \frac{\mu_0 I_0}{2\pi r}} \qquad (\text{for } r \geq R) \qquad \textbf{(30.14)}$$

which is identical in form to Equation 30.5. Note how much easier it is to use Ampère's law than to use the Biot–Savart law. This is often the case in highly symmetric situations.

Now consider the interior of the wire, where $r < R$. Here the current I passing through the plane of circle 2 is less than the total current I_0. Because the current is uniform over the cross-section of the wire, the fraction of the current enclosed by circle 2 must equal the ratio of the area πr^2 enclosed by circle 2 to the cross-sectional area πR^2 of the wire:[2]

$$\frac{I}{I_0} = \frac{\pi r^2}{\pi R^2}$$

$$I = \frac{r^2}{R^2} I_0$$

Following the same procedure as for circle 1, we apply Ampère's law to circle 2:

$$\oint \mathbf{B} \cdot d\mathbf{s} = B(2\pi r) = \mu_0 I = \mu_0\left(\frac{r^2}{R^2} I_0\right)$$

$$\boxed{B = \left(\frac{\mu_0 I_0}{2\pi R^2}\right) r} \qquad (\text{for } r < R) \qquad \textbf{(30.15)}$$

This result is similar in form to the expression for the electric field inside a uniformly charged sphere (see Example 24.5). The magnitude of the magnetic field versus r for this configuration is plotted in Figure 30.12. Note that inside the wire, $B \rightarrow 0$ as $r \rightarrow 0$. Note also that Equations 30.14 and 30.15 give the same value of the magnetic field at $r = R$, demonstrating that the magnetic field is continuous at the surface of the wire.

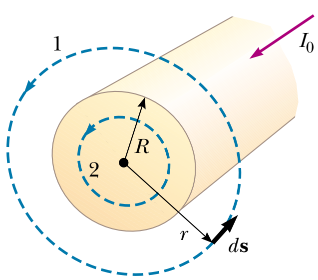


Figure 30.11 A long, straight wire of radius R carrying a steady current I_0 uniformly distributed across the cross-section of the wire. The magnetic field at any point can be calculated from Ampère's law using a circular path of radius r, concentric with the wire.

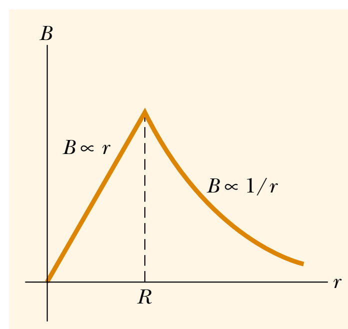


Figure 30.12 Magnitude of the magnetic field versus r for the wire shown in Figure 30.11. The field is proportional to r inside the wire and varies as $1/r$ outside the wire.

EXAMPLE 30.5 The Magnetic Field Created by a Toroid

A device called a *toroid* (Fig. 30.13) is often used to create an almost uniform magnetic field in some enclosed area. The device consists of a conducting wire wrapped around a ring (a *torus*) made of a nonconducting material. For a toroid having N closely spaced turns of wire, calculate the magnetic field in the region occupied by the torus, a distance r from the center.

[2] Another way to look at this problem is to see that the current enclosed by circle 2 must equal the product of the current density $J = I_0/\pi R^2$ and the area πr^2 of this circle.

Solution To calculate this field, we must evaluate $\oint \mathbf{B} \cdot d\mathbf{s}$ over the circle of radius r in Figure 30.13. By symmetry, we see that the magnitude of the field is constant on this circle and tangent to it, so $\mathbf{B} \cdot d\mathbf{s} = B\,ds$. Furthermore, note that the circular closed path surrounds N loops of wire, each of which carries a current I. Therefore, the right side of Equation 30.13 is $\mu_0 NI$ in this case.

Ampère's law applied to the circle gives

$$\oint \mathbf{B} \cdot d\mathbf{s} = B \oint ds = B(2\pi r) = \mu_0 NI$$

$$\boxed{B = \frac{\mu_0 NI}{2\pi r}} \qquad \textbf{(30.16)}$$

This result shows that B varies as $1/r$ and hence is nonuniform in the region occupied by the torus. However, if r is very large compared with the cross-sectional radius of the torus, then the field is approximately uniform inside the torus.

For an ideal toroid, in which the turns are closely spaced, the external magnetic field is zero. This can be seen by noting that the net current passing through any circular path lying outside the toroid (including the region of the "hole in the doughnut") is zero. Therefore, from Ampère's law we find that $B = 0$ in the regions exterior to the torus.

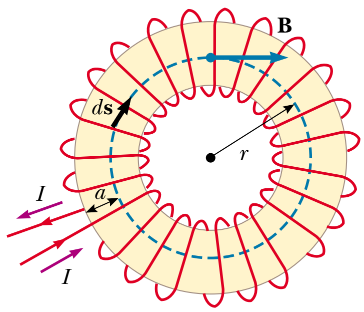


Figure 30.13 A toroid consisting of many turns of wire. If the turns are closely spaced, the magnetic field in the interior of the torus (the gold-shaded region) is tangent to the dashed circle and varies as $1/r$. The field outside the toroid is zero. The dimension a is the cross-sectional radius of the torus.

EXAMPLE 30.6 Magnetic Field Created by an Infinite Current Sheet

So far we have imagined currents through wires of small cross-section. Let us now consider an example in which a current exists in an extended object. A thin, infinitely large sheet lying in the yz plane carries a current of linear current density $\mathbf{J}_s$. The current is in the y direction, and J_s represents the current per unit length measured along the z axis. Find the magnetic field near the sheet.

Solution This situation brings to mind similar calculations involving Gauss's law (see Example 24.8). You may recall that the electric field due to an infinite sheet of charge does not depend on distance from the sheet. Thus, we might expect a similar result here for the magnetic field.

To evaluate the line integral in Ampère's law, let us take a rectangular path through the sheet, as shown in Figure 30.14. The rectangle has dimensions ℓ and w, with the sides of length ℓ parallel to the sheet surface. The net current passing through the plane of the rectangle is $J_s\ell$. We apply Ampère's law over the rectangle and note that the two sides of length w do not contribute to the line integral because the component of $\mathbf{B}$ along the direction of these paths is zero. By symmetry, we can argue that the magnetic field is constant over the sides of length ℓ because every point on the infinitely large sheet is equivalent, and hence the field should not vary from point to point. The only choices of field direction that are reasonable for the symmetry are perpendicular or parallel to the sheet, and a perpendicular field would pass *through* the current, which is inconsistent with the Biot–Savart law. Assuming a field that is constant in magnitude and parallel to the plane of the sheet, we obtain

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I = \mu_0 J_s \ell$$

$$2B\ell = \mu_0 J_s \ell$$

$$\boxed{B = \mu_0 \frac{J_s}{2}}$$

This result shows that *the magnetic field is independent of distance from the current sheet,* as we suspected.

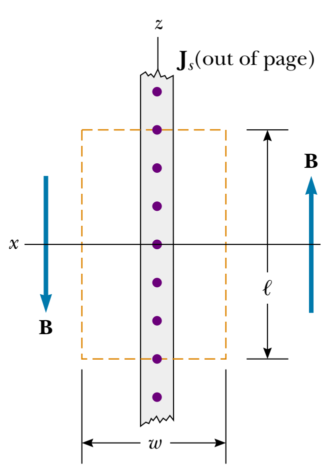


Figure 30.14 End view of an infinite current sheet lying in the yz plane, where the current is in the y direction (out of the page). This view shows the direction of $\mathbf{B}$ on both sides of the sheet.

EXAMPLE 30.7 The Magnetic Force on a Current Segment

Wire 1 in Figure 30.15 is oriented along the y axis and carries a steady current I_1. A rectangular loop located to the right of the wire and in the xy plane carries a current I_2. Find the magnetic force exerted by wire 1 on the top wire of length b in the loop, labeled "Wire 2" in the figure.

Solution You may be tempted to use Equation 30.12 to obtain the force exerted on a small segment of length dx of wire 2. However, this equation applies only to two *parallel* wires and cannot be used here. The correct approach is to consider the force exerted by wire 1 on a small segment $d\mathbf{s}$ of wire 2 by using Equation 29.4. This force is given by $d\mathbf{F}_B = I\,d\mathbf{s} \times \mathbf{B}$, where $I = I_2$ and $\mathbf{B}$ is the magnetic field created by the current in wire 1 at the position of $d\mathbf{s}$. From Ampère's law, the field at a distance x from wire 1 (see Eq. 30.14) is

$$\mathbf{B} = \frac{\mu_0 I_1}{2\pi x}(-\mathbf{k})$$

where the unit vector $-\mathbf{k}$ is used to indicate that the field at $d\mathbf{s}$ points into the page. Because wire 2 is along the x axis, $d\mathbf{s} = dx\,\mathbf{i}$, and we find that

$$d\mathbf{F}_B = \frac{\mu_0 I_1 I_2}{2\pi x}[\mathbf{i} \times (-\mathbf{k})]\,dx = \frac{\mu_0 I_1 I_2}{2\pi}\frac{dx}{x}\,\mathbf{j}$$

Integrating over the limits $x = a$ to $x = a + b$ gives

$$\mathbf{F}_B = \frac{\mu_0 I_1 I_2}{2\pi}\ln x\Big]_a^{a+b}\mathbf{j} = \boxed{\frac{\mu_0 I_1 I_2}{2\pi}\ln\left(1 + \frac{b}{a}\right)\mathbf{j}}$$

The force points in the positive y direction, as indicated by the unit vector $\mathbf{j}$ and as shown in Figure 30.15.

Exercise What are the magnitude and direction of the force exerted on the bottom wire of length b?

Answer The force has the same magnitude as the force on wire 2 but is directed downward.

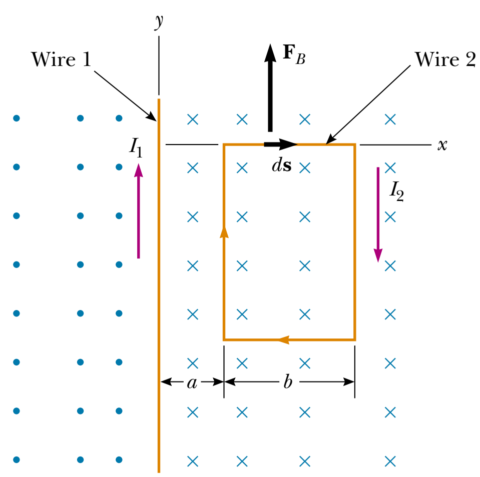


Figure 30.15

Quick Quiz 30.5

Is a net force acting on the current loop in Example 30.7? A net torque?

30.4 THE MAGNETIC FIELD OF A SOLENOID

A **solenoid** is a long wire wound in the form of a helix. With this configuration, a reasonably uniform magnetic field can be produced in the space surrounded by the turns of wire—which we shall call the *interior* of the solenoid—when the solenoid carries a current. When the turns are closely spaced, each can be approximated as a circular loop, and the net magnetic field is the vector sum of the fields resulting from all the turns.

Figure 30.16 shows the magnetic field lines surrounding a loosely wound solenoid. Note that the field lines in the interior are nearly parallel to one another, are uniformly distributed, and are close together, indicating that the field in this space is uniform and strong. The field lines between current elements on two adjacent turns tend to cancel each other because the field vectors from the two elements are in opposite directions. The field at exterior points such as P is weak because the field due to current elements on the right-hand portion of a turn tends to cancel the field due to current elements on the left-hand portion.

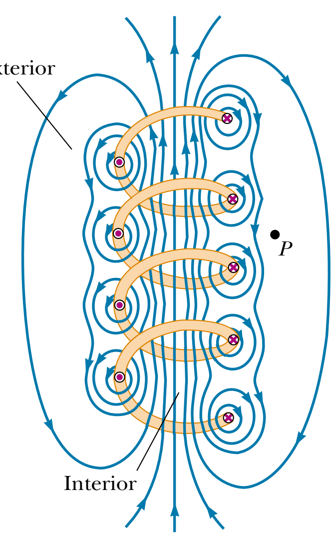


Figure 30.16 The magnetic field lines for a loosely wound solenoid.

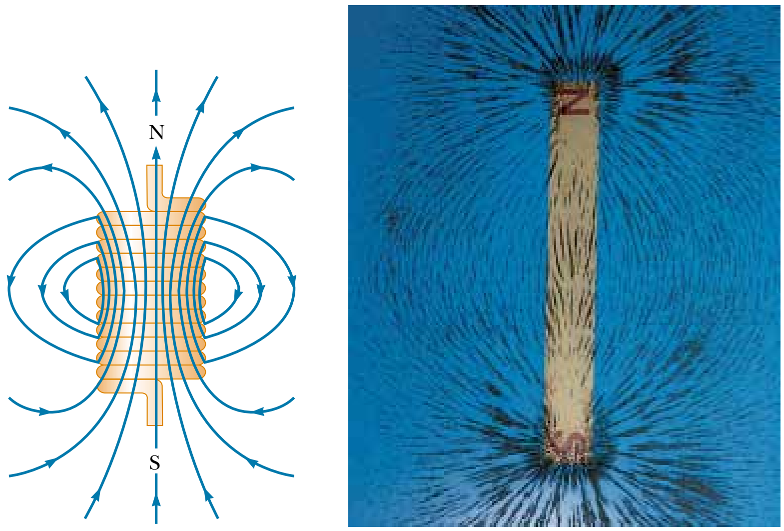


Figure 30.17 (a) Magnetic field lines for a tightly wound solenoid of finite length, carrying a steady current. The field in the interior space is nearly uniform and strong. Note that the field lines resemble those of a bar magnet, meaning that the solenoid effectively has north and south poles. (b) The magnetic field pattern of a bar magnet, displayed with small iron filings on a sheet of paper.

A technician studies the scan of a patient's head. The scan was obtained using a medical diagnostic technique known as magnetic resonance imaging (MRI). This instrument makes use of strong magnetic fields produced by superconducting solenoids.

If the turns are closely spaced and the solenoid is of finite length, the magnetic field lines are as shown in Figure 30.17a. This field line distribution is similar to that surrounding a bar magnet (see Fig. 30.17b). Hence, one end of the solenoid behaves like the north pole of a magnet, and the opposite end behaves like the south pole. As the length of the solenoid increases, the interior field becomes more uniform and the exterior field becomes weaker. An *ideal solenoid* is approached when the turns are closely spaced and the length is much greater than the radius of the turns. In this case, the external field is zero, and the interior field is uniform over a great volume.

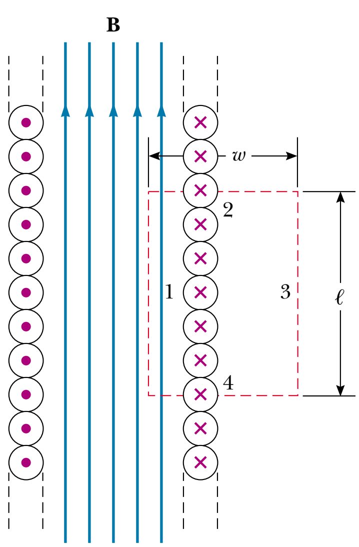


Figure 30.18 Cross-sectional view of an ideal solenoid, where the interior magnetic field is uniform and the exterior field is zero. Ampère's law applied to the red dashed path can be used to calculate the magnitude of the interior field.

We can use Ampère's law to obtain an expression for the interior magnetic field in an ideal solenoid. Figure 30.18 shows a longitudinal cross-section of part of such a solenoid carrying a current I. Because the solenoid is ideal, $\mathbf{B}$ in the interior space is uniform and parallel to the axis, and $\mathbf{B}$ in the exterior space is zero. Consider the rectangular path of length ℓ and width w shown in Figure 30.18. We can apply Ampère's law to this path by evaluating the integral of $\mathbf{B} \cdot d\mathbf{s}$ over each side of the rectangle. The contribution along side 3 is zero because $B = 0$ in this region. The contributions from sides 2 and 4 are both zero because $\mathbf{B}$ is perpendicular to $d\mathbf{s}$ along these paths. Side 1 gives a contribution $B\ell$ to the integral because along this path $\mathbf{B}$ is uniform and parallel to $d\mathbf{s}$. The integral over the closed rectangular path is therefore

$$\oint \mathbf{B} \cdot d\mathbf{s} = \int_{\text{path } 1} \mathbf{B} \cdot d\mathbf{s} = B \int_{\text{path } 1} ds = B\ell$$

The right side of Ampère's law involves the total current passing through the area bounded by the path of integration. In this case, the total current through the rectangular path equals the current through each turn multiplied by the number of turns. If N is the number of turns in the length ℓ, the total current through the rectangle is NI. Therefore, Ampère's law applied to this path gives

$$\oint \mathbf{B} \cdot d\mathbf{s} = B\ell = \mu_0 NI$$

$$B = \mu_0 \frac{N}{\ell} I = \mu_0 n I \qquad \textbf{(30.17)}$$

Magnetic field inside a solenoid

where $n = N/\ell$ is the number of turns per unit length.

We also could obtain this result by reconsidering the magnetic field of a toroid (see Example 30.5). If the radius r of the torus in Figure 30.13 containing N turns is much greater than the toroid's cross-sectional radius a, a short section of the toroid approximates a solenoid for which $n = N/2\pi r$. In this limit, Equation 30.16 agrees with Equation 30.17.

Equation 30.17 is valid only for points near the center (that is, far from the ends) of a very long solenoid. As you might expect, the field near each end is smaller than the value given by Equation 30.17. At the very end of a long solenoid, the magnitude of the field is one-half the magnitude at the center.

QuickLab

Wrap a few turns of wire around a compass, essentially putting the compass inside a solenoid. Hold the ends of the wire to the two terminals of a flashlight battery. What happens to the compass? Is the effect as strong when the compass is outside the turns of wire?

web

For a more detailed discussion of the magnetic field along the axis of a solenoid, visit **www.saunderscollege.com/physics/**

30.5 MAGNETIC FLUX

12.5 The flux associated with a magnetic field is defined in a manner similar to that used to define electric flux (see Eq. 24.3). Consider an element of area dA on an arbitrarily shaped surface, as shown in Figure 30.19. If the magnetic field at this element is $\mathbf{B}$, the magnetic flux through the element is $\mathbf{B} \cdot d\mathbf{A}$, where $d\mathbf{A}$ is a vector that is perpendicular to the surface and has a magnitude equal to the area dA. Hence, the total magnetic flux Φ_B through the surface is

$$\Phi_B \equiv \int \mathbf{B} \cdot d\mathbf{A} \qquad \textbf{(30.18)}$$

Definition of magnetic flux

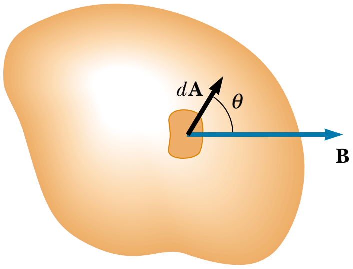


Figure 30.19 The magnetic flux through an area element dA is $\mathbf{B}\cdot d\mathbf{A} = BdA\cos\theta$, where $d\mathbf{A}$ is a vector perpendicular to the surface.

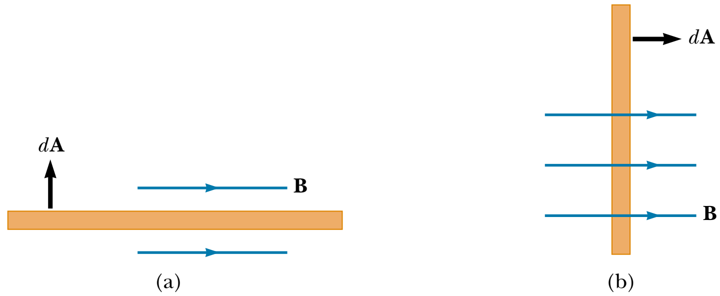


Figure 30.20 Magnetic flux through a plane lying in a magnetic field. (a) The flux through the plane is zero when the magnetic field is parallel to the plane surface. (b) The flux through the plane is a maximum when the magnetic field is perpendicular to the plane.

Consider the special case of a plane of area A in a uniform field $\mathbf{B}$ that makes an angle θ with $d\mathbf{A}$. The magnetic flux through the plane in this case is

$$\Phi_B = BA\cos\theta \tag{30.19}$$

If the magnetic field is parallel to the plane, as in Figure 30.20a, then $\theta = 90°$ and the flux is zero. If the field is perpendicular to the plane, as in Figure 30.20b, then $\theta = 0$ and the flux is BA (the maximum value).

The unit of flux is the $\mathrm{T\cdot m^2}$, which is defined as a *weber* (Wb); $1\ \mathrm{Wb} = 1\ \mathrm{T\cdot m^2}$.

EXAMPLE 30.8 Magnetic Flux Through a Rectangular Loop

A rectangular loop of width a and length b is located near a long wire carrying a current I (Fig. 30.21). The distance between the wire and the closest side of the loop is c. The wire is parallel to the long side of the loop. Find the total magnetic flux through the loop due to the current in the wire.

Solution From Equation 30.14, we know that the magnitude of the magnetic field created by the wire at a distance r from the wire is

$$B = \frac{\mu_0 I}{2\pi r}$$

The factor $1/r$ indicates that the field varies over the loop, and Figure 30.21 shows that the field is directed into the page. Because $\mathbf{B}$ is parallel to $d\mathbf{A}$ at any point within the loop, the magnetic flux through an area element dA is

$$\Phi_B = \int B\,dA = \int \frac{\mu_0 I}{2\pi r}\,dA$$

(Because B is not uniform but depends on r, it cannot be removed from the integral.)

To integrate, we first express the area element (the tan region in Fig. 30.21) as $dA = b\,dr$. Because r is now the only variable in the integral, we have

$$\Phi_B = \frac{\mu_0 Ib}{2\pi}\int_c^{a+c}\frac{dr}{r} = \frac{\mu_0 Ib}{2\pi}\ln r\Big]_c^{a+c}$$

$$= \frac{\mu_0 Ib}{2\pi}\ln\left(\frac{a+c}{c}\right) = \boxed{\frac{\mu_0 Ib}{2\pi}\ln\left(1+\frac{a}{c}\right)}$$

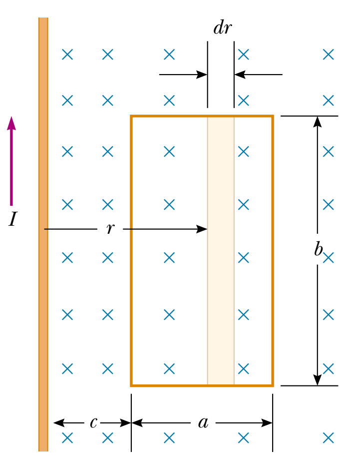


Figure 30.21 The magnetic field due to the wire carrying a current I is not uniform over the rectangular loop.

Exercise Apply the series expansion formula for $\ln(1+x)$ (see Appendix B.5) to this equation to show that it gives a reasonable result when the loop is far from the wire relative to the loop dimensions (in other words, when $c \gg a$).

Answer $\Phi_B \rightarrow 0$.

30.6 GAUSS'S LAW IN MAGNETISM

12.5 In Chapter 24 we found that the electric flux through a closed surface surrounding a net charge is proportional to that charge (Gauss's law). In other words, the number of electric field lines leaving the surface depends only on the net charge within it. This property is based on the fact that electric field lines originate and terminate on electric charges.

The situation is quite different for magnetic fields, which are continuous and form closed loops. In other words, magnetic field lines do not begin or end at any point—as illustrated by the magnetic field lines of the bar magnet in Figure 30.22. Note that for any closed surface, such as the one outlined by the dashed red line in Figure 30.22, the number of lines entering the surface equals the number leaving the surface; thus, the net magnetic flux is zero. In contrast, for a closed surface surrounding one charge of an electric dipole (Fig. 30.23), the net electric flux is not zero.

Gauss's law in magnetism states that

the net magnetic flux through any closed surface is always zero:

$$\oint \mathbf{B} \cdot d\mathbf{A} = 0 \quad (30.20)$$

Gauss's law for magnetism

This statement is based on the experimental fact, mentioned in the opening of Chapter 29, that **isolated magnetic poles (monopoles) have never been detected and perhaps do not exist.** Nonetheless, scientists continue the search be-

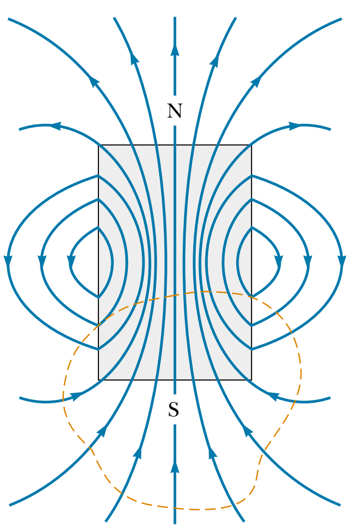


Figure 30.22 The magnetic field lines of a bar magnet form closed loops. Note that the net magnetic flux through the closed surface (dashed red line) surrounding one of the poles (or any other closed surface) is zero.

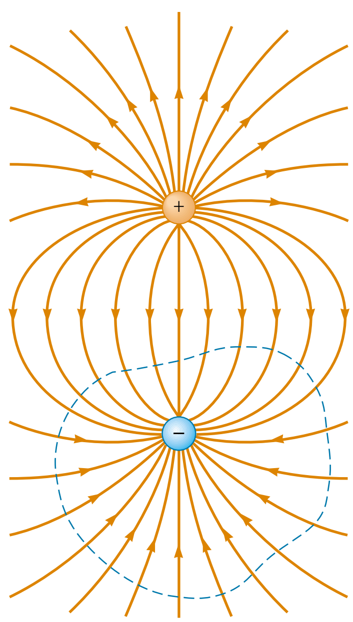


Figure 30.23 The electric field lines surrounding an electric dipole begin on the positive charge and terminate on the negative charge. The electric flux through a closed surface surrounding one of the charges is not zero.

cause certain theories that are otherwise successful in explaining fundamental physical behavior suggest the possible existence of monopoles.

30.7 DISPLACEMENT CURRENT AND THE GENERAL FORM OF AMPÈRE'S LAW

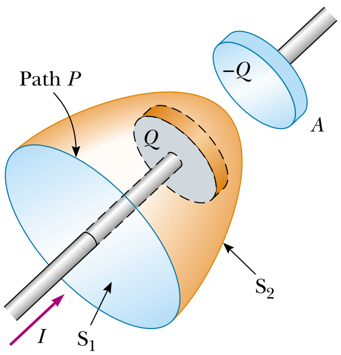


Figure 30.24 Two surfaces S_1 and S_2 near the plate of a capacitor are bounded by the same path P. The conduction current in the wire passes only through S_1. This leads to a contradiction in Ampère's law that is resolved only if one postulates a displacement current through S_2.

We have seen that charges in motion produce magnetic fields. When a current-carrying conductor has high symmetry, we can use Ampère's law to calculate the magnetic field it creates. In Equation 30.13, $\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I$, the line integral is over any closed path through which the conduction current passes, and the conduction current is defined by the expression $I = dq/dt$. (In this section we use the term *conduction current* to refer to the current carried by the wire, to distinguish it from a new type of current that we shall introduce shortly.) We now show that **Ampère's law in this form is valid only if any electric fields present are constant in time.** Maxwell recognized this limitation and modified Ampère's law to include time-varying electric fields.

We can understand the problem by considering a capacitor that is being charged as illustrated in Figure 30.24. When a conduction current is present, the charge on the positive plate changes but *no conduction current passes across the gap between the plates.* Now consider the two surfaces S_1 and S_2 in Figure 30.24, bounded by the same path P. Ampère's law states that $\oint \mathbf{B} \cdot d\mathbf{s}$ around this path must equal $\mu_0 I$, where I is the total current through any surface bounded by the path P.

12.9 When the path P is considered as bounding S_1, $\oint \mathbf{B} \cdot d\mathbf{s}$ is $\mu_0 I$ because the conduction current passes through S_1. When the path is considered as bounding S_2, however, $\oint \mathbf{B} \cdot d\mathbf{s} = 0$ because no conduction current passes through S_2. Thus, we arrive at a contradictory situation that arises from the discontinuity of the current! Maxwell solved this problem by postulating an additional term on the right side of Equation 30.13, which includes a factor called the **displacement current** I_d, defined as[3]

Displacement current

$$I_d \equiv \epsilon_0 \frac{d\Phi_E}{dt} \tag{30.21}$$

where ϵ_0 is the permittivity of free space (see Section 23.3) and $\Phi_E = \int \mathbf{E} \cdot d\mathbf{A}$ is the electric flux (see Eq. 24.3).

As the capacitor is being charged (or discharged), the changing electric field between the plates may be considered equivalent to a current that acts as a continuation of the conduction current in the wire. When the expression for the displacement current given by Equation 30.21 is added to the conduction current on the right side of Ampère's law, the difficulty represented in Figure 30.24 is resolved. No matter which surface bounded by the path P is chosen, either conduction current or displacement current passes through it. With this new term I_d, we can express the general form of Ampère's law (sometimes called the **Ampère–Maxwell law**) as[4]

Ampère–Maxwell law

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0(I + I_d) = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \tag{30.22}$$

[3] *Displacement* in this context does not have the meaning it does in Chapter 2. Despite the inaccurate implications, the word is historically entrenched in the language of physics, so we continue to use it.

[4] Strictly speaking, this expression is valid only in a vacuum. If a magnetic material is present, one must change μ_0 and ϵ_0 on the right-hand side of Equation 30.22 to the permeability μ_m and permittivity ϵ characteristic of the material. Alternatively, one may include a magnetizing current I_m on the righthand side of Equation 30.22 to make Ampère's law fully general. On a microscopic scale, I_m is as real as I.

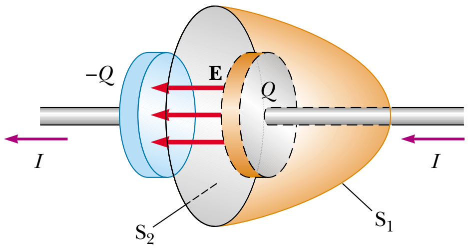


Figure 30.25 Because it exists only in the wires attached to the capacitor plates, the conduction current $I = dQ/dt$ passes through S_1 but not through S_2. Only the displacement current $I_d = \epsilon_0\, d\Phi_E/dt$ passes through S_2. The two currents must be equal for continuity.

We can understand the meaning of this expression by referring to Figure 30.25. The electric flux through surface S_2 is $\Phi_E = \int \mathbf{E} \cdot d\mathbf{A} = EA$, where A is the area of the capacitor plates and E is the magnitude of the uniform electric field between the plates. If Q is the charge on the plates at any instant, then $E = Q/\epsilon_0 A$ (see Section 26.2). Therefore, the electric flux through S_2 is simply

$$\Phi_E = EA = \frac{Q}{\epsilon_0}$$

Hence, the displacement current through S_2 is

$$I_d = \epsilon_0 \frac{d\Phi_E}{dt} = \frac{dQ}{dt} \tag{30.23}$$

That is, the displacement current through S_2 is precisely equal to the conduction current I through S_1!

By considering surface S_2, we can identify the displacement current as the source of the magnetic field on the surface boundary. The displacement current has its physical origin in the time-varying electric field. The central point of this formalism, then, is that

magnetic fields are produced both by conduction currents and by time-varying electric fields.

This result was a remarkable example of theoretical work by Maxwell, and it contributed to major advances in the understanding of electromagnetism.

Quick Quiz 30.6

What is the displacement current for a fully charged 3-μF capacitor?

Example 30.9 Displacement Current in a Capacitor

A sinusoidally varying voltage is applied across an 8.00-μF capacitor. The frequency of the voltage is 3.00 kHz, and the voltage amplitude is 30.0 V. Find the displacement current between the plates of the capacitor.

Solution The angular frequency of the source, from Equation 13.6, is $\omega = 2\pi f = 2\pi(3.00 \times 10^3 \text{ Hz}) = 1.88 \times 10^4 \text{ s}^{-1}$. Hence, the voltage across the capacitor in terms of t is

$$\Delta V = \Delta V_{\max} \sin \omega t = (30.0 \text{ V}) \sin(1.88 \times 10^4 t)$$

We can use Equation 30.23 and the fact that the charge on the capacitor is $Q = C\,\Delta V$ to find the displacement current:

$$I_d = \frac{dQ}{dt} = \frac{d}{dt}(C\,\Delta V) = C\frac{d}{dt}(\Delta V)$$

$$= (8.00 \times 10^{-6} \text{ F})\frac{d}{dt}[(30.0 \text{ V}) \sin(1.88 \times 10^4 t)]$$

$$= \boxed{(4.52 \text{ A}) \cos(1.88 \times 10^4 t)}$$

The displacement current varies sinusoidally with time and has a maximum value of 4.52 A.

Optional Section

30.8 MAGNETISM IN MATTER

The magnetic field produced by a current in a coil of wire gives us a hint as to what causes certain materials to exhibit strong magnetic properties. Earlier we found that a coil like the one shown in Figure 30.17 has a north pole and a south pole. In general, *any* current loop has a magnetic field and thus has a magnetic dipole moment, including the atomic-level current loops described in some models of the atom. Thus, the magnetic moments in a magnetized substance may be described as arising from these atomic-level current loops. For the Bohr model of the atom, these current loops are associated with the movement of electrons around the nucleus in circular orbits. In addition, a magnetic moment is intrinsic to electrons, protons, neutrons, and other particles; it arises from a property called *spin*.

The Magnetic Moments of Atoms

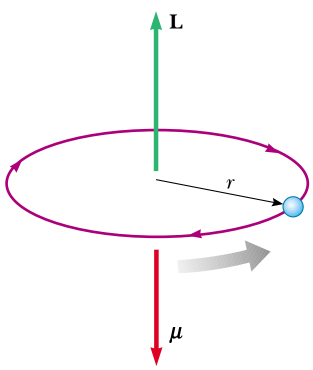


Figure 30.26 An electron moving in a circular orbit of radius r has an angular momentum $\mathbf{L}$ in one direction and a magnetic moment $\boldsymbol{\mu}$ in the opposite direction.

It is instructive to begin our discussion with a classical model of the atom in which electrons move in circular orbits around the much more massive nucleus. In this model, an orbiting electron constitutes a tiny current loop (because it is a moving charge), and the magnetic moment of the electron is associated with this orbital motion. Although this model has many deficiencies, its predictions are in good agreement with the correct theory, which is expressed in terms of quantum physics.

Consider an electron moving with constant speed v in a circular orbit of radius r about the nucleus, as shown in Figure 30.26. Because the electron travels a distance of $2\pi r$ (the circumference of the circle) in a time T, its orbital speed is $v = 2\pi r/T$. The current I associated with this orbiting electron is its charge e divided by T. Using $T = 2\pi/\omega$ and $\omega = v/r$, we have

$$I = \frac{e}{T} = \frac{e\omega}{2\pi} = \frac{ev}{2\pi r}$$

The magnetic moment associated with this current loop is $\mu = IA$, where $A = \pi r^2$ is the area enclosed by the orbit. Therefore,

$$\mu = IA = \left(\frac{ev}{2\pi r}\right)\pi r^2 = \tfrac{1}{2}evr \qquad \textbf{(30.24)}$$

Because the magnitude of the orbital angular momentum of the electron is $L = m_e vr$ (Eq. 11.16 with $\phi = 90°$), the magnetic moment can be written as

Orbital magnetic moment

$$\mu = \left(\frac{e}{2m_e}\right)L \qquad \textbf{(30.25)}$$

This result demonstrates that **the magnetic moment of the electron is proportional to its orbital angular momentum.** Note that because the electron is negatively charged, the vectors $\boldsymbol{\mu}$ and $\mathbf{L}$ point in opposite directions. Both vectors are perpendicular to the plane of the orbit, as indicated in Figure 30.26.

Angular momentum is quantized

A fundamental outcome of quantum physics is that orbital angular momentum is quantized and is equal to multiples of $\hbar = h/2\pi = 1.05 \times 10^{-34}\ \text{J}\cdot\text{s}$, where h is Planck's constant. The smallest nonzero value of the electron's magnetic moment resulting from its orbital motion is

$$\mu = \sqrt{2}\,\frac{e}{2m_e}\,\hbar \qquad \textbf{(30.26)}$$

We shall see in Chapter 42 how expressions such as Equation 30.26 arise.

Because all substances contain electrons, you may wonder why not all substances are magnetic. The main reason is that in most substances, the magnetic moment of one electron in an atom is canceled by that of another electron orbiting in the opposite direction. The net result is that, for most materials, **the magnetic effect produced by the orbital motion of the electrons is either zero or very small.**

In addition to its orbital magnetic moment, an electron has an intrinsic property called **spin** that also contributes to its magnetic moment. In this regard, the electron can be viewed as spinning about its axis while it orbits the nucleus, as shown in Figure 30.27. (Warning: This classical description should not be taken literally because spin arises from relativistic dynamics that must be incorporated into a quantum-mechanical analysis.) The magnitude of the angular momentum S associated with spin is of the same order of magnitude as the angular momentum L due to the orbital motion. The magnitude of the spin angular momentum predicted by quantum theory is

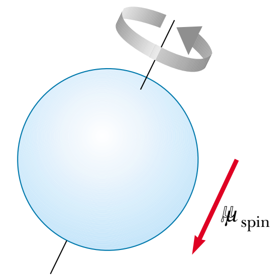


Figure 30.27 Classical model of a spinning electron. This model gives an incorrect magnitude for the magnetic moment, incorrect quantum numbers, and too many degrees of freedom.

$$S = \frac{\sqrt{3}}{2}\hbar$$

Spin angular momentum

The magnetic moment characteristically associated with the spin of an electron has the value

$$\mu_{\text{spin}} = \frac{e\hbar}{2m_e} \tag{30.27}$$

This combination of constants is called the **Bohr magneton:**

$$\mu_{\text{B}} = \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24}\ \text{J/T} \tag{30.28}$$

Bohr magneton

Thus, atomic magnetic moments can be expressed as multiples of the Bohr magneton. (Note that $1\ \text{J/T} = 1\ \text{A}\cdot\text{m}^2$.)

In atoms containing many electrons, the electrons usually pair up with their spins opposite each other; thus, the spin magnetic moments cancel. However, atoms containing an odd number of electrons must have at least one unpaired electron and therefore some spin magnetic moment. The total magnetic moment of an atom is the vector sum of the orbital and spin magnetic moments, and a few examples are given in Table 30.1. Note that helium and neon have zero moments because their individual spin and orbital moments cancel.

TABLE 30.1
Magnetic Moments of Some Atoms and Ions

Atom or Ion	Magnetic Moment (10^{-24} J/T)
H	9.27
He	0
Ne	0
Ce^{3+}	19.8
Yb^{3+}	37.1

The nucleus of an atom also has a magnetic moment associated with its constituent protons and neutrons. However, the magnetic moment of a proton or neutron is much smaller than that of an electron and can usually be neglected. We can understand this by inspecting Equation 30.28 and replacing the mass of the electron with the mass of a proton or a neutron. Because the masses of the proton and neutron are much greater than that of the electron, their magnetic moments are on the order of 10^3 times smaller than that of the electron.

Magnetization Vector and Magnetic Field Strength

The magnetic state of a substance is described by a quantity called the **magnetization vector M. The magnitude of this vector is defined as the magnetic moment per unit volume of the substance.** As you might expect, the total magnetic field **B** at a point within a substance depends on both the applied (external) field $\mathbf{B}_0$ and the magnetization of the substance.

Magnetization vector **M**

To understand the problems involved in measuring the total magnetic field **B** in such situations, consider this: Scientists use small probes that utilize the Hall ef-

fect (see Section 29.6) to measure magnetic fields. What would such a probe read if it were positioned inside the solenoid mentioned in the QuickLab on page 951 when you inserted the compass? Because the compass is a magnetic material, the probe would measure a total magnetic field $\mathbf{B}$ that is the sum of the solenoid (external) field $\mathbf{B}_0$ and the (magnetization) field $\mathbf{B}_m$ due to the compass. This tells us that we need a way to distinguish between magnetic fields originating from currents and those originating from magnetic materials. Consider a region in which a magnetic field $\mathbf{B}_0$ is produced by a current-carrying conductor. If we now fill that region with a magnetic substance, the total magnetic field $\mathbf{B}$ in the region is $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_m$, where $\mathbf{B}_m$ is the field produced by the magnetic substance. We can express this contribution in terms of the magnetization vector of the substance as $\mathbf{B}_m = \mu_0\mathbf{M}$; hence, the total magnetic field in the region becomes

$$\mathbf{B} = \mathbf{B}_0 + \mu_0\mathbf{M} \tag{30.29}$$

Magnetic field strength **H**

When analyzing magnetic fields that arise from magnetization, it is convenient to introduce a field quantity, called the **magnetic field strength H** within the substance. The magnetic field strength represents the effect of the conduction currents in wires on a substance. To emphasize the distinction between the field strength $\mathbf{H}$ and the field $\mathbf{B}$, the latter is often called the *magnetic flux density* or the *magnetic induction.* The magnetic field strength is a vector defined by the relationship $\mathbf{H} = \mathbf{B}_0/\mu_0 = (\mathbf{B}/\mu_0) - \mathbf{M}$. Thus, Equation 30.29 can be written

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \tag{30.30}$$

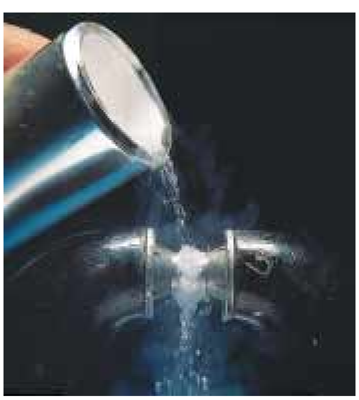
The quantities $\mathbf{H}$ and $\mathbf{M}$ have the same units. In SI units, because $\mathbf{M}$ is magnetic moment per unit volume, the units are (ampere)(meter)^2/(meter)^3, or amperes per meter.

To better understand these expressions, consider the torus region of a toroid that carries a current I. If this region is a vacuum, $\mathbf{M} = 0$ (because no magnetic material is present), the total magnetic field is that arising from the current alone, and $\mathbf{B} = \mathbf{B}_0 = \mu_0\mathbf{H}$. Because $B_0 = \mu_0 nI$ in the torus region, where n is the number of turns per unit length of the toroid, $H = B_0/\mu_0 = \mu_0 nI/\mu_0$, or

$$H = nI \tag{30.31}$$

In this case, the magnetic field B in the torus region is due only to the current in the windings of the toroid.

If the torus is now made of some substance and the current I is kept constant, $\mathbf{H}$ in the torus region remains unchanged (because it depends on the current only) and has magnitude nI. The total field $\mathbf{B}$, however, is different from that when the torus region was a vacuum. From Equation 30.30, we see that part of $\mathbf{B}$ arises from the term $\mu_0\mathbf{H}$ associated with the current in the toroid, and part arises from the term $\mu_0\mathbf{M}$ due to the magnetization of the substance of which the torus is made.



Oxygen, a paramagnetic substance, is attracted to a magnetic field. The liquid oxygen in this photograph is suspended between the poles of the magnet.

Classification of Magnetic Substances

Substances can be classified as belonging to one of three categories, depending on their magnetic properties. **Paramagnetic** and **ferromagnetic** materials are those made of atoms that have permanent magnetic moments. **Diamagnetic** materials are those made of atoms that do not have permanent magnetic moments.

For paramagnetic and diamagnetic substances, the magnetization vector $\mathbf{M}$ is proportional to the magnetic field strength $\mathbf{H}$. For these substances placed in an external magnetic field, we can write

$$\mathbf{M} = \chi\mathbf{H} \tag{30.32}$$

TABLE 30.2 Magnetic Susceptibilities of Some Paramagnetic and Diamagnetic Substances at 300 K

Paramagnetic Substance	χ	Diamagnetic Substance	χ
Aluminum	2.3×10^{-5}	Bismuth	-1.66×10^{-5}
Calcium	1.9×10^{-5}	Copper	-9.8×10^{-6}
Chromium	2.7×10^{-4}	Diamond	-2.2×10^{-5}
Lithium	2.1×10^{-5}	Gold	-3.6×10^{-5}
Magnesium	1.2×10^{-5}	Lead	-1.7×10^{-5}
Niobium	2.6×10^{-4}	Mercury	-2.9×10^{-5}
Oxygen	2.1×10^{-6}	Nitrogen	-5.0×10^{-9}
Platinum	2.9×10^{-4}	Silver	-2.6×10^{-5}
Tungsten	6.8×10^{-5}	Silicon	-4.2×10^{-6}

where χ (Greek letter chi) is a dimensionless factor called the **magnetic susceptibility.** For paramagnetic substances, χ is positive and **M** is in the same direction as **H**. For diamagnetic substances, χ is negative and **M** is opposite **H**. (It is important to note that this linear relationship between **M** and **H** does not apply to ferromagnetic substances.) The susceptibilities of some substances are given in Table 30.2.

Magnetic susceptibility χ

Substituting Equation 30.32 for **M** into Equation 30.30 gives

$$\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = \mu_0(\mathbf{H} + \chi\mathbf{H}) = \mu_0(1 + \chi)\mathbf{H}$$

or

$$\mathbf{B} = \mu_m\mathbf{H} \tag{30.33}$$

where the constant μ_m is called the **magnetic permeability** of the substance and is related to the susceptibility by

$$\mu_m = \mu_0(1 + \chi) \tag{30.34}$$

Magnetic permeability μ_m

Substances may be classified in terms of how their magnetic permeability μ_m compares with μ_0 (the permeability of free space), as follows:

Paramagnetic	$\mu_m > \mu_0$
Diamagnetic	$\mu_m < \mu_0$

Because χ is very small for paramagnetic and diamagnetic substances (see Table 30.2), μ_m is nearly equal to μ_0 for these substances. For ferromagnetic substances, however, μ_m is typically several thousand times greater than μ_0 (meaning that χ is very great for ferromagnetic substances).

Although Equation 30.33 provides a simple relationship between **B** and **H**, we must interpret it with care when dealing with ferromagnetic substances. As mentioned earlier, **M** is not a linear function of **H** for ferromagnetic substances. This is because the value of μ_m is not only a characteristic of the ferromagnetic substance but also depends on the previous state of the substance and on the process it underwent as it moved from its previous state to its present one. We shall investigate this more deeply after the following example.

EXAMPLE 30.10 An Iron-Filled Toroid

A toroid wound with 60.0 turns/m of wire carries a current of 5.00 A. The torus is iron, which has a magnetic permeability of $\mu_m = 5\,000\mu_0$ under the given conditions. Find H and B inside the iron.

Solution Using Equations 30.31 and 30.33, we obtain

$$H = nI = \left(60.0\ \frac{\text{turns}}{\text{m}}\right)(5.00\ \text{A}) = \boxed{300\ \frac{\text{A}\cdot\text{turns}}{\text{m}}}$$

$$B = \mu_m H = 5\,000\mu_0 H$$
$$= 5\,000\left(4\pi \times 10^{-7}\ \frac{\text{T}\cdot\text{m}}{\text{A}}\right)\left(300\ \frac{\text{A}\cdot\text{turns}}{\text{m}}\right) = \boxed{1.88\ \text{T}}$$

This value of B is 5 000 times the value in the absence of iron!

Exercise Determine the magnitude of the magnetization vector inside the iron torus.

Answer $M = 1.5 \times 10^6\ \text{A/m}$.

Quick Quiz 30.7

A current in a solenoid having air in the interior creates a magnetic field $\mathbf{B} = \mu_0\mathbf{H}$. Describe qualitatively what happens to the magnitude of $\mathbf{B}$ as (a) aluminum, (b) copper, and (c) iron are placed in the interior.

Ferromagnetism

A small number of crystalline substances in which the atoms have permanent magnetic moments exhibit strong magnetic effects called **ferromagnetism.** Some examples of ferromagnetic substances are iron, cobalt, nickel, gadolinium, and dysprosium. These substances contain atomic magnetic moments that tend to align parallel to each other even in a weak external magnetic field. Once the moments are aligned, the substance remains magnetized after the external field is removed. This permanent alignment is due to a strong coupling between neighboring moments, a coupling that can be understood only in quantum-mechanical terms.

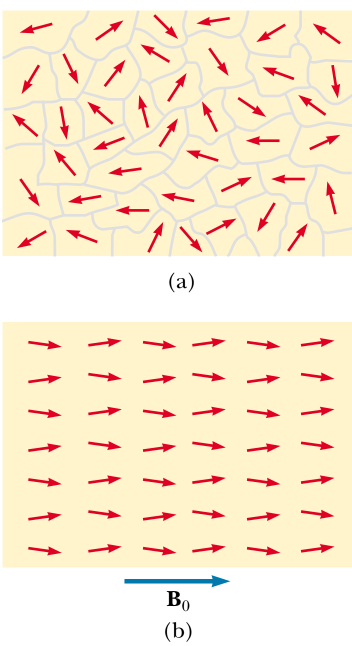


Figure 30.28 (a) Random orientation of atomic magnetic moments in an unmagnetized substance. (b) When an external field $\mathbf{B}_0$ is applied, the atomic magnetic moments tend to align with the field, giving the sample a net magnetization vector $\mathbf{M}$.

All ferromagnetic materials are made up of microscopic regions called **domains,** regions within which all magnetic moments are aligned. These domains have volumes of about 10^{-12} to 10^{-8} m^3 and contain 10^{17} to 10^{21} atoms. The boundaries between the various domains having different orientations are called **domain walls.** In an unmagnetized sample, the domains are randomly oriented so that the net magnetic moment is zero, as shown in Figure 30.28a. When the sample is placed in an external magnetic field, the magnetic moments of the atoms tend to align with the field, which results in a magnetized sample, as in Figure 30.28b. Observations show that domains initially oriented along the external field grow larger at the expense of the less favorably oriented domains. When the external field is removed, the sample may retain a net magnetization in the direction of the original field. At ordinary temperatures, thermal agitation is not sufficient to disrupt this preferred orientation of magnetic moments.

A typical experimental arrangement that is used to measure the magnetic properties of a ferromagnetic material consists of a torus made of the material wound with N turns of wire, as shown in Figure 30.29, where the windings are represented in black and are referred to as the *primary coil*. This apparatus is sometimes referred to as a **Rowland ring.** A *secondary coil* (the red wires in Fig. 30.29) connected to a galvanometer is used to measure the total magnetic flux through the torus. The magnetic field $\mathbf{B}$ in the torus is measured by increasing the current in the toroid from zero to I. As the current changes, the magnetic flux through

the secondary coil changes by an amount BA, where A is the cross-sectional area of the toroid. As we shall find in Chapter 31, because of this changing flux, an emf that is proportional to the rate of change in magnetic flux is induced in the secondary coil. If the galvanometer is properly calibrated, a value for **B** corresponding to any value of the current in the primary coil can be obtained. The magnetic field **B** is measured first in the absence of the torus and then with the torus in place. The magnetic properties of the torus material are then obtained from a comparison of the two measurements.

Now consider a torus made of unmagnetized iron. If the current in the primary coil is increased from zero to some value I, the magnitude of the magnetic field strength H increases linearly with I according to the expression $H = nI$. Furthermore, the magnitude of the total field B also increases with increasing current, as shown by the curve from point O to point a in Figure 30.30. At point O, the domains in the iron are randomly oriented, corresponding to $B_m = 0$. As the increasing current in the primary coil causes the external field $\mathbf{B}_0$ to increase, the domains become more aligned until all of them are nearly aligned at point a. At this point the iron core is approaching *saturation,* which is the condition in which all domains in the iron are aligned.

Next, suppose that the current is reduced to zero, and the external field is consequently eliminated. The B versus H curve, called a **magnetization curve,** now follows the path ab in Figure 30.30. Note that at point b, **B** is not zero even though the external field is $\mathbf{B}_0 = 0$. The reason is that the iron is now magnetized due to the alignment of a large number of its domains (that is, $\mathbf{B} = \mathbf{B}_m$). At this point, the iron is said to have a *remanent* magnetization.

If the current in the primary coil is reversed so that the direction of the external magnetic field is reversed, the domains reorient until the sample is again unmagnetized at point c, where $B = 0$. An increase in the reverse current causes the iron to be magnetized in the opposite direction, approaching saturation at point d in Figure 30.30. A similar sequence of events occurs as the current is reduced to zero and then increased in the original (positive) direction. In this case the magnetization curve follows the path def. If the current is increased sufficiently, the magnetization curve returns to point a, where the sample again has its maximum magnetization.

The effect just described, called **magnetic hysteresis,** shows that the magnetization of a ferromagnetic substance depends on the history of the substance as well as on the magnitude of the applied field. (The word *hysteresis* means "lagging behind.") It is often said that a ferromagnetic substance has a "memory" because it remains magnetized after the external field is removed. The closed loop in Figure 30.30 is referred to as a hysteresis loop. Its shape and size depend on the proper-

QuickLab

You've probably done this experiment before. Magnetize a nail by repeatedly dragging it across a bar magnet. Test the strength of the nail's magnetic field by picking up some paper clips. Now hit the nail several times with a hammer, and again test the strength of its magnetism. Explain what happens in terms of domains in the steel of the nail.

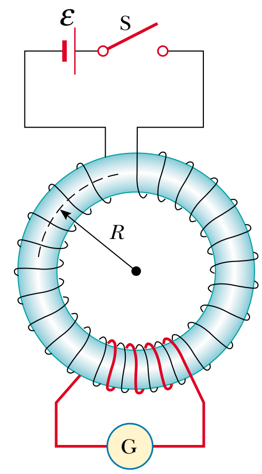


Figure 30.29 A toroidal winding arrangement used to measure the magnetic properties of a material. The torus is made of the material under study, and the circuit containing the galvanometer measures the magnetic flux.

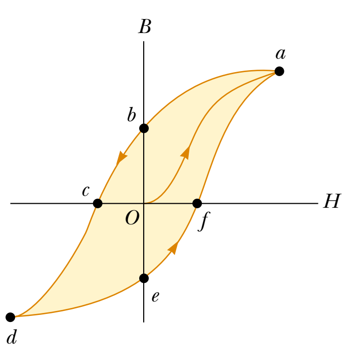


Figure 30.30 Magnetization curve for a ferromagnetic material.

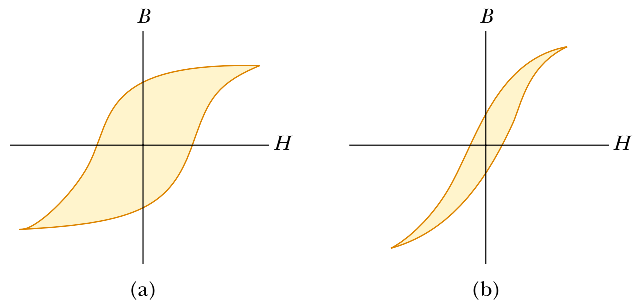


Figure 30.31 Hysteresis loops for (a) a hard ferromagnetic material and (b) a soft ferromagnetic material.

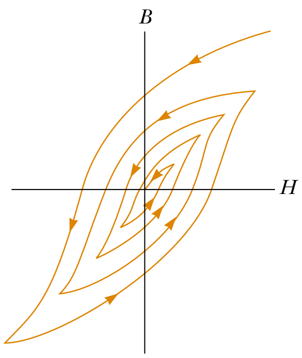


Figure 30.32 Demagnetizing a ferromagnetic material by carrying it through successive hysteresis loops.

ties of the ferromagnetic substance and on the strength of the maximum applied field. The hysteresis loop for "hard" ferromagnetic materials is characteristically wide like the one shown in Figure 30.31a, corresponding to a large remanent magnetization. Such materials cannot be easily demagnetized by an external field. "Soft" ferromagnetic materials, such as iron, have a very narrow hysteresis loop and a small remanent magnetization (Fig. 30.31b.) Such materials are easily magnetized and demagnetized. An ideal soft ferromagnet would exhibit no hysteresis and hence would have no remanent magnetization. A ferromagnetic substance can be demagnetized by being carried through successive hysteresis loops, due to a decreasing applied magnetic field, as shown in Figure 30.32.

Quick Quiz 30.8

Which material would make a better permanent magnet, one whose hysteresis loop looks like Figure 30.31a or one whose loop looks like Figure 30.31b?

The magnetization curve is useful for another reason: **The area enclosed by the magnetization curve represents the work required to take the material through the hysteresis cycle.** The energy acquired by the material in the magnetization process originates from the source of the external field—that is, the emf in the circuit of the toroidal coil. When the magnetization cycle is repeated, dissipative processes within the material due to realignment of the domains result in a transformation of magnetic energy into internal energy, which is evidenced by an increase in the temperature of the substance. For this reason, devices subjected to alternating fields (such as ac adapters for cell phones, power tools, and so on) use cores made of soft ferromagnetic substances, which have narrow hysteresis loops and correspondingly little energy loss per cycle.

Magnetic computer disks store information by alternating the direction of **B** for portions of a thin layer of ferromagnetic material. Floppy disks have the layer on a circular sheet of plastic. Hard disks have several rigid platters with magnetic coatings on each side. Audio tapes and videotapes work the same way as floppy disks except that the ferromagnetic material is on a very long strip of plastic. Tiny coils of wire in a recording head are placed close to the magnetic material (which is moving rapidly past the head). Varying the current through the coils creates a magnetic field that magnetizes the recording material. To retrieve the information, the magnetized material is moved past a playback coil. The changing magnetism of the material induces a current in the coil, as we shall discuss in Chapter 31. This current is then amplified by audio or video equipment, or it is processed by computer circuitry.

Paramagnetism

Paramagnetic substances have a small but positive magnetic susceptibility ($0 < \chi \ll 1$) resulting from the presence of atoms (or ions) that have permanent magnetic moments. These moments interact only weakly with each other and are randomly oriented in the absence of an external magnetic field. When a paramagnetic substance is placed in an external magnetic field, its atomic moments tend to line up with the field. However, this alignment process must compete with thermal motion, which tends to randomize the magnetic moment orientations.

Pierre Curie (1859–1906) and others since him have found experimentally that, under a wide range of conditions, the magnetization of a paramagnetic substance is proportional to the applied magnetic field and inversely proportional to the absolute temperature:

$$M = C\frac{B_0}{T} \tag{30.35}$$

This relationship is known as **Curie's law** after its discoverer, and the constant C is called **Curie's constant.** The law shows that when $B_0 = 0$, the magnetization is zero, corresponding to a random orientation of magnetic moments. As the ratio of magnetic field to temperature becomes great, the magnetization approaches its saturation value, corresponding to a complete alignment of its moments, and Equation 30.35 is no longer valid.

When the temperature of a ferromagnetic substance reaches or exceeds a critical temperature called the **Curie temperature,** the substance loses its residual magnetization and becomes paramagnetic (Fig. 30.33). Below the Curie temperature, the magnetic moments are aligned and the substance is ferromagnetic. Above the Curie temperature, the thermal agitation is great enough to cause a random orientation of the moments, and the substance becomes paramagnetic. Curie temperatures for several ferromagnetic substances are given in Table 30.3.

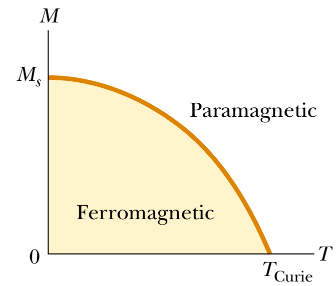


Figure 30.33 Magnetization versus absolute temperature for a ferromagnetic substance. The magnetic moments are aligned below the Curie temperature T_{Curie}, where the substance is ferromagnetic. The substance becomes paramagnetic (magnetic moments unaligned) above T_{Curie}.

TABLE 30.3
Curie Temperatures for Several Ferromagnetic Substances

Substance	T_{Curie} (K)
Iron	1 043
Cobalt	1 394
Nickel	631
Gadolinium	317
Fe_2O_3	893

Diamagnetism

When an external magnetic field is applied to a diamagnetic substance, a weak magnetic moment is induced in the direction opposite the applied field. This causes diamagnetic substances to be weakly repelled by a magnet. Although diamagnetism is present in all matter, its effects are much smaller than those of paramagnetism or ferromagnetism, and are evident only when those other effects do not exist.

We can attain some understanding of diamagnetism by considering a classical model of two atomic electrons orbiting the nucleus in opposite directions but with the same speed. The electrons remain in their circular orbits because of the attractive electrostatic force exerted by the positively charged nucleus. Because the magnetic moments of the two electrons are equal in magnitude and opposite in direction, they cancel each other, and the magnetic moment of the atom is zero. When an external magnetic field is applied, the electrons experience an additional force $q\mathbf{v} \times \mathbf{B}$. This added force combines with the electrostatic force to increase the orbital speed of the electron whose magnetic moment is antiparallel to the field and to decrease the speed of the electron whose magnetic moment is parallel to the field. As a result, the two magnetic moments of the electrons no longer cancel, and the substance acquires a net magnetic moment that is opposite the applied field.

web

Visit **www.exploratorium.edu/snacks/diamagnetism_www/index.html** for an experiment showing that grapes are repelled by magnets!

Figure 30.34 A small permanent magnet levitated above a disk of the superconductor $YBa_2Cu_3O_7$ cooled to liquid nitrogen temperature (77 K).

web
For a more detailed description of the unusual properties of superconductors, visit **www.saunderscollege.com/physics/**

As you recall from Chapter 27, a superconductor is a substance in which the electrical resistance is zero below some critical temperature. Certain types of superconductors also exhibit perfect diamagnetism in the superconducting state. As a result, an applied magnetic field is expelled by the superconductor so that the field is zero in its interior. This phenomenon of flux expulsion is known as the **Meissner effect.** If a permanent magnet is brought near a superconductor, the two objects repel each other. This is illustrated in Figure 30.34, which shows a small permanent magnet levitated above a superconductor maintained at 77 K.

EXAMPLE 30.11 Saturation Magnetization

Estimate the saturation magnetization in a long cylinder of iron, assuming one unpaired electron spin per atom.

Solution The saturation magnetization is obtained when all the magnetic moments in the sample are aligned. If the sample contains n atoms per unit volume, then the saturation magnetization M_s has the value

$$M_s = n\mu$$

where μ is the magnetic moment per atom. Because the molar mass of iron is 55 g/mol and its density is 7.9 g/cm^3, the value of n for iron is 8.6×10^{28} atoms/m^3. Assuming that each atom contributes one Bohr magneton (due to one unpaired spin) to the magnetic moment, we obtain

$$M_s = \left(8.6 \times 10^{28} \frac{\text{atoms}}{\text{m}^3}\right)\left(9.27 \times 10^{-24} \frac{\text{A}\cdot\text{m}^2}{\text{atom}}\right)$$

$$= \boxed{8.0 \times 10^5 \text{ A/m}}$$

This is about one-half the experimentally determined saturation magnetization for iron, which indicates that actually two unpaired electron spins are present per atom.

Optional Section

30.9 THE MAGNETIC FIELD OF THE EARTH

When we speak of a compass magnet having a north pole and a south pole, we should say more properly that it has a "north-seeking" pole and a "south-seeking" pole. By this we mean that one pole of the magnet seeks, or points to, the north geographic pole of the Earth. Because the north pole of a magnet is attracted toward the north geographic pole of the Earth, we conclude that **the Earth's south magnetic pole is located near the north geographic pole, and the Earth's north magnetic pole is located near the south geographic pole.** In fact, the configuration of the Earth's magnetic field, pictured in Figure 30.35, is very much like the one that would be achieved by burying a gigantic bar magnet deep in the interior of the Earth.

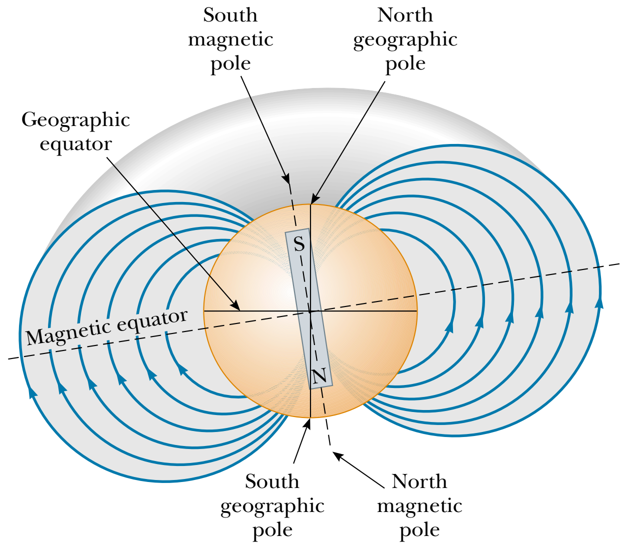
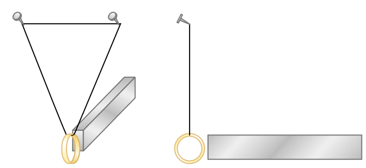


Figure 30.35 The Earth's magnetic field lines. Note that a south magnetic pole is near the north geographic pole, and a north magnetic pole is near the south geographic pole.

QuickLab

A gold ring is very weakly repelled by a magnet. To see this, suspend a 14- or 18-karat gold ring on a long loop of thread, as shown in (a). Gently tap the ring and estimate its period of oscillation. Now bring the ring to rest, letting it hang for a few moments so that you can verify that it is not moving. Quickly bring a very strong magnet to within a few millimeters of the ring, taking care not to bump it, as shown in (b). Now pull the magnet away. Repeat this action many times, matching the oscillation period you estimated earlier. This is just like pushing a child on a swing. A small force applied at the resonant frequency results in a large-amplitude oscillation. If you have a platinum ring, you will be able to see a similar effect except that platinum is weakly attracted to a magnet because it is paramagnetic.



If a compass needle is suspended in bearings that allow it to rotate in the vertical plane as well as in the horizontal plane, the needle is horizontal with respect to the Earth's surface only near the equator. As the compass is moved northward, the needle rotates so that it points more and more toward the surface of the Earth. Finally, at a point near Hudson Bay in Canada, the north pole of the needle points directly downward. This site, first found in 1832, is considered to be the location of the south magnetic pole of the Earth. It is approximately 1 300 mi from the Earth's geographic North Pole, and its exact position varies slowly with time. Similarly, the north magnetic pole of the Earth is about 1 200 mi away from the Earth's geographic South Pole.

Because of this distance between the north geographic and south magnetic poles, it is only approximately correct to say that a compass needle points north. The difference between true north, defined as the geographic North Pole, and north indicated by a compass varies from point to point on the Earth, and the difference is referred to as *magnetic declination.* For example, along a line through Florida and the Great Lakes, a compass indicates true north, whereas in Washington state, it aligns 25° east of true north.

The north end of a compass needle points to the *south* magnetic pole of the Earth. The "north" compass direction varies from true geographic north depending on the magnetic declination at that point on the Earth's surface.

Quick Quiz 30.9

If we wanted to cancel the Earth's magnetic field by running an enormous current loop around the equator, which way would the current have to flow: east to west or west to east?

Although the magnetic field pattern of the Earth is similar to the one that would be set up by a bar magnet deep within the Earth, it is easy to understand why the source of the Earth's magnetic field cannot be large masses of permanently magnetized material. The Earth does have large deposits of iron ore deep beneath its surface, but the high temperatures in the Earth's core prevent the iron from retaining any permanent magnetization. Scientists consider it more likely that the true source of the Earth's magnetic field is charge-carrying convection currents in the Earth's core. Charged ions or electrons circulating in the liquid interior could produce a magnetic field just as a current loop does. There is also strong evidence that the magnitude of a planet's magnetic field is related to the planet's rate of rotation. For example, Jupiter rotates faster than the Earth, and space probes indicate that Jupiter's magnetic field is stronger than ours. Venus, on the other hand, rotates more slowly than the Earth, and its magnetic field is found to be weaker. Investigation into the cause of the Earth's magnetism is ongoing.

There is an interesting sidelight concerning the Earth's magnetic field. It has been found that the direction of the field has been reversed several times during the last million years. Evidence for this is provided by basalt, a type of rock that contains iron and that forms from material spewed forth by volcanic activity on the ocean floor. As the lava cools, it solidifies and retains a picture of the Earth's magnetic field direction. The rocks are dated by other means to provide a timeline for these periodic reversals of the magnetic field.

SUMMARY

The **Biot–Savart law** says that the magnetic field $d\mathbf{B}$ at a point P due to a length element $d\mathbf{s}$ that carries a steady current I is

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{I\,d\mathbf{s} \times \hat{\mathbf{r}}}{r^2} \qquad \textbf{(30.1)}$$

where $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T\cdot m/A}$ is the **permeability of free space,** r is the distance from the element to the point P, and $\hat{\mathbf{r}}$ is a unit vector pointing from $d\mathbf{s}$ to point P. We find the total field at P by integrating this expression over the entire current distribution.

The magnetic field at a distance a from a long, straight wire carrying an electric current I is

$$B = \frac{\mu_0 I}{2\pi a} \qquad \textbf{(30.5)}$$

The field lines are circles concentric with the wire.

The magnetic force per unit length between two parallel wires separated by a distance a and carrying currents I_1 and I_2 has a magnitude

$$\frac{F_B}{\ell} = \frac{\mu_0 I_1 I_2}{2\pi a} \qquad \textbf{(30.12)}$$

The force is attractive if the currents are in the same direction and repulsive if they are in opposite directions.

Ampère's law says that the line integral of $\mathbf{B} \cdot d\mathbf{s}$ around any closed path equals $\mu_0 I$, where I is the total steady current passing through any surface bounded by the closed path:

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I \tag{30.13}$$

Using Ampère's law, one finds that the fields inside a toroid and solenoid are

$$B = \frac{\mu_0 N I}{2\pi r} \qquad \text{(toroid)} \tag{30.16}$$

$$B = \mu_0 \frac{N}{\ell} I = \mu_0 n I \qquad \text{(solenoid)} \tag{30.17}$$

where N is the total number of turns.

The **magnetic flux Φ_B** through a surface is defined by the surface integral

$$\Phi_B \equiv \int \mathbf{B} \cdot d\mathbf{A} \tag{30.18}$$

Gauss's law of magnetism states that the net magnetic flux through any closed surface is zero.

The general form of Ampère's law, which is also called the **Ampère-Maxwell law,** is

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \tag{30.22}$$

This law describes the fact that magnetic fields are produced both by conduction currents and by changing electric fields.

QUESTIONS

1. Is the magnetic field created by a current loop uniform? Explain.

2. A current in a conductor produces a magnetic field that can be calculated using the Biot–Savart law. Because current is defined as the rate of flow of charge, what can you conclude about the magnetic field produced by stationary charges? What about that produced by moving charges?

3. Two parallel wires carry currents in opposite directions. Describe the nature of the magnetic field created by the two wires at points (a) between the wires and (b) outside the wires, in a plane containing them.

4. Explain why two parallel wires carrying currents in opposite directions repel each other.

5. When an electric circuit is being assembled, a common practice is to twist together two wires carrying equal currents in opposite directions. Why does this technique reduce stray magnetic fields?

6. Is Ampère's law valid for all closed paths surrounding a conductor? Why is it not useful for calculating **B** for all such paths?

7. Compare Ampère's law with the Biot–Savart law. Which is more generally useful for calculating **B** for a current-carrying conductor?

8. Is the magnetic field inside a toroid uniform? Explain.

9. Describe the similarities between Ampère's law in magnetism and Gauss's law in electrostatics.

10. A hollow copper tube carries a current along its length. Why does $\mathbf{B} = 0$ inside the tube? Is **B** nonzero outside the tube?

11. Why is **B** nonzero outside a solenoid? Why does $\mathbf{B} = 0$ outside a toroid? (Remember that the lines of **B** must form closed paths.)

12. Describe the change in the magnetic field in the interior of a solenoid carrying a steady current I (a) if the length of the solenoid is doubled but the number of turns remains the same and (b) if the number of turns is doubled but the length remains the same.

13. A flat conducting loop is positioned in a uniform magnetic field directed along the x axis. For what orientation of the loop is the flux through it a maximum? A minimum?

14. What new concept does Maxwell's general form of Ampère's law include?

15. Many loops of wire are wrapped around a nail and then connected to a battery. Identify the source of **M**, of **H**, and of **B**.

16. A magnet attracts a piece of iron. The iron can then attract another piece of iron. On the basis of domain alignment, explain what happens in each piece of iron.

17. You are stranded on a planet that does not have a magnetic field, with no test equipment. You have two bars of iron in your possession; one is magnetized, and one is not. How can you determine which is which?

18. Why does hitting a magnet with a hammer cause the magnetism to be reduced?

19. Is a nail attracted to either pole of a magnet? Explain what is happening inside the nail when it is placed near the magnet.

20. A Hindu ruler once suggested that he be entombed in a magnetic coffin with the polarity arranged so that he would be forever suspended between heaven and Earth. Is such magnetic levitation possible? Discuss.

21. Why does **M** = 0 in a vacuum? What is the relationship between **B** and **H** in a vacuum?

22. Explain why some atoms have permanent magnetic moments and others do not.

23. What factors contribute to the total magnetic moment of an atom?

24. Why is the magnetic susceptibility of a diamagnetic substance negative?

25. Why can the effect of diamagnetism be neglected in a paramagnetic substance?

26. Explain the significance of the Curie temperature for a ferromagnetic substance.

27. Discuss the differences among ferromagnetic, paramagnetic, and diamagnetic substances.

28. What is the difference between hard and soft ferromagnetic materials?

29. Should the surface of a computer disk be made from a hard or a soft ferromagnetic substance?

30. Explain why it is desirable to use hard ferromagnetic materials to make permanent magnets.

31. Would you expect the tape from a tape recorder to be attracted to a magnet? (Try it, but not with a recording you wish to save.)

32. Given only a strong magnet and a screwdriver, how would you first magnetize and then demagnetize the screwdriver?

33. Figure Q30.33 shows two permanent magnets, each having a hole through its center. Note that the upper magnet is levitated above the lower one. (a) How does this occur? (b) What purpose does the pencil serve? (c) What can you say about the poles of the magnets on the basis of this observation? (d) What do you suppose would happen if the upper magnet were inverted?

Figure Q30.33 Magnetic levitation using two ceramic magnets.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 30.1 The Biot–Savart Law

1. In Niels Bohr's 1913 model of the hydrogen atom, an electron circles the proton at a distance of 5.29×10^{-11} m with a speed of 2.19×10^{6} m/s. Compute the magnitude of the magnetic field that this motion produces at the location of the proton.

2. A current path shaped as shown in Figure P30.2 produces a magnetic field at P, the center of the arc. If the arc subtends an angle of 30.0° and the radius of the arc is 0.600 m, what are the magnitude and direction of the field produced at P if the current is 3.00 A?

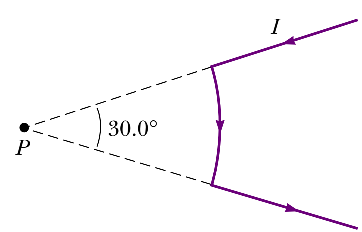


Figure P30.2

3. (a) A conductor in the shape of a square of edge length $\ell = 0.400$ m carries a current $I = 10.0$ A (Fig. P30.3). Calculate the magnitude and direction of the magnetic field at the center of the square. (b) If this conductor is formed into a single circular turn and carries the same current, what is the value of the magnetic field at the center?

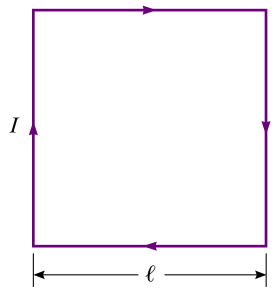


Figure P30.3

4. Calculate the magnitude of the magnetic field at a point 100 cm from a long, thin conductor carrying a current of 1.00 A.

WEB 5. Determine the magnetic field at a point P located a distance x from the corner of an infinitely long wire bent at a right angle, as shown in Figure P30.5. The wire carries a steady current I.

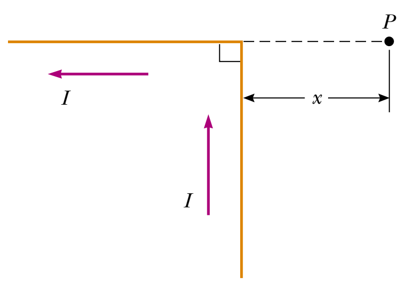


Figure P30.5

6. A wire carrying a current of 5.00 A is to be formed into a circular loop of one turn. If the required value of the magnetic field at the center of the loop is 10.0 μT, what is the required radius?

7. A conductor consists of a circular loop of radius R = 0.100 m and two straight, long sections, as shown in Figure P30.7. The wire lies in the plane of the paper and carries a current of I = 7.00 A. Determine the magnitude and direction of the magnetic field at the center of the loop.

8. A conductor consists of a circular loop of radius R and two straight, long sections, as shown in Figure P30.7. The wire lies in the plane of the paper and carries a current I. Determine the magnitude and direction of the magnetic field at the center of the loop.

9. The segment of wire in Figure P30.9 carries a current of I = 5.00 A, where the radius of the circular arc is R = 3.00 cm. Determine the magnitude and direction of the magnetic field at the origin.

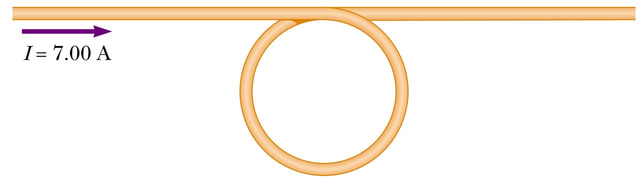


Figure P30.7 Problems 7 and 8.

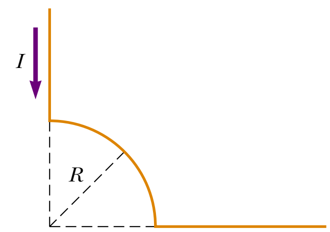


Figure P30.9

10. Consider a flat, circular current loop of radius R carrying current I. Choose the x axis to be along the axis of the loop, with the origin at the center of the loop. Graph the ratio of the magnitude of the magnetic field at coordinate x to that at the origin, for $x = 0$ to $x = 5R$. It may be helpful to use a programmable calculator or a computer to solve this problem.

11. Consider the current-carrying loop shown in Figure P30.11, formed of radial lines and segments of circles whose centers are at point P. Find the magnitude and direction of **B** at P.

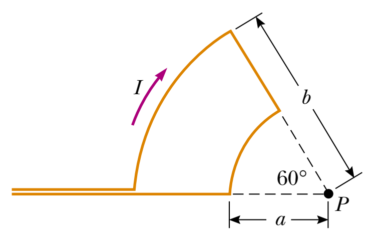


Figure P30.11

12. Determine the magnetic field (in terms of I, a, and d) at the origin due to the current loop shown in Figure P30.12.

13. The loop in Figure P30.13 carries a current I. Determine the magnetic field at point A in terms of I, R, and L.

14. Three long, parallel conductors carry currents of I = 2.00 A. Figure P30.14 is an end view of the conductors, with each current coming out of the page. If a = 1.00 cm, determine the magnitude and direction of the magnetic field at points A, B, and C.

15. Two long, parallel conductors carry currents I_1 = 3.00 A and I_2 = 3.00 A, both directed into the page in

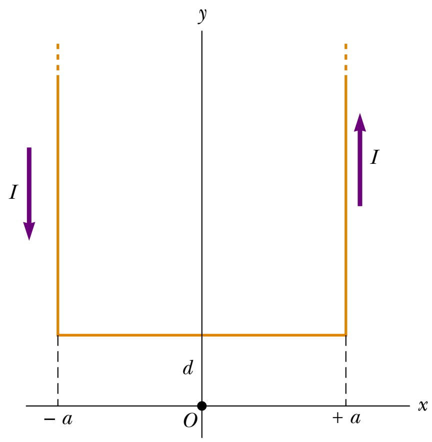


Figure P30.12

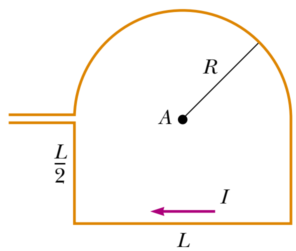


Figure P30.13

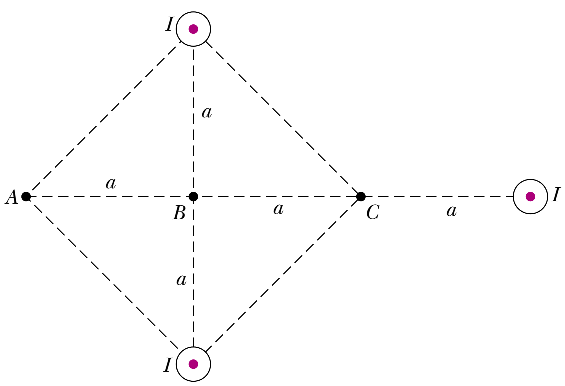


Figure P30.14

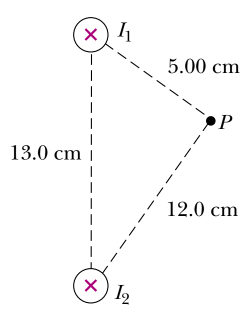


Figure P30.15

Figure P30.15. Determine the magnitude and direction of the resultant magnetic field at P.

Section 30.2 The Magnetic Force Between Two Parallel Conductors

16. Two long, parallel conductors separated by 10.0 cm carry currents in the same direction. The first wire carries current $I_1 = 5.00$ A, and the second carries $I_2 = 8.00$ A. (a) What is the magnitude of the magnetic field created by I_1 and acting on I_2? (b) What is the force per unit length exerted on I_2 by I_1? (c) What is the magnitude of the magnetic field created by I_2 at the location of I_1? (d) What is the force per unit length exerted by I_2 on I_1?

17. In Figure P30.17, the current in the long, straight wire is $I_1 = 5.00$ A, and the wire lies in the plane of the rectangular loop, which carries 10.0 A. The dimensions are $c = 0.100$ m, $a = 0.150$ m, and $\ell = 0.450$ m. Find the magnitude and direction of the net force exerted on the loop by the magnetic field created by the wire.

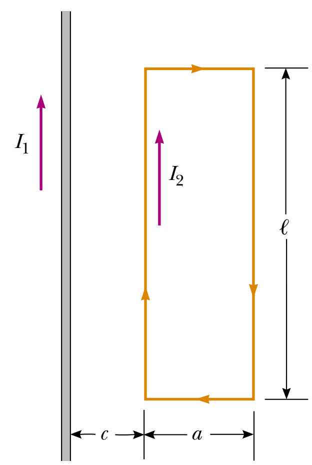


Figure P30.17

18. The unit of magnetic flux is named for Wilhelm Weber. The practical-size unit of magnetic field is named for Johann Karl Friedrich Gauss. Both were scientists at Göttingen, Germany. In addition to their individual accomplishments, they built a telegraph together in 1833. It consisted of a battery and switch that were positioned at one end of a transmission line 3 km long and operated an electromagnet at the other end. (Andre Ampère suggested electrical signaling in 1821; Samuel Morse built a telegraph line between Baltimore and Washington in 1844.) Suppose that Weber and Gauss's transmission line was as diagrammed in Figure P30.18. Two long, parallel wires, each having a mass per unit length of 40.0 g/m, are supported in a horizontal plane by strings 6.00 cm long. When both wires carry the same current I, the wires repel each other so that the angle θ

between the supporting strings is 16.0°. (a) Are the currents in the same direction or in opposite directions? (b) Find the magnitude of the current.

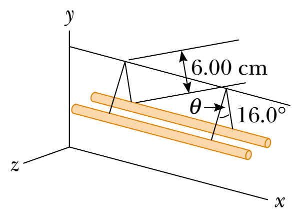


Figure P30.18

Section 30.3 Ampère's Law

WEB **19.** Four long, parallel conductors carry equal currents of $I = 5.00$ A. Figure P30.19 is an end view of the conductors. The direction of the current is into the page at points *A* and *B* (indicated by the crosses) and out of the page at *C* and *D* (indicated by the dots). Calculate the magnitude and direction of the magnetic field at point *P*, located at the center of the square with an edge length of 0.200 m.

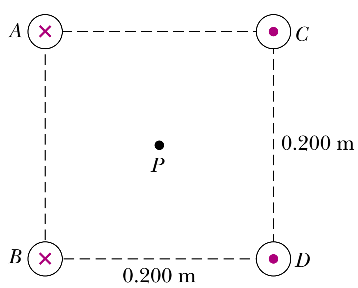


Figure P30.19

20. A long, straight wire lies on a horizontal table and carries a current of 1.20 μA. In a vacuum, a proton moves parallel to the wire (opposite the current) with a constant velocity of 2.30×10^4 m/s at a distance *d* above the wire. Determine the value of *d*. You may ignore the magnetic field due to the Earth.

21. Figure P30.21 is a cross-sectional view of a coaxial cable. The center conductor is surrounded by a rubber layer, which is surrounded by an outer conductor, which is surrounded by another rubber layer. In a particular application, the current in the inner conductor is 1.00 A out of the page, and the current in the outer conductor is 3.00 A into the page. Determine the magnitude and direction of the magnetic field at points *a* and *b*.

22. The magnetic field 40.0 cm away from a long, straight wire carrying current 2.00 A is 1.00 μT. (a) At what distance is it 0.100 μT? (b) At one instant, the two conductors in a long household extension cord carry equal

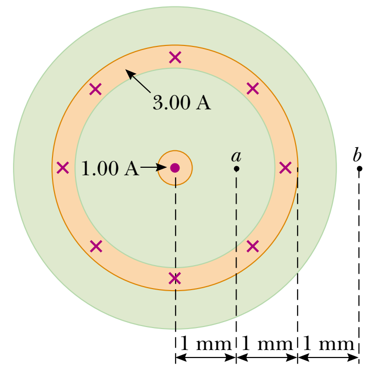


Figure P30.21

2.00-A currents in opposite directions. The two wires are 3.00 mm apart. Find the magnetic field 40.0 cm away from the middle of the straight cord, in the plane of the two wires. (c) At what distance is it one-tenth as large? (d) The center wire in a coaxial cable carries current 2.00 A in one direction, and the sheath around it carries current 2.00 A in the opposite direction. What magnetic field does the cable create at points outside?

23. The magnetic coils of a tokamak fusion reactor are in the shape of a toroid having an inner radius of 0.700 m and an outer radius of 1.30 m. If the toroid has 900 turns of large-diameter wire, each of which carries a current of 14.0 kA, find the magnitude of the magnetic field inside the toroid (a) along the inner radius and (b) along the outer radius.

24. A cylindrical conductor of radius $R = 2.50$ cm carries a current of $I = 2.50$ A along its length; this current is uniformly distributed throughout the cross-section of the conductor. (a) Calculate the magnetic field midway along the radius of the wire (that is, at $r = R/2$). (b) Find the distance beyond the surface of the conductor at which the magnitude of the magnetic field has the same value as the magnitude of the field at $r = R/2$.

WEB **25.** A packed bundle of 100 long, straight, insulated wires forms a cylinder of radius $R = 0.500$ cm. (a) If each wire carries 2.00 A, what are the magnitude and direction of the magnetic force per unit length acting on a wire located 0.200 cm from the center of the bundle? (b) Would a wire on the outer edge of the bundle experience a force greater or less than the value calculated in part (a)?

26. Niobium metal becomes a superconductor when cooled below 9 K. If superconductivity is destroyed when the surface magnetic field exceeds 0.100 T, determine the maximum current a 2.00-mm-diameter niobium wire can carry and remain superconducting, in the absence of any external magnetic field.

27. A long, cylindrical conductor of radius *R* carries a current *I*, as shown in Figure P30.27. The current density *J*, however, is not uniform over the cross-section of the

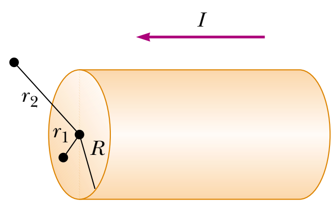


Figure P30.27

conductor but is a function of the radius according to $J = br$, where b is a constant. Find an expression for the magnetic field B (a) at a distance $r_1 < R$ and (b) at a distance $r_2 > R$, measured from the axis.

28. In Figure P30.28, both currents are in the negative x direction. (a) Sketch the magnetic field pattern in the yz plane. (b) At what distance d along the z axis is the magnetic field a maximum?

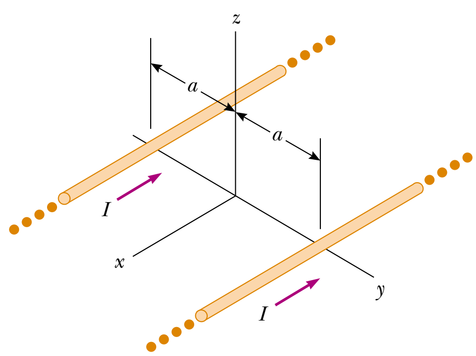


Figure P30.28

Section 30.4 The Magnetic Field of a Solenoid

WEB **29.** What current is required in the windings of a long solenoid that has 1 000 turns uniformly distributed over a length of 0.400 m, to produce at the center of the solenoid a magnetic field of magnitude 1.00×10^{-4} T?

30. A superconducting solenoid is meant to generate a magnetic field of 10.0 T. (a) If the solenoid winding has 2 000 turns/m, what current is required? (b) What force per unit length is exerted on the windings by this magnetic field?

31. A solenoid of radius $R = 5.00$ cm is made of a long piece of wire of radius $r = 2.00$ mm, length $\ell = 10.0$ m ($\ell \gg R$) and resistivity $\rho = 1.70 \times 10^{-8}\ \Omega \cdot \text{m}$. Find the magnetic field at the center of the solenoid if the wire is connected to a battery having an emf $\mathcal{E} = 20.0$ V.

32. A single-turn square loop of wire with an edge length of 2.00 cm carries a clockwise current of 0.200 A. The loop is inside a solenoid, with the plane of the loop perpendicular to the magnetic field of the solenoid. The solenoid has 30 turns/cm and carries a clockwise current of 15.0 A. Find the force on each side of the loop and the torque acting on the loop.

Section 30.5 Magnetic Flux

33. A cube of edge length $\ell = 2.50$ cm is positioned as shown in Figure P30.33. A uniform magnetic field given by $\mathbf{B} = (5.00\mathbf{i} + 4.00\mathbf{j} + 3.00\mathbf{k})$ T exists throughout the region. (a) Calculate the flux through the shaded face. (b) What is the total flux through the six faces?

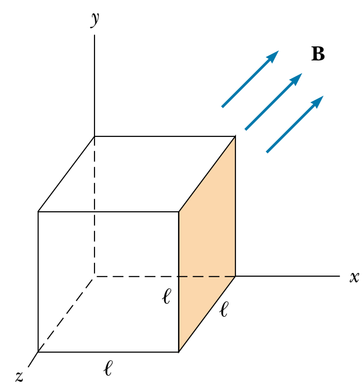


Figure P30.33

34. A solenoid 2.50 cm in diameter and 30.0 cm long has 300 turns and carries 12.0 A. (a) Calculate the flux through the surface of a disk of radius 5.00 cm that is positioned perpendicular to and centered on the axis of the solenoid, as in Figure P30.34a. (b) Figure P30.34b shows an enlarged end view of the same solenoid. Calculate the flux through the blue area, which is defined by an annulus that has an inner radius of 0.400 cm and outer radius of 0.800 cm.

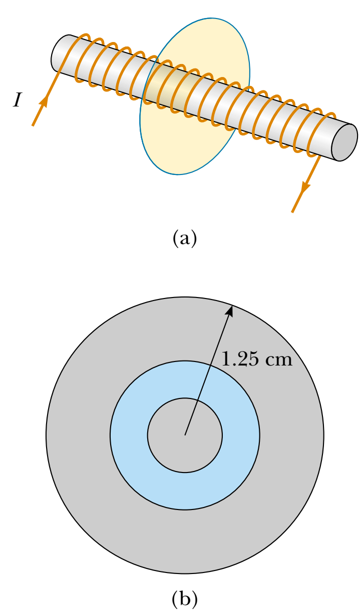


Figure P30.34

35. Consider the hemispherical closed surface in Figure P30.35. If the hemisphere is in a uniform magnetic field that makes an angle θ with the vertical, calculate the magnetic flux (a) through the flat surface S_1 and (b) through the hemispherical surface S_2.

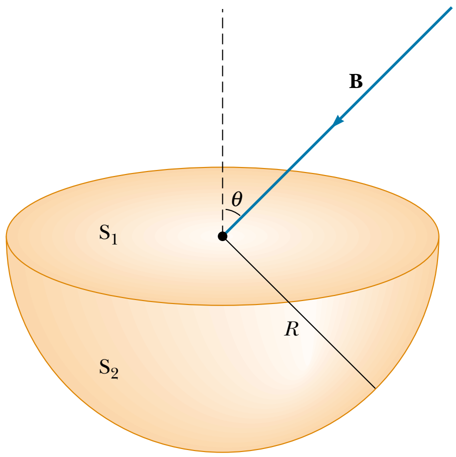


Figure P30.35

Section 30.6 Gauss's Law in Magnetism

Section 30.7 Displacement Current and the General Form of Ampère's Law

36. A 0.200-A current is charging a capacitor that has circular plates 10.0 cm in radius. If the plate separation is 4.00 mm, (a) what is the time rate of increase of electric field between the plates? (b) What is the magnetic field between the plates 5.00 cm from the center?

37. A 0.100-A current is charging a capacitor that has square plates 5.00 cm on each side. If the plate separation is 4.00 mm, find (a) the time rate of change of electric flux between the plates and (b) the displacement current between the plates.

(Optional)
Section 30.8 Magnetism in Matter

38. In Bohr's 1913 model of the hydrogen atom, the electron is in a circular orbit of radius 5.29×10^{-11} m, and its speed is 2.19×10^6 m/s. (a) What is the magnitude of the magnetic moment due to the electron's motion? (b) If the electron orbits counterclockwise in a horizontal circle, what is the direction of this magnetic moment vector?

39. A toroid with a mean radius of 20.0 cm and 630 turns (see Fig. 30.29) is filled with powdered steel whose magnetic susceptibility χ is 100. If the current in the windings is 3.00 A, find B (assumed uniform) inside the toroid.

40. A magnetic field of 1.30 T is to be set up in an iron-core toroid. The toroid has a mean radius of 10.0 cm and magnetic permeability of $5\,000\mu_0$. What current is required if there are 470 turns of wire in the winding? The thickness of the iron ring is small compared to 10 cm, so the field in the material is nearly uniform.

41. A coil of 500 turns is wound on an iron ring ($\mu_m = 750\mu_0$) with a 20.0-cm mean radius and an 8.00-cm^2 cross-sectional area. Calculate the magnetic flux Φ_B in this Rowland ring when the current in the coil is 0.500 A.

42. A uniform ring with a radius of 2.00 cm and a total charge of 6.00 μC rotates with a constant angular speed of 4.00 rad/s around an axis perpendicular to the plane of the ring and passing through its center. What is the magnetic moment of the rotating ring?

43. Calculate the magnetic field strength H of a magnetized substance in which the magnetization is 880 kA/m and the magnetic field has a magnitude of 4.40 T.

44. At saturation, the alignment of spins in iron can contribute as much as 2.00 T to the total magnetic field B. If each electron contributes a magnetic moment of $9.27 \times 10^{-24}\ \text{A}\cdot\text{m}^2$ (one Bohr magneton), how many electrons per atom contribute to the saturated field of iron? (*Hint:* Iron contains 8.50×10^{28} atoms/m^3.)

45. (a) Show that Curie's law can be stated in the following way: The magnetic susceptibility of a paramagnetic substance is inversely proportional to the absolute temperature, according to $\chi = C\mu_0/T$, where C is Curie's constant. (b) Evaluate Curie's constant for chromium.

(Optional)
Section 30.9 The Magnetic Field of the Earth

46. A circular coil of 5 turns and a diameter of 30.0 cm is oriented in a vertical plane with its axis perpendicular to the horizontal component of the Earth's magnetic field. A horizontal compass placed at the center of the coil is made to deflect 45.0° from magnetic north by a current of 0.600 A in the coil. (a) What is the horizontal component of the Earth's magnetic field? (b) The current in the coil is switched off. A "dip needle" is a magnetic compass mounted so that it can rotate in a vertical north-south plane. At this location a dip needle makes an angle of 13.0° from the vertical. What is the total magnitude of the Earth's magnetic field at this location?

47. The magnetic moment of the Earth is approximately $8.00 \times 10^{22}\ \text{A}\cdot\text{m}^2$. (a) If this were caused by the complete magnetization of a huge iron deposit, how many unpaired electrons would this correspond to? (b) At two unpaired electrons per iron atom, how many kilograms of iron would this correspond to? (Iron has a density of 7 900 kg/m^3 and approximately 8.50×10^{28} atoms/m^3.)

ADDITIONAL PROBLEMS

48. A lightning bolt may carry a current of 1.00×10^4 A for a short period of time. What is the resultant magnetic

field 100 m from the bolt? Suppose that the bolt extends far above and below the point of observation.

49. The magnitude of the Earth's magnetic field at either pole is approximately 7.00×10^{-5} T. Suppose that the field fades away, before its next reversal. Scouts, sailors, and wire merchants around the world join together in a program to replace the field. One plan is to use a current loop around the equator, without relying on magnetization of any materials inside the Earth. Determine the current that would generate such a field if this plan were carried out. (Take the radius of the Earth as $R_E = 6.37 \times 10^6$ m.)

50. Two parallel conductors carry current in opposite directions, as shown in Figure P30.50. One conductor carries a current of 10.0 A. Point A is at the midpoint between the wires, and point C is a distance $d/2$ to the right of the 10.0-A current. If $d = 18.0$ cm and I is adjusted so that the magnetic field at C is zero, find (a) the value of the current I and (b) the value of the magnetic field at A.

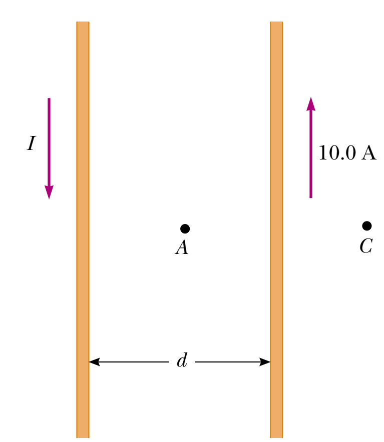


Figure P30.50

51. Suppose you install a compass on the center of the dashboard of a car. Compute an order-of-magnitude estimate for the magnetic field that is produced at this location by the current when you switch on the headlights. How does your estimate compare with the Earth's magnetic field? You may suppose the dashboard is made mostly of plastic.

52. Imagine a long, cylindrical wire of radius R that has a current density $J(r) = J_0(1 - r^2/R^2)$ for $r \le R$ and $J(r) = 0$ for $r > R$, where r is the distance from the axis of the wire. (a) Find the resulting magnetic field inside ($r \le R$) and outside ($r > R$) the wire. (b) Plot the magnitude of the magnetic field as a function of r. (c) Find the location where the magnitude of the magnetic field is a maximum, and the value of that maximum field.

53. A very long, thin strip of metal of width w carries a current I along its length, as shown in Figure P30.53. Find the magnetic field at point P in the diagram. Point P is in the plane of the strip at a distance b away from the strip.

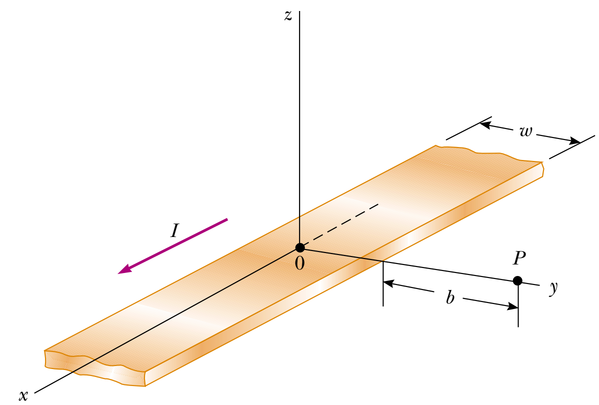


Figure P30.53

54. For a research project, a student needs a solenoid that produces an interior magnetic field of 0.030 0 T. She decides to use a current of 1.00 A and a wire 0.500 mm in diameter. She winds the solenoid in layers on an insulating form 1.00 cm in diameter and 10.0 cm long. Determine the number of layers of wire she needs and the total length of the wire.

WEB **55.** A nonconducting ring with a radius of 10.0 cm is uniformly charged with a total positive charge of 10.0 μC. The ring rotates at a constant angular speed of 20.0 rad/s about an axis through its center, perpendicular to the plane of the ring. What is the magnitude of the magnetic field on the axis of the ring, 5.00 cm from its center?

56. A nonconducting ring of radius R is uniformly charged with a total positive charge q. The ring rotates at a constant angular speed ω about an axis through its center, perpendicular to the plane of the ring. What is the magnitude of the magnetic field on the axis of the ring a distance $R/2$ from its center?

57. Two circular coils of radius R are each perpendicular to a common axis. The coil centers are a distance R apart, and a steady current I flows in the same direction around each coil, as shown in Figure P30.57. (a) Show that the magnetic field on the axis at a distance x from the center of one coil is

$$B = \frac{\mu_0 I R^2}{2}\left[\frac{1}{(R^2 + x^2)^{3/2}} + \frac{1}{(2R^2 + x^2 - 2Rx)^{3/2}}\right]$$

(b) Show that dB/dx and d^2B/dx^2 are both zero at a point midway between the coils. This means that the magnetic field in the region midway between the coils is uniform. Coils in this configuration are called **Helmholtz coils.**

58. Two identical, flat, circular coils of wire each have 100 turns and a radius of 0.500 m. The coils are arranged as

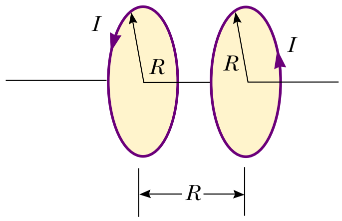


Figure P30.57 Problems 57 and 58.

a set of Helmholtz coils (see Fig. P30.57), parallel and with a separation of 0.500 m. If each coil carries a current of 10.0 A, determine the magnitude of the magnetic field at a point on the common axis of the coils and halfway between them.

59. Two circular loops are parallel, coaxial, and almost in contact, 1.00 mm apart (Fig. P30.59). Each loop is 10.0 cm in radius. The top loop carries a clockwise current of 140 A. The bottom loop carries a counterclockwise current of 140 A. (a) Calculate the magnetic force that the bottom loop exerts on the top loop. (b) The upper loop has a mass of 0.021 0 kg. Calculate its acceleration, assuming that the only forces acting on it are the force in part (a) and its weight. (*Hint:* Think about how one loop looks to a bug perched on the other loop.)

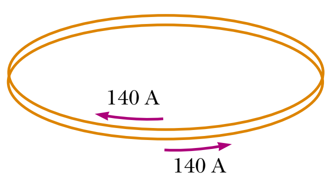


Figure P30.59

60. What objects experience a force in an electric field? Chapter 23 gives the answer: any electric charge, stationary or moving, other than the charge that created the field. What creates an electric field? Any electric charge, stationary or moving, also as discussed in Chapter 23. What objects experience a force in a magnetic field? An electric current or a moving electric charge other than the current or charge that created the field, as discovered in Chapter 29. What creates a magnetic field? An electric current, as you found in Section 30.11, or a moving electric charge, as in this problem. (a) To display how a moving charge creates a magnetic field, consider a charge q moving with velocity $\mathbf{v}$. Define the unit vector $\hat{\mathbf{r}} = \mathbf{r}/r$ to point from the charge to some location. Show that the magnetic field at that location is

$$\mathbf{B} = \frac{\mu_0}{4\pi} \frac{q\mathbf{v} \times \hat{\mathbf{r}}}{r^2}$$

(b) Find the magnitude of the magnetic field 1.00 mm to the side of a proton moving at 2.00×10^7 m/s. (c) Find the magnetic force on a second proton at this point, moving with the same speed in the opposite direction. (d) Find the electric force on the second proton.

61. *Rail guns* have been suggested for launching projectiles into space without chemical rockets, and for ground-to-air antimissile weapons of war. A tabletop model rail gun (Fig. P30.61) consists of two long parallel horizontal rails 3.50 cm apart, bridged by a bar *BD* of mass 3.00 g. The bar is originally at rest at the midpoint of the rails and is free to slide without friction. When the switch is closed, electric current is very quickly established in the circuit *ABCDEA*. The rails and bar have low electrical resistance, and the current is limited to a constant 24.0 A by the power supply. (a) Find the magnitude of the magnetic field 1.75 cm from a single very long, straight wire carrying current 24.0 A. (b) Find the vector magnetic field at point *C* in the diagram, the midpoint of the bar, immediately after the switch is closed. (*Hint:* Consider what conclusions you can draw from the Biot–Savart law.) (c) At other points along the bar *BD*, the field is in the same direction as at point *C*, but greater in magnitude. Assume that the average effective magnetic field along *BD* is five times larger than the field at *C*. With this assumption, find the vector force on the bar. (d) Find the vector acceleration with which the bar starts to move. (e) Does the bar move with constant acceleration? (f) Find the velocity of the bar after it has traveled 130 cm to the end of the rails.

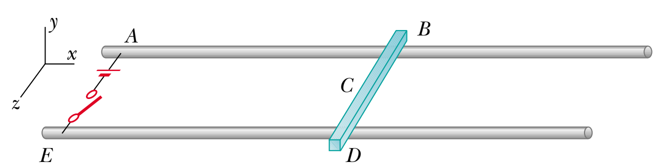


Figure P30.61

62. Two long, parallel conductors carry currents in the same direction, as shown in Figure P30.62. Conductor A carries a current of 150 A and is held firmly in position. Conductor B carries a current I_B and is allowed to slide freely up and down (parallel to A) between a set of nonconducting guides. If the mass per unit length of conductor B is 0.100 g/cm, what value of current I_B will result in equilibrium when the distance between the two conductors is 2.50 cm?

63. Charge is sprayed onto a large nonconducting belt above the left-hand roller in Figure P30.63. The belt carries the charge, with a uniform surface charge density σ, as it moves with a speed v between the rollers as shown. The charge is removed by a wiper at the right-hand roller. Consider a point just above the surface of the moving belt. (a) Find an expression for the magni-

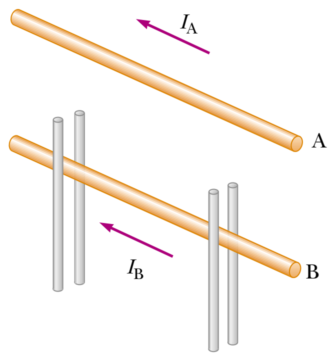


Figure P30.62

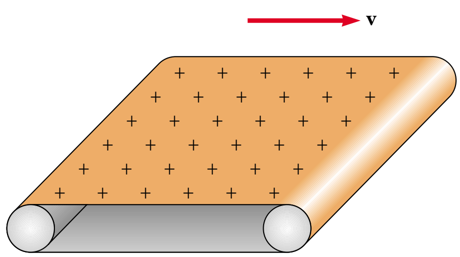


Figure P30.63

tude of the magnetic field **B** at this point. (b) If the belt is positively charged, what is the direction of **B**? (Note that the belt may be considered as an infinite sheet.)

64. A particular paramagnetic substance achieves 10.0% of its saturation magnetization when placed in a magnetic field of 5.00 T at a temperature of 4.00 K. The density of magnetic atoms in the sample is 8.00×10^{27} atoms/m^3, and the magnetic moment per atom is 5.00 Bohr magnetons. Calculate the Curie constant for this substance.

65. A bar magnet (mass = 39.4 g, magnetic moment = 7.65 J/T, length = 10.0 cm) is connected to the ceiling by a string. A uniform external magnetic field is applied horizontally, as shown in Figure P30.65. The magnet is in equilibrium, making an angle θ with the horizontal. If $\theta = 5.00°$, determine the magnitude of the applied magnetic field.

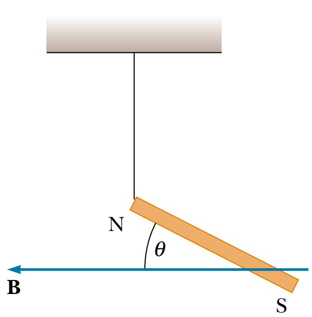


Figure P30.65

66. An infinitely long, straight wire carrying a current I_1 is partially surrounded by a loop, as shown in Figure P30.66. The loop has a length L and a radius R and carries a current I_2. The axis of the loop coincides with the wire. Calculate the force exerted on the loop.

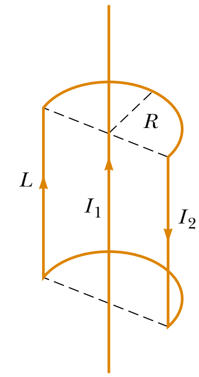


Figure P30.66

67. A wire is bent into the shape shown in Figure P30.67a, and the magnetic field is measured at P_1 when the current in the wire is I. The same wire is then formed into the shape shown in Figure P30.67b, and the magnetic field is measured at point P_2 when the current is again I. If the total length of wire is the same in each case, what is the ratio of B_1/B_2?

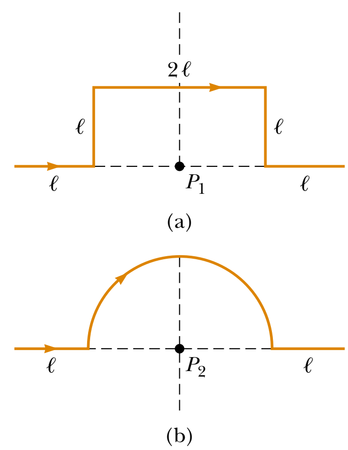


Figure P30.67

68. Measurements of the magnetic field of a large tornado were made at the Geophysical Observatory in Tulsa, Oklahoma, in 1962. If the tornado's field was B = 15.0 nT pointing north when the tornado was 9.00 km east of the observatory, what current was carried up or down the funnel of the tornado, modeled as a long straight wire?

69. A wire is formed into a square of edge length L (Fig. P30.69). Show that when the current in the loop is I, the magnetic field at point P, a distance x from the center of the square along its axis, is

$$B = \frac{\mu_0 I L^2}{2\pi(x^2 + L^2/4)\sqrt{x^2 + L^2/2}}$$

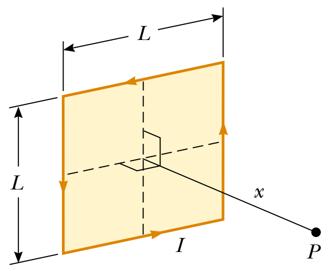


Figure P30.69

70. The force on a magnetic dipole $\boldsymbol{\mu}$ aligned with a nonuniform magnetic field in the x direction is given by $F_x = |\boldsymbol{\mu}|\, dB/dx$. Suppose that two flat loops of wire each have radius R and carry current I. (a) If the loops are arranged coaxially and separated by variable distance x, which is great compared to R, show that the magnetic force between them varies as $1/x^4$. (b) Evaluate the magnitude of this force if $I = 10.0$ A, $R = 0.500$ cm, and $x = 5.00$ cm.

71. A wire carrying a current I is bent into the shape of an exponential spiral $r = e^\theta$ from $\theta = 0$ to $\theta = 2\pi$, as in Figure P30.71. To complete a loop, the ends of the spiral are connected by a straight wire along the x axis. Find the magnitude and direction of $\mathbf{B}$ at the origin. *Hints:* Use the Biot–Savart law. The angle β between a radial line and its tangent line at any point on the curve $r = f(\theta)$ is related to the function in the following way:

$$\tan \beta = \frac{r}{dr/d\theta}$$

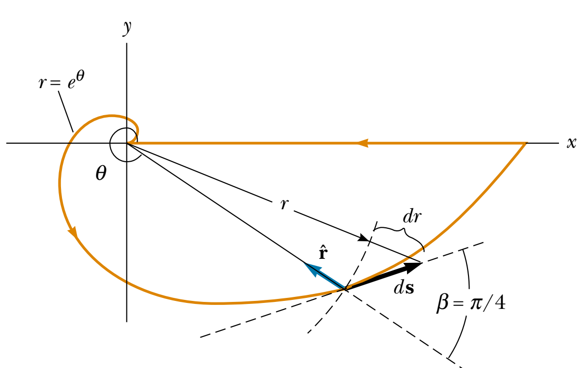


Figure P30.71

Thus, in this case $r = e^\theta$, $\tan \beta = 1$, and $\beta = \pi/4$. Therefore, the angle between $d\mathbf{s}$ and $\hat{\mathbf{r}}$ is $\pi - \beta = 3\pi/4$. Also,

$$ds = \frac{dr}{\sin \pi/4} = \sqrt{2}\, dr$$

72. Table P30.72 contains data taken for a ferromagnetic material. (a) Construct a magnetization curve from the data. Remember that $\mathbf{B} = \mathbf{B}_0 + \mu_0\mathbf{M}$. (b) Determine the ratio B/B_0 for each pair of values of B and B_0, and construct a graph of B/B_0 versus B_0. (The fraction B/B_0 is called the relative permeability and is a measure of the induced magnetic field.)

TABLE P30.72

B(T)	B_0 (T)
0.2	4.8×10^{-5}
0.4	7.0×10^{-5}
0.6	8.8×10^{-5}
0.8	1.2×10^{-4}
1.0	1.8×10^{-4}
1.2	3.1×10^{-4}
1.4	8.7×10^{-4}
1.6	3.4×10^{-3}
1.8	1.2×10^{-1}

73. **Review Problem.** A sphere of radius R has a constant volume charge density ρ. Determine the magnetic field at the center of the sphere when it rotates as a rigid body with angular velocity ω about an axis through its center (Fig. P30.73).

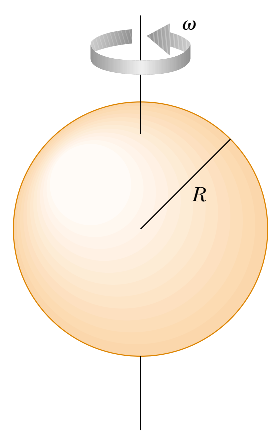


Figure P30.73 Problems 73 and 74.

74. **Review Problem.** A sphere of radius R has a constant volume charge density ρ. Determine the magnetic di-

pole moment of the sphere when it rotates as a rigid body with angular velocity ω about an axis through its center (see Fig. P30.73).

75. A long, cylindrical conductor of radius a has two cylindrical cavities of diameter a through its entire length, as shown in cross-section in Figure P30.75. A current I is directed out of the page and is uniform through a cross section of the conductor. Find the magnitude and direction of the magnetic field in terms of μ_0, I, r, and a (a) at point P_1 and (b) at point P_2.

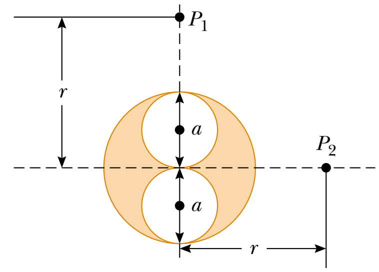


Figure P30.75

ANSWERS TO QUICK QUIZZES

30.1 (c) $F_1 = F_2$ because of Newton's third law. Another way to arrive at this answer is to realize that Equation 30.11 gives the same result whether the multiplication of currents is (2 A)(6 A) or (6 A)(2 A).

30.2 Closer together; the coils act like wires carrying parallel currents and hence attract one another.

30.3 *b, d, a, c.* Equation 30.13 indicates that the value of the line integral depends only on the net current through each closed path. Path *b* encloses 1 A, path *d* encloses 3 A, path *a* encloses 4 A, and path *c* encloses 6 A.

30.4 *b*, then $a = c = d$. Paths *a, c*, and *d* all give the same nonzero value $\mu_0 I$ because the size and shape of the paths do not matter. Path *b* does not enclose the current, and hence its line integral is zero.

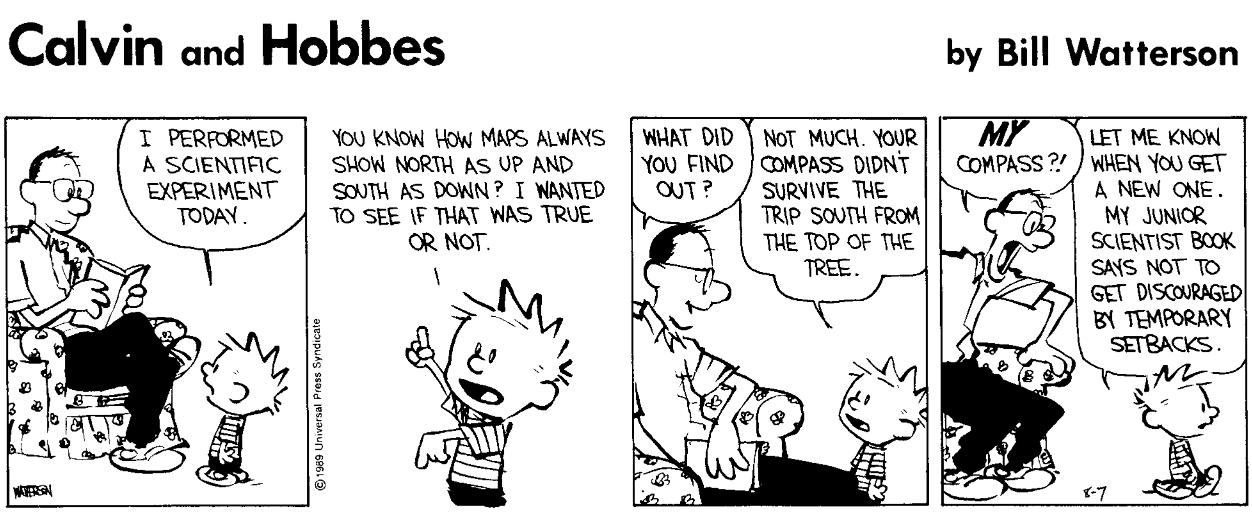
30.5 Net force, yes; net torque, no. The forces on the top and bottom of the loop cancel because they are equal in magnitude but opposite in direction. The current in the left side of the loop is parallel to I_1, and hence the force F_L exerted by I_1 on this side is attractive. The current in the right side of the loop is antiparallel to I_1, and hence the force F_R exerted by I_1 on this side of the loop is repulsive. Because the left side is closer to wire 1, $F_L > F_R$ and a net force is directed toward wire 1. Because the forces on all four sides of the loop lie in the plane of the loop, there is no net torque.

30.6 Zero; no charges flow into a fully charged capacitor, so no change occurs in the amount of charge on the plates, and the electric field between the plates is constant. It is only when the electric field is changing that a displacement current exists.

30.7 (a) Increases slightly; (b) decreases slightly; (c) increases greatly. Equations 30.33 and 30.34 indicate that, when each metal is in place, the total field is $\mathbf{B} = \mu_0(1 + \chi)\mathbf{H}$. Table 30.2 indicates that $\mu_0(1 + \chi)\mathbf{H}$ is slightly greater than $\mu_0\mathbf{H}$ for aluminum and slightly less for copper. For iron, the field can be made thousands of times stronger, as we saw in Example 30.10.

30.8 One whose loop looks like Figure 30.31a because the remanent magnetization at the point corresponding to point *b* in Figure 30.30 is greater.

30.9 West to east. The lines of the Earth's magnetic field enter the planet in Hudson Bay and emerge from Antarctica; thus, the field lines resulting from the current would have to go in the opposite direction. Compare Figure 30.6a with Figure 30.35.



PUZZLER

Before this vending machine will deliver its product, it conducts several tests on the coins being inserted. How can it determine what material the coins are made of without damaging them and without making the customer wait a long time for the results? *(George Semple)*

chapter 31

Faraday's Law

Chapter Outline

The focus of our studies in electricity and magnetism so far has been the electric fields produced by stationary charges and the magnetic fields produced by moving charges. This chapter deals with electric fields produced by changing magnetic fields.

Experiments conducted by Michael Faraday in England in 1831 and independently by Joseph Henry in the United States that same year showed that an emf can be induced in a circuit by a changing magnetic field. As we shall see, an emf (and therefore a current as well) can be induced in many ways—for instance, by moving a closed loop of wire into a region where a magnetic field exists. The results of these experiments led to a very basic and important law of electromagnetism known as *Faraday's law of induction.* This law states that the magnitude of the emf induced in a circuit equals the time rate of change of the magnetic flux through the circuit.

With the treatment of Faraday's law, we complete our introduction to the fundamental laws of electromagnetism. These laws can be summarized in a set of four equations called *Maxwell's equations.* Together with the *Lorentz force law,* which we discuss briefly, they represent a complete theory for describing the interaction of charged objects. Maxwell's equations relate electric and magnetic fields to each other and to their ultimate source, namely, electric charges.

31.1 FARADAY'S LAW OF INDUCTION

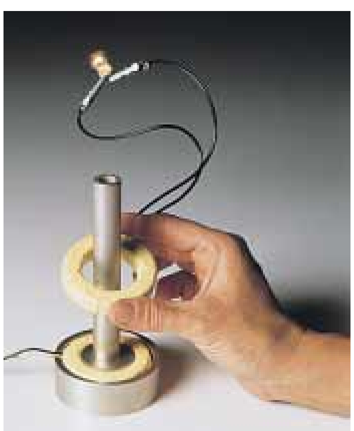
12.6 & 12.7

To see how an emf can be induced by a changing magnetic field, let us consider a loop of wire connected to a galvanometer, as illustrated in Figure 31.1. When a magnet is moved toward the loop, the galvanometer needle deflects in one direction, arbitrarily shown to the right in Figure 31.1a. When the magnet is moved away from the loop, the needle deflects in the opposite direction, as shown in Figure 31.1c. When the magnet is held stationary relative to the loop (Fig. 31.1b), no deflection is observed. Finally, if the magnet is held stationary and the loop is moved either toward or away from it, the needle deflects. From these observations, we conclude that the loop "knows" that the magnet is moving relative to it because it experiences a change in magnetic field. Thus, it seems that a relationship exists between current and changing magnetic field.

These results are quite remarkable in view of the fact that **a current is set up even though no batteries are present in the circuit!** We call such a current an *induced current* and say that it is produced by an *induced emf.*

Now let us describe an experiment conducted by Faraday[1] and illustrated in Figure 31.2. A primary coil is connected to a switch and a battery. The coil is wrapped around a ring, and a current in the coil produces a magnetic field when the switch is closed. A secondary coil also is wrapped around the ring and is connected to a galvanometer. No battery is present in the secondary circuit, and the secondary coil is not connected to the primary coil. Any current detected in the secondary circuit must be induced by some external agent.

Initially, you might guess that no current is ever detected in the secondary circuit. However, something quite amazing happens when the switch in the primary



A demonstration of electromagnetic induction. A changing potential difference is applied to the lower coil. An emf is induced in the upper coil as indicated by the illuminated lamp. What happens to the lamp's intensity as the upper coil is moved over the vertical tube? *(Courtesy of Central Scientific Company)*

[1] A physicist named J. D. Colladon was the first to perform the moving-magnet experiment. To minimize the effect of the changing magnetic field on his galvanometer, he placed the meter in an adjacent room. Thus, as he moved the magnet in the loop, he could not see the meter needle deflecting. By the time he returned next door to read the galvanometer, the needle was back to zero because he had stopped moving the magnet. Unfortunately for Colladon, there must be relative motion between the loop and the magnet for an induced emf and a corresponding induced current to be observed. Thus, physics students learn Faraday's law of induction rather than "Colladon's law of induction."

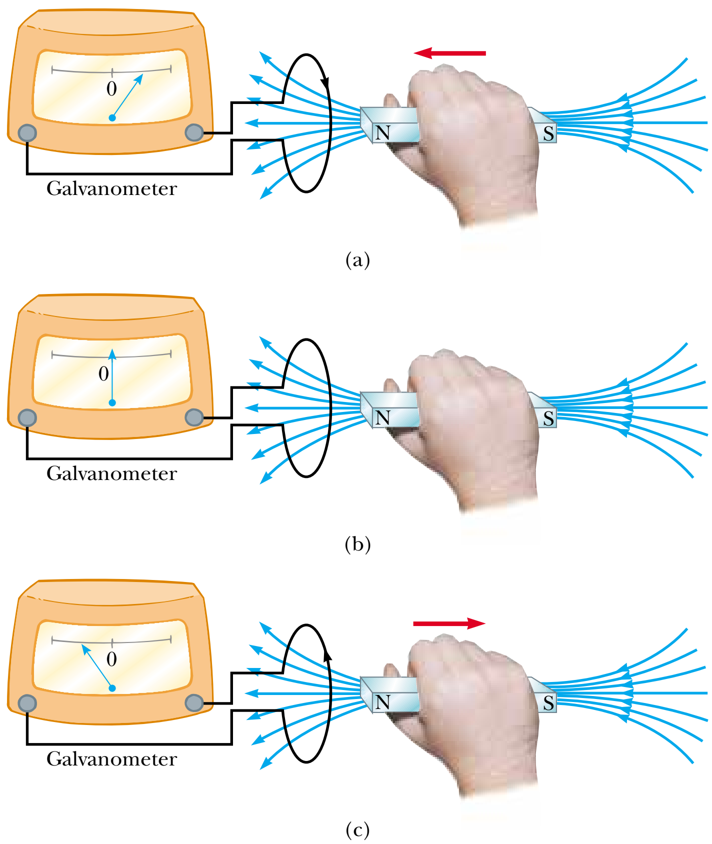


Figure 31.1 (a) When a magnet is moved toward a loop of wire connected to a galvanometer, the galvanometer deflects as shown, indicating that a current is induced in the loop. (b) When the magnet is held stationary, there is no induced current in the loop, even when the magnet is inside the loop. (c) When the magnet is moved away from the loop, the galvanometer deflects in the opposite direction, indicating that the induced current is opposite that shown in part (a). Changing the direction of the magnet's motion changes the direction of the current induced by that motion.

circuit is either suddenly closed or suddenly opened. At the instant the switch is closed, the galvanometer needle deflects in one direction and then returns to zero. At the instant the switch is opened, the needle deflects in the opposite direction and again returns to zero. Finally, the galvanometer reads zero when there is either a steady current or no current in the primary circuit. The key to under-

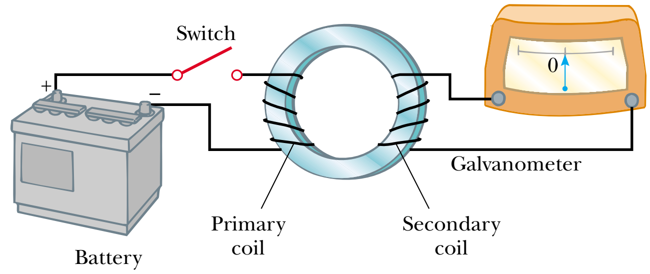


Figure 31.2 Faraday's experiment. When the switch in the primary circuit is closed, the galvanometer in the secondary circuit deflects momentarily. The emf induced in the secondary circuit is caused by the changing magnetic field through the secondary coil.

Michael Faraday (1791–1867)
Faraday, a British physicist and chemist, is often regarded as the greatest experimental scientist of the 1800s. His many contributions to the study of electricity include the invention of the electric motor, electric generator, and transformer, as well as the discovery of electromagnetic induction and the laws of electrolysis. Greatly influenced by religion, he refused to work on the development of poison gas for the British military.
(By kind permission of the President and Council of the Royal Society)

standing what happens in this experiment is to first note that when the switch is closed, the current in the primary circuit produces a magnetic field in the region of the circuit, and it is this magnetic field that penetrates the secondary circuit. Furthermore, when the switch is closed, the magnetic field produced by the current in the primary circuit changes from zero to some value over some finite time, and it is this changing field that induces a current in the secondary circuit.

As a result of these observations, Faraday concluded that **an electric current can be induced in a circuit (the secondary circuit in our setup) by a changing magnetic field.** The induced current exists for only a short time while the magnetic field through the secondary coil is changing. Once the magnetic field reaches a steady value, the current in the secondary coil disappears. In effect, the secondary circuit behaves as though a source of emf were connected to it for a short time. It is customary to say that **an induced emf is produced in the secondary circuit by the changing magnetic field.**

The experiments shown in Figures 31.1 and 31.2 have one thing in common: In each case, an emf is induced in the circuit when the magnetic flux through the circuit changes with time. In general,

the emf induced in a circuit is directly proportional to the time rate of change of the magnetic flux through the circuit.

This statement, known as **Faraday's law of induction,** can be written

Faraday's law

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{31.1}$$

where $\Phi_B = \int \mathbf{B} \cdot d\mathbf{A}$ is the magnetic flux through the circuit (see Section 30.5).

If the circuit is a coil consisting of N loops all of the same area and if Φ_B is the flux through one loop, an emf is induced in every loop; thus, the total induced emf in the coil is given by the expression

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} \tag{31.2}$$

The negative sign in Equations 31.1 and 31.2 is of important physical significance, which we shall discuss in Section 31.3.

Suppose that a loop enclosing an area A lies in a uniform magnetic field $\mathbf{B}$, as shown in Figure 31.3. The magnetic flux through the loop is equal to $BA\cos\theta$;

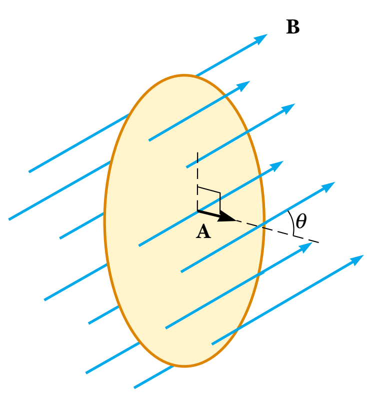


Figure 31.3 A conducting loop that encloses an area A in the presence of a uniform magnetic field $\mathbf{B}$. The angle between $\mathbf{B}$ and the normal to the loop is θ.

hence, the induced emf can be expressed as

$$\mathcal{E} = -\frac{d}{dt}(BA\cos\theta) \tag{31.3}$$

From this expression, we see that an emf can be induced in the circuit in several ways:

- The magnitude of **B** can change with time.
- The area enclosed by the loop can change with time.
- The angle θ between **B** and the normal to the loop can change with time.
- Any combination of the above can occur.

QuickLab

A cassette tape is made up of tiny particles of metal oxide attached to a long plastic strip. A current in a small conducting loop magnetizes the particles in a pattern related to the music being recorded. During playback, the tape is moved past a second small loop (inside the playback head) and induces a current that is then amplified. Pull a strip of tape out of a cassette (one that you don't mind recording over) and see if it is attracted or repelled by a refrigerator magnet. If you don't have a cassette, try this with an old floppy disk you are ready to trash.

Quick Quiz 31.1

Equation 31.3 can be used to calculate the emf induced when the north pole of a magnet is moved toward a loop of wire, along the axis perpendicular to the plane of the loop passing through its center. What changes are necessary in the equation when the south pole is moved toward the loop?

Some Applications of Faraday's Law

The ground fault interrupter (GFI) is an interesting safety device that protects users of electrical appliances against electric shock. Its operation makes use of Faraday's law. In the GFI shown in Figure 31.4, wire 1 leads from the wall outlet to the appliance to be protected, and wire 2 leads from the appliance back to the wall outlet. An iron ring surrounds the two wires, and a sensing coil is wrapped around part of the ring. Because the currents in the wires are in opposite directions, the net magnetic flux through the sensing coil due to the currents is zero. However, if the return current in wire 2 changes, the net magnetic flux through the sensing coil is no longer zero. (This can happen, for example, if the appliance gets wet, enabling current to leak to ground.) Because household current is alternating (meaning that its direction keeps reversing), the magnetic flux through the sensing coil changes with time, inducing an emf in the coil. This induced emf is used to trigger a circuit breaker, which stops the current before it is able to reach a harmful level.

Another interesting application of Faraday's law is the production of sound in an electric guitar (Fig. 31.5). The coil in this case, called the *pickup coil,* is placed near the vibrating guitar string, which is made of a metal that can be magnetized. A permanent magnet inside the coil magnetizes the portion of the string nearest

This electric range cooks food on the basis of the principle of induction. An oscillating current is passed through a coil placed below the cooking surface, which is made of a special glass. The current produces an oscillating magnetic field, which induces a current in the cooking utensil. Because the cooking utensil has some electrical resistance, the electrical energy associated with the induced current is transformed to internal energy, causing the utensil and its contents to become hot. *(Courtesy of Corning, Inc.)*

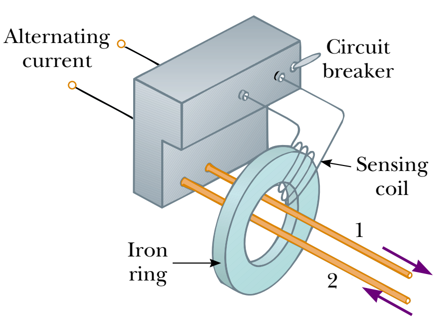
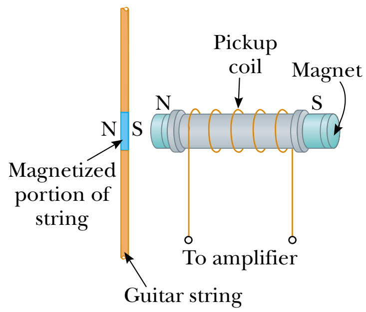


Figure 31.4 Essential components of a ground fault interrupter.



(a)

(b)

Figure 31.5 (a) In an electric guitar, a vibrating string induces an emf in a pickup coil. (b) The circles beneath the metallic strings of this electric guitar detect the notes being played and send this information through an amplifier and into speakers. (A switch on the guitar allows the musician to select which set of six is used.) How does a guitar "pickup" sense what music is being played? *(b, Charles D. Winters)*

the coil. When the string vibrates at some frequency, its magnetized segment produces a changing magnetic flux through the coil. The changing flux induces an emf in the coil that is fed to an amplifier. The output of the amplifier is sent to the loudspeakers, which produce the sound waves we hear.

EXAMPLE 31.1 One Way to Induce an emf in a Coil

A coil consists of 200 turns of wire having a total resistance of 2.0 Ω. Each turn is a square of side 18 cm, and a uniform magnetic field directed perpendicular to the plane of the coil is turned on. If the field changes linearly from 0 to 0.50 T in 0.80 s, what is the magnitude of the induced emf in the coil while the field is changing?

Solution The area of one turn of the coil is $(0.18\ \text{m})^2 = 0.032\ 4\ \text{m}^2$. The magnetic flux through the coil at $t = 0$ is zero because $B = 0$ at that time. At $t = 0.80$ s, the magnetic flux through one turn is $\Phi_B = BA = (0.50\ \text{T})(0.032\ 4\ \text{m}^2) = 0.016\ 2\ \text{T}\cdot\text{m}^2$. Therefore, the magnitude of the induced emf is, from Equation 31.2,

$$|\mathcal{E}| = \frac{N\,\Delta\Phi_B}{\Delta t} = \frac{200(0.016\ 2\ \text{T}\cdot\text{m}^2 - 0\ \text{T}\cdot\text{m}^2)}{0.80\ \text{s}}$$

$$= 4.1\ \text{T}\cdot\text{m}^2/\text{s} = \boxed{4.1\ \text{V}}$$

You should be able to show that $1\ \text{T}\cdot\text{m}^2/\text{s} = 1$ V.

Exercise What is the magnitude of the induced current in the coil while the field is changing?

Answer 2.0 A.

EXAMPLE 31.2 An Exponentially Decaying B Field

A loop of wire enclosing an area A is placed in a region where the magnetic field is perpendicular to the plane of the loop. The magnitude of **B** varies in time according to the expression $B = B_{\text{max}}e^{-at}$, where a is some constant. That is, at $t = 0$ the field is B_{max}, and for $t > 0$, the field decreases exponentially (Fig. 31.6). Find the induced emf in the loop as a function of time.

Solution Because **B** is perpendicular to the plane of the loop, the magnetic flux through the loop at time $t > 0$ is

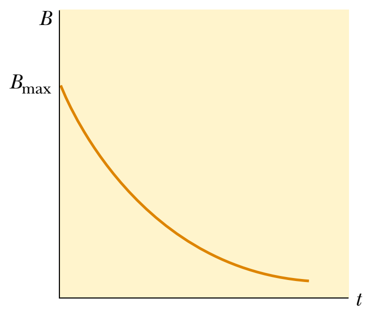


Figure 31.6 Exponential decrease in the magnitude of the magnetic field with time. The induced emf and induced current vary with time in the same way.

$$\Phi_B = BA\cos 0 = AB_{max}e^{-at}$$

Because AB_{max} and a are constants, the induced emf calculated from Equation 31.1 is

$$\boldsymbol{\mathcal{E}} = -\frac{d\Phi_B}{dt} = -AB_{max}\frac{d}{dt}e^{-at} = \boxed{aAB_{max}e^{-at}}$$

This expression indicates that the induced emf decays exponentially in time. Note that the maximum emf occurs at $t = 0$, where $\boldsymbol{\mathcal{E}}_{max} = aAB_{max}$. The plot of $\boldsymbol{\mathcal{E}}$ versus t is similar to the B-versus-t curve shown in Figure 31.6.

CONCEPTUAL EXAMPLE 31.3 What Is Connected to What?

Two bulbs are connected to opposite sides of a loop of wire, as shown in Figure 31.7. A decreasing magnetic field (confined to the circular area shown in the figure) induces an emf in the loop that causes the two bulbs to light. What happens to the brightness of the bulbs when the switch is closed?

Solution Bulb 1 glows brighter, and bulb 2 goes out. Once the switch is closed, bulb 1 is in the large loop consisting of the wire to which it is attached and the wire connected to the switch. Because the changing magnetic flux is completely enclosed within this loop, a current exists in bulb 1. Bulb 1 now glows brighter than before the switch was closed because it is now the only resistance in the loop. As a result, the current in bulb 1 is greater than when bulb 2 was also in the loop.

Once the switch is closed, bulb 2 is in the loop consisting of the wires attached to it and those connected to the switch. There is no changing magnetic flux through this loop and hence no induced emf.

Exercise What would happen if the switch were in a wire located to the left of bulb 1?

Answer Bulb 1 would go out, and bulb 2 would glow brighter.

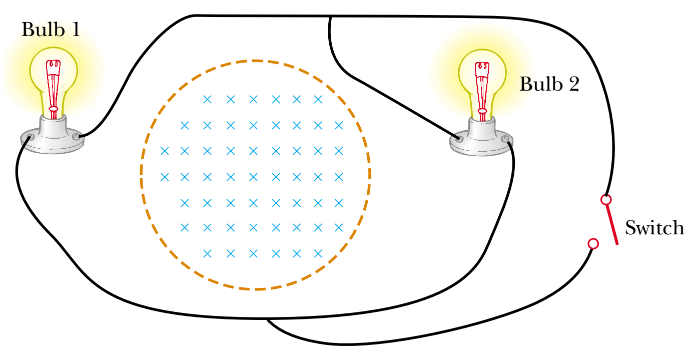


Figure 31.7

31.2 MOTIONAL EMF

In Examples 31.1 and 31.2, we considered cases in which an emf is induced in a stationary circuit placed in a magnetic field when the field changes with time. In this section we describe what is called **motional emf,** which is the emf induced in a conductor moving through a constant magnetic field.

The straight conductor of length ℓ shown in Figure 31.8 is moving through a uniform magnetic field directed into the page. For simplicity, we assume that the conductor is moving in a direction perpendicular to the field with constant veloc-

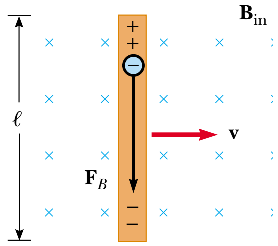


Figure 31.8 A straight electrical conductor of length ℓ moving with a velocity $\mathbf{v}$ through a uniform magnetic field $\mathbf{B}$ directed perpendicular to $\mathbf{v}$. A potential difference $\Delta V = B\ell v$ is maintained between the ends of the conductor.

ity under the influence of some external agent. The electrons in the conductor experience a force $\mathbf{F}_B = q\mathbf{v} \times \mathbf{B}$ that is directed along the length ℓ, perpendicular to both $\mathbf{v}$ and $\mathbf{B}$ (Eq. 29.1). Under the influence of this force, the electrons move to the lower end of the conductor and accumulate there, leaving a net positive charge at the upper end. As a result of this charge separation, an electric field is produced inside the conductor. The charges accumulate at both ends until the downward magnetic force qvB is balanced by the upward electric force qE. At this point, electrons stop moving. The condition for equilibrium requires that

$$qE = qvB \qquad \text{or} \qquad E = vB$$

The electric field produced in the conductor (once the electrons stop moving and E is constant) is related to the potential difference across the ends of the conductor according to the relationship $\Delta V = E\ell$ (Eq. 25.6). Thus,

$$\Delta V = E\ell = B\ell v \tag{31.4}$$

where the upper end is at a higher electric potential than the lower end. Thus, **a potential difference is maintained between the ends of the conductor as long as the conductor continues to move through the uniform magnetic field.** If the direction of the motion is reversed, the polarity of the potential difference also is reversed.

A more interesting situation occurs when the moving conductor is part of a closed conducting path. This situation is particularly useful for illustrating how a changing magnetic flux causes an induced current in a closed circuit. Consider a circuit consisting of a conducting bar of length ℓ sliding along two fixed parallel conducting rails, as shown in Figure 31.9a.

For simplicity, we assume that the bar has zero resistance and that the stationary part of the circuit has a resistance R. A uniform and constant magnetic field $\mathbf{B}$ is applied perpendicular to the plane of the circuit. As the bar is pulled to the right with a velocity $\mathbf{v}$, under the influence of an applied force $\mathbf{F}_{\text{app}}$, free charges in the bar experience a magnetic force directed along the length of the bar. This force sets up an induced current because the charges are free to move in the closed conducting path. In this case, the rate of change of magnetic flux through the loop and the corresponding induced motional emf across the moving bar are proportional to the change in area of the loop. As we shall see, if the bar is pulled to the right with a constant velocity, the work done by the applied force appears as internal energy in the resistor R (see Section 27.6).

Because the area enclosed by the circuit at any instant is ℓx, where x is the width of the circuit at any instant, the magnetic flux through that area is

$$\Phi_B = B\ell x$$

Using Faraday's law, and noting that x changes with time at a rate $dx/dt = v$, we find that the induced motional emf is

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}(B\ell x) = -B\ell\frac{dx}{dt}$$

Motional emf

$$\boxed{\mathcal{E} = -B\ell v} \tag{31.5}$$

Because the resistance of the circuit is R, the magnitude of the induced current is

$$I = \frac{|\mathcal{E}|}{R} = \frac{B\ell v}{R} \tag{31.6}$$

The equivalent circuit diagram for this example is shown in Figure 31.9b.

Let us examine the system using energy considerations. Because no battery is in the circuit, we might wonder about the origin of the induced current and the electrical energy in the system. We can understand the source of this current and energy by noting that the applied force does work on the conducting bar, thereby moving charges through a magnetic field. Their movement through the field causes the charges to move along the bar with some average drift velocity, and hence a current is established. Because energy must be conserved, the work done by the applied force on the bar during some time interval must equal the electrical energy supplied by the induced emf during that same interval. Furthermore, if the bar moves with constant speed, the work done on it must equal the energy delivered to the resistor during this time interval.

As it moves through the uniform magnetic field **B**, the bar experiences a magnetic force $\mathbf{F}_B$ of magnitude $I\ell B$ (see Section 29.2). The direction of this force is opposite the motion of the bar, to the left in Figure 31.9a. Because the bar moves with constant velocity, the applied force must be equal in magnitude and opposite in direction to the magnetic force, or to the right in Figure 31.9a. (If $\mathbf{F}_B$ acted in the direction of motion, it would cause the bar to accelerate. Such a situation would violate the principle of conservation of energy.) Using Equation 31.6 and the fact that $F_{\text{app}} = I\ell B$, we find that the power delivered by the applied force is

$$\mathcal{P} = F_{\text{app}}v = (I\ell B)v = \frac{B^2\ell^2v^2}{R} = \frac{\mathcal{E}^2}{R} \qquad \textbf{(31.7)}$$

From Equation 27.23, we see that this power is equal to the rate at which energy is delivered to the resistor I^2R, as we would expect. It is also equal to the power $I\mathcal{E}$ supplied by the motional emf. This example is a clear demonstration of the conversion of mechanical energy first to electrical energy and finally to internal energy in the resistor.

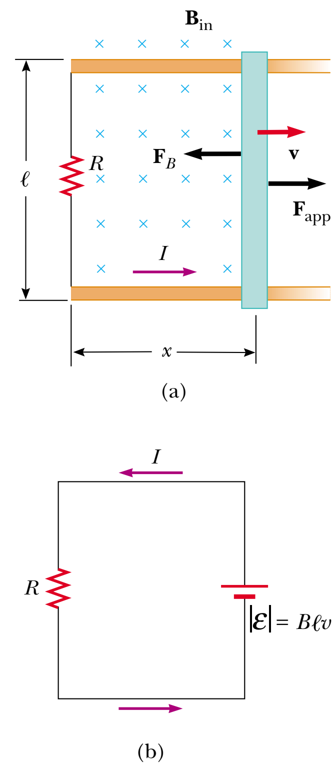


Figure 31.9 (a) A conducting bar sliding with a velocity **v** along two conducting rails under the action of an applied force $\mathbf{F}_{\text{app}}$. The magnetic force $\mathbf{F}_B$ opposes the motion, and a counterclockwise current I is induced in the loop. (b) The equivalent circuit diagram for the setup shown in part (a).

Quick Quiz 31.2

As an airplane flies from Los Angeles to Seattle, it passes through the Earth's magnetic field. As a result, a motional emf is developed between the wingtips. Which wingtip is positively charged?

Example 31.4 Motional emf Induced in a Rotating Bar

A conducting bar of length ℓ rotates with a constant angular speed ω about a pivot at one end. A uniform magnetic field **B** is directed perpendicular to the plane of rotation, as shown in Figure 31.10. Find the motional emf induced between the ends of the bar.

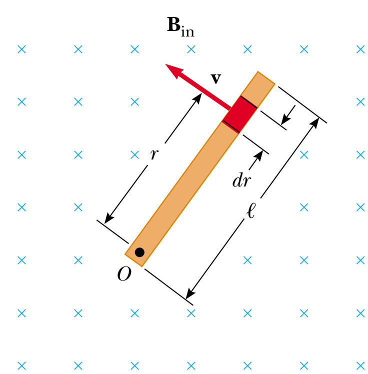


Figure 31.10 A conducting bar rotating around a pivot at one end in a uniform magnetic field that is perpendicular to the plane of rotation. A motional emf is induced across the ends of the bar.

Solution Consider a segment of the bar of length dr having a velocity **v**. According to Equation 31.5, the magnitude of the emf induced in this segment is

$$d\mathcal{E} = Bv\,dr$$

Because every segment of the bar is moving perpendicular to **B**, an emf $d\mathcal{E}$ of the same form is generated across each. Summing the emfs induced across all segments, which are in series, gives the total emf between the ends of

the bar:

$$\mathcal{E} = \int Bv\,dr$$

To integrate this expression, we must note that the linear speed of an element is related to the angular speed ω through the relationship $v = r\omega$. Therefore, because B and ω are constants, we find that

$$\mathcal{E} = B\int v\,dr = B\omega\int_0^\ell r\,dr = \boxed{\tfrac{1}{2}B\omega\ell^2}$$

EXAMPLE 31.5 Magnetic Force Acting on a Sliding Bar

The conducting bar illustrated in Figure 31.11, of mass m and length ℓ, moves on two frictionless parallel rails in the presence of a uniform magnetic field directed into the page. The bar is given an initial velocity $\mathbf{v}_i$ to the right and is released at $t = 0$. Find the velocity of the bar as a function of time.

Solution The induced current is counterclockwise, and the magnetic force is $F_B = -I\ell B$, where the negative sign denotes that the force is to the left and retards the motion. This is the only horizontal force acting on the bar, and hence Newton's second law applied to motion in the horizontal direction gives

$$F_x = ma = m\frac{dv}{dt} = -I\ell B$$

From Equation 31.6, we know that $I = B\ell v/R$, and so we can write this expression as

$$m\frac{dv}{dt} = -\frac{B^2\ell^2}{R}v$$

$$\frac{dv}{v} = -\left(\frac{B^2\ell^2}{mR}\right)dt$$

Integrating this equation using the initial condition that $v = v_i$ at $t = 0$, we find that

$$\int_{v_i}^{v}\frac{dv}{v} = \frac{-B^2\ell^2}{mR}\int_0^t dt$$

$$\ln\left(\frac{v}{v_i}\right) = -\left(\frac{B^2\ell^2}{mR}\right)t = -\frac{t}{\tau}$$

where the constant $\tau = mR/B^2\ell^2$. From this result, we see that the velocity can be expressed in the exponential form

$$\boxed{v = v_i e^{-t/\tau}}$$

This expression indicates that the velocity of the bar decreases exponentially with time under the action of the magnetic retarding force.

Exercise Find expressions for the induced current and the magnitude of the induced emf as functions of time for the bar in this example.

Answer $I = \dfrac{B\ell v_i}{R}e^{-t/\tau}$; $\mathcal{E} = B\ell v_i e^{-t/\tau}$. (They both decrease exponentially with time.)

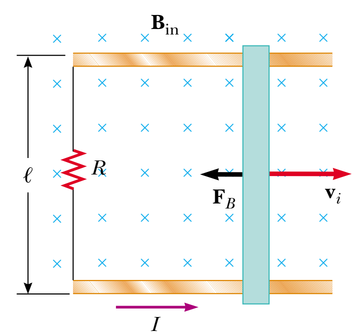


Figure 31.11 A conducting bar of length ℓ sliding on two fixed conducting rails is given an initial velocity $\mathbf{v}_i$ to the right.

31.3 LENZ'S LAW

12.7 Faraday's law (Eq. 31.1) indicates that the induced emf and the change in flux have opposite algebraic signs. This has a very real physical interpretation that has come to be known as **Lenz's law**[2]:

[2] Developed by the German physicist Heinrich Lenz (1804–1865).

The polarity of the induced emf is such that it tends to produce a current that creates a magnetic flux to oppose the change in magnetic flux through the area enclosed by the current loop.

That is, the induced current tends to keep the original magnetic flux through the circuit from changing. As we shall see, this law is a consequence of the law of conservation of energy.

To understand Lenz's law, let us return to the example of a bar moving to the right on two parallel rails in the presence of a uniform magnetic field that we shall refer to as the *external* magnetic field (Fig. 31.12a). As the bar moves to the right, the magnetic flux through the area enclosed by the circuit increases with time because the area increases. Lenz's law states that the induced current must be directed so that the magnetic flux it produces opposes the change in the external magnetic flux. Because the external magnetic flux is increasing into the page, the induced current, if it is to oppose this change, must produce a flux directed out of the page. Hence, the induced current must be directed counterclockwise when the bar moves to the right. (Use the right-hand rule to verify this direction.) If the bar is moving to the left, as shown in Figure 31.12b, the external magnetic flux through the area enclosed by the loop decreases with time. Because the flux is directed into the page, the direction of the induced current must be clockwise if it is to produce a flux that also is directed into the page. In either case, the induced current tends to maintain the original flux through the area enclosed by the current loop.

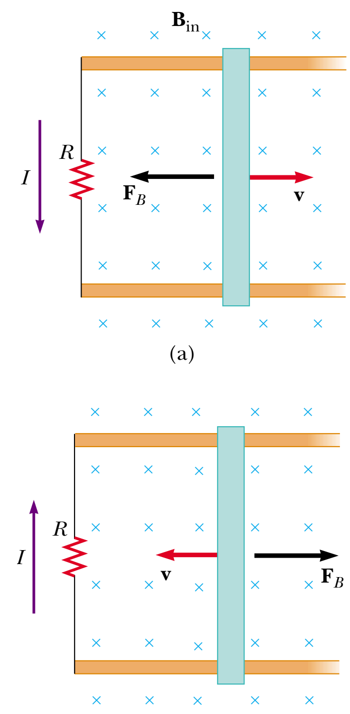


Figure 31.12 (a) As the conducting bar slides on the two fixed conducting rails, the magnetic flux through the area enclosed by the loop increases in time. By Lenz's law, the induced current must be counterclockwise so as to produce a counteracting magnetic flux directed out of the page. (b) When the bar moves to the left, the induced current must be clockwise. Why?

Let us examine this situation from the viewpoint of energy considerations. Suppose that the bar is given a slight push to the right. In the preceding analysis, we found that this motion sets up a counterclockwise current in the loop. Let us see what happens if we assume that the current is clockwise, such that the direction of the magnetic force exerted on the bar is to the right. This force would accelerate the rod and increase its velocity. This, in turn, would cause the area enclosed by the loop to increase more rapidly; this would result in an increase in the induced current, which would cause an increase in the force, which would produce an increase in the current, and so on. In effect, the system would acquire energy with no additional input of energy. This is clearly inconsistent with all experience and with the law of conservation of energy. Thus, we are forced to conclude that the current must be counterclockwise.

Let us consider another situation, one in which a bar magnet moves toward a stationary metal loop, as shown in Figure 31.13a. As the magnet moves to the right toward the loop, the external magnetic flux through the loop increases with time. To counteract this increase in flux to the right, the induced current produces a flux to the left, as illustrated in Figure 31.13b; hence, the induced current is in the direction shown. Note that the magnetic field lines associated with the induced current oppose the motion of the magnet. Knowing that like magnetic poles repel each other, we conclude that the left face of the current loop is in essence a north pole and that the right face is a south pole.

If the magnet moves to the left, as shown in Figure 31.13c, its flux through the area enclosed by the loop, which is directed to the right, decreases in time. Now the induced current in the loop is in the direction shown in Figure 31.13d because this current direction produces a magnetic flux in the same direction as the external flux. In this case, the left face of the loop is a south pole and the right face is a north pole.

QuickLab

This experiment takes steady hands, a dime, and a strong magnet. After verifying that a dime is not attracted to the magnet, carefully balance the coin on its edge. (This won't work with other coins because they require too much force to topple them.) Hold one pole of the magnet within a millimeter of the face of the dime, but don't bump it. Now very rapidly pull the magnet straight back away from the coin. Which way does the dime tip? Does the coin fall the same way most of the time? Explain what is going on in terms of Lenz's law. You may want to refer to Figure 31.13.

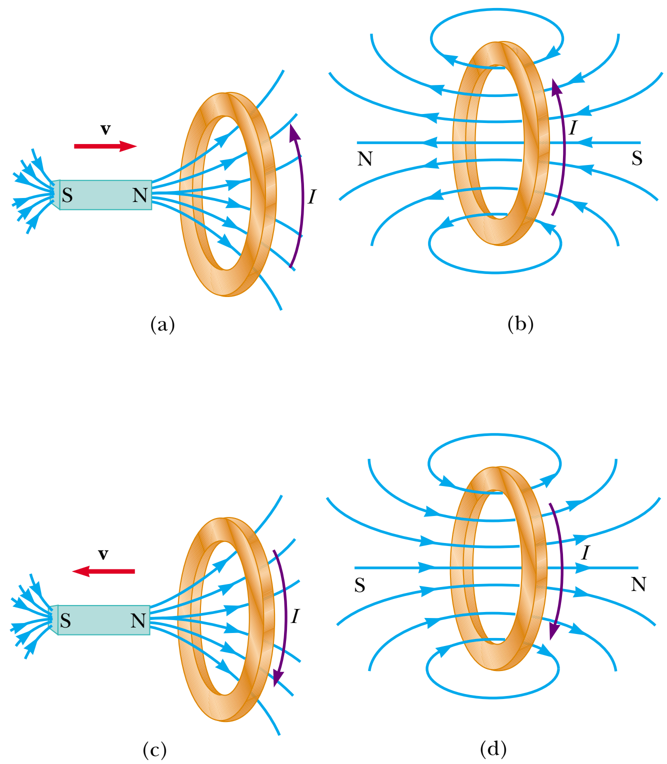


Figure 31.13 (a) When the magnet is moved toward the stationary conducting loop, a current is induced in the direction shown. (b) This induced current produces its own magnetic flux that is directed to the left and so counteracts the increasing external flux to the right. (c) When the magnet is moved away from the stationary conducting loop, a current is induced in the direction shown. (d) This induced current produces a magnetic flux that is directed to the right and so counteracts the decreasing external flux to the right.

Quick Quiz 31.3

Figure 31.14 shows a magnet being moved in the vicinity of a solenoid connected to a galvanometer. The south pole of the magnet is the pole nearest the solenoid, and the gal-

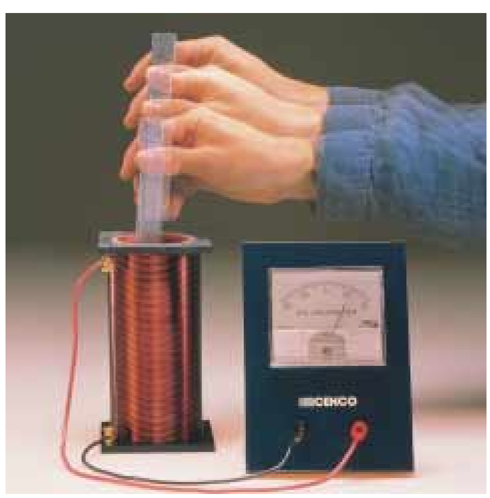


Figure 31.14 When a magnet is moved toward or away from a solenoid attached to a galvanometer, an electric current is induced, indicated by the momentary deflection of the galvanometer needle. *(Richard Megna/Fundamental Photographs)*

vanometer indicates a clockwise (viewed from above) current in the solenoid. Is the person inserting the magnet or pulling it out?

CONCEPTUAL EXAMPLE 31.6 Application of Lenz's Law

A metal ring is placed near a solenoid, as shown in Figure 31.15a. Find the direction of the induced current in the ring (a) at the instant the switch in the circuit containing the solenoid is thrown closed, (b) after the switch has been closed for several seconds, and (c) at the instant the switch is thrown open.

Solution (a) At the instant the switch is thrown closed, the situation changes from one in which no magnetic flux passes through the ring to one in which flux passes through in the direction shown in Figure 31.15b. To counteract this change in the flux, the current induced in the ring must set up a magnetic field directed from left to right in Figure 31.15b. This requires a current directed as shown.

(b) After the switch has been closed for several seconds, no change in the magnetic flux through the loop occurs; hence, the induced current in the ring is zero.

(c) Opening the switch changes the situation from one in which magnetic flux passes through the ring to one in which there is no magnetic flux. The direction of the induced current is as shown in Figure 31.15c because current in this direction produces a magnetic field that is directed right to left and so counteracts the decrease in the field produced by the solenoid.

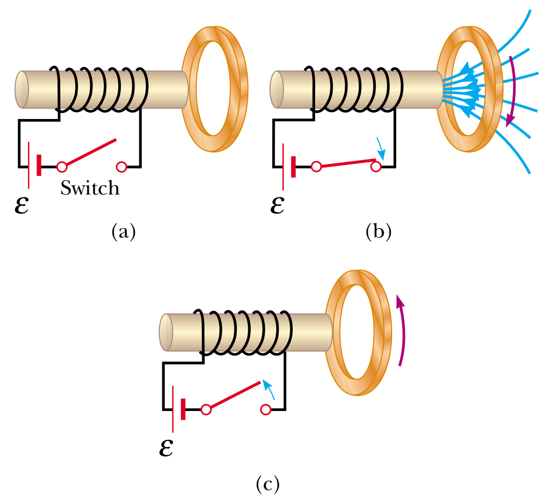


Figure 31.15

CONCEPTUAL EXAMPLE 31.7 A Loop Moving Through a Magnetic Field

A rectangular metallic loop of dimensions ℓ and w and resistance R moves with constant speed v to the right, as shown in Figure 31.16a, passing through a uniform magnetic field **B** directed into the page and extending a distance $3w$ along the x axis. Defining x as the position of the right side of the loop along the x axis, plot as functions of x (a) the magnetic flux through the area enclosed by the loop, (b) the induced motional emf, and (c) the external applied force necessary to counter the magnetic force and keep v constant.

Solution (a) Figure 31.16b shows the flux through the area enclosed by the loop as a function x. Before the loop enters the field, the flux is zero. As the loop enters the field, the flux increases linearly with position until the left edge of the loop is just inside the field. Finally, the flux through the loop decreases linearly to zero as the loop leaves the field.

(b) Before the loop enters the field, no motional emf is induced in it because no field is present (Fig. 31.16c). As the right side of the loop enters the field, the magnetic flux directed into the page increases. Hence, according to Lenz's law, the induced current is counterclockwise because it must produce a magnetic field directed out of the page. The motional emf $-B\ell v$ (from Eq. 31.5) arises from the magnetic force experienced by charges in the right side of the loop. When the loop is entirely in the field, the change in magnetic flux is zero, and hence the motional emf vanishes. This happens because, once the left side of the loop enters the field, the motional emf induced in it cancels the motional emf present in the right side of the loop. As the right side of the loop leaves the field, the flux inward begins to decrease, a clockwise current is induced, and the induced emf is $B\ell v$. As soon as the left side leaves the field, the emf decreases to zero.

(c) The external force that must be applied to the loop to maintain this motion is plotted in Figure 31.16d. Before the loop enters the field, no magnetic force acts on it; hence, the applied force must be zero if v is constant. When the right side of the loop enters the field, the applied force necessary to maintain constant speed must be equal in magnitude and opposite in direction to the magnetic force exerted on that side: $F_B = -I\ell B = -B^2\ell^2 v/R$. When the loop is entirely in the field, the flux through the loop is not changing with time. Hence, the net emf induced in the loop is zero, and the current also is zero. Therefore, no external force is needed to maintain the motion. Finally, as the right side leaves the field, the applied force must be equal in magnitude and opposite

in direction to the magnetic force acting on the left side of the loop.

From this analysis, we conclude that power is supplied only when the loop is either entering or leaving the field. Furthermore, this example shows that the motional emf induced in the loop can be zero even when there is motion through the field! A motional emf is induced only when the magnetic flux through the loop *changes in time.*

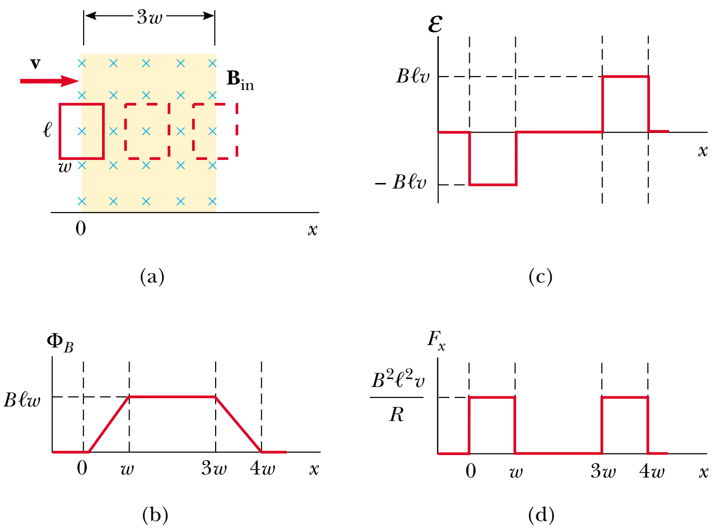


Figure 31.16 (a) A conducting rectangular loop of width w and length ℓ moving with a velocity $\mathbf{v}$ through a uniform magnetic field extending a distance $3w$. (b) Magnetic flux through the area enclosed by the loop as a function of loop position. (c) Induced emf as a function of loop position. (d) Applied force required for constant velocity as a function of loop position.

31.4 INDUCED EMF AND ELECTRIC FIELDS

12.8 We have seen that a changing magnetic flux induces an emf and a current in a conducting loop. Therefore, we must conclude that **an electric field is created in the conductor as a result of the changing magnetic flux.** However, this induced electric field has two important properties that distinguish it from the electrostatic field produced by stationary charges: The induced field is nonconservative and can vary in time.

We can illustrate this point by considering a conducting loop of radius r situated in a uniform magnetic field that is perpendicular to the plane of the loop, as shown in Figure 31.17. If the magnetic field changes with time, then, according to Faraday's law (Eq. 31.1), an emf $\mathcal{E} = -d\Phi_B/dt$ is induced in the loop. The induction of a current in the loop implies the presence of an induced electric field $\mathbf{E}$, which must be tangent to the loop because all points on the loop are equivalent. The work done in moving a test charge q once around the loop is equal to $q\mathcal{E}$. Because the electric force acting on the charge is $q\mathbf{E}$, the work done by this force in moving the charge once around the loop is $qE(2\pi r)$, where $2\pi r$ is the circumference of the loop. These two expressions for the work must be equal; therefore, we see that

$$q\mathcal{E} = qE(2\pi r)$$

$$E = \frac{\mathcal{E}}{2\pi r}$$

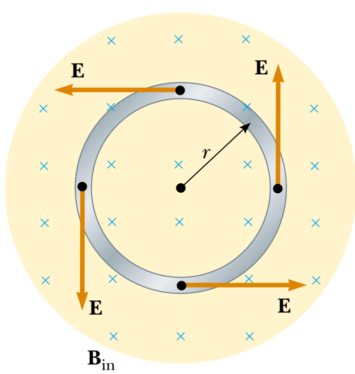


Figure 31.17 A conducting loop of radius r in a uniform magnetic field perpendicular to the plane of the loop. If $\mathbf{B}$ changes in time, an electric field is induced in a direction tangent to the circumference of the loop.

Using this result, along with Equation 31.1 and the fact that $\Phi_B = BA = \pi r^2 B$ for a

circular loop, we find that the induced electric field can be expressed as

$$E = -\frac{1}{2\pi r}\frac{d\Phi_B}{dt} = -\frac{r}{2}\frac{dB}{dt} \tag{31.8}$$

If the time variation of the magnetic field is specified, we can easily calculate the induced electric field from Equation 31.8. The negative sign indicates that the induced electric field opposes the change in the magnetic field.

The emf for any closed path can be expressed as the line integral of $\mathbf{E} \cdot d\mathbf{s}$ over that path: $\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{s}$. In more general cases, E may not be constant, and the path may not be a circle. Hence, Faraday's law of induction, $\mathcal{E} = -d\Phi_B/dt$, can be written in the general form

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \tag{31.9}$$

Faraday's law in general form

It is important to recognize that **the induced electric field E in Equation 31.9 is a nonconservative field that is generated by a changing magnetic field.** The field $\mathbf{E}$ that satisfies Equation 31.9 cannot possibly be an electrostatic field for the following reason: If the field were electrostatic, and hence conservative, the line integral of $\mathbf{E} \cdot d\mathbf{s}$ over a closed loop would be zero; this would be in contradiction to Equation 31.9.

EXAMPLE 31.8 Electric Field Induced by a Changing Magnetic Field in a Solenoid

A long solenoid of radius R has n turns of wire per unit length and carries a time-varying current that varies sinusoidally as $I = I_{max} \cos \omega t$, where I_{max} is the maximum current and ω is the angular frequency of the alternating current source (Fig. 31.18). (a) Determine the magnitude of the induced electric field outside the solenoid, a distance $r > R$ from its long central axis.

Solution First let us consider an external point and take the path for our line integral to be a circle of radius r centered on the solenoid, as illustrated in Figure 31.18. By symmetry we see that the magnitude of $\mathbf{E}$ is constant on this path and that $\mathbf{E}$ is tangent to it. The magnetic flux through the area enclosed by this path is $BA = B\pi R^2$; hence, Equation 31.9 gives

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d}{dt}(B\pi R^2) = -\pi R^2 \frac{dB}{dt}$$

$$(1) \qquad \oint \mathbf{E} \cdot d\mathbf{s} = E(2\pi r) = -\pi R^2 \frac{dB}{dt}$$

The magnetic field inside a long solenoid is given by Equation 30.17, $B = \mu_0 n I$. When we substitute $I = I_{max} \cos \omega t$ into this equation and then substitute the result into Equation (1), we find that

$$E(2\pi r) = -\pi R^2 \mu_0 n I_{max} \frac{d}{dt}(\cos \omega t) = \pi R^2 \mu_0 n I_{max} \omega \sin \omega t$$

$$(2) \qquad E = \frac{\mu_0 n I_{max} \omega R^2}{2r} \sin \omega t \qquad (\text{for } r > R)$$

Hence, the electric field varies sinusoidally with time and its amplitude falls off as $1/r$ outside the solenoid.

(b) What is the magnitude of the induced electric field inside the solenoid, a distance r from its axis?

Solution For an interior point ($r < R$), the flux threading an integration loop is given by $B\pi r^2$. Using the same proce-

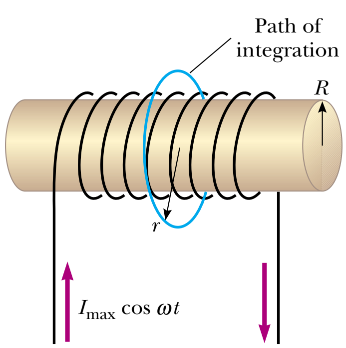


Figure 31.18 A long solenoid carrying a time-varying current given by $I = I_0 \cos \omega t$. An electric field is induced both inside and outside the solenoid.

dure as in part (a), we find that

$$E(2\pi r) = -\pi r^2 \frac{dB}{dt} = \pi r^2 \mu_0 n I_{\text{max}} \omega \sin \omega t$$

$$(3) \qquad \boxed{E = \frac{\mu_0 n I_{\text{max}} \omega}{2} r \sin \omega t} \qquad (\text{for } r < R)$$

This shows that the amplitude of the electric field induced inside the solenoid by the changing magnetic flux through the solenoid increases linearly with r and varies sinusoidally with time.

Exercise Show that Equations (2) and (3) for the exterior and interior regions of the solenoid match at the boundary, $r = R$.

Exercise Would the electric field be different if the solenoid had an iron core?

Answer Yes, it could be much stronger because the maximum magnetic field (and thus the change in flux) through the solenoid could be thousands of times larger. (See Example 30.10.)

Optional Section

31.5 GENERATORS AND MOTORS

Turbines turn generators at a hydroelectric power plant. *(Luis Castaneda/The Image Bank)*

Electric generators are used to produce electrical energy. To understand how they work, let us consider the **alternating current** (ac) **generator,** a device that converts mechanical energy to electrical energy. In its simplest form, it consists of a loop of wire rotated by some external means in a magnetic field (Fig. 31.19a).

In commercial power plants, the energy required to rotate the loop can be derived from a variety of sources. For example, in a hydroelectric plant, falling water directed against the blades of a turbine produces the rotary motion; in a coal-fired plant, the energy released by burning coal is used to convert water to steam, and this steam is directed against the turbine blades. As a loop rotates in a magnetic field, the magnetic flux through the area enclosed by the loop changes with time; this induces an emf and a current in the loop according to Faraday's law. The ends of the loop are connected to slip rings that rotate with the loop. Connections from these slip rings, which act as output terminals of the generator, to the external circuit are made by stationary brushes in contact with the slip rings.

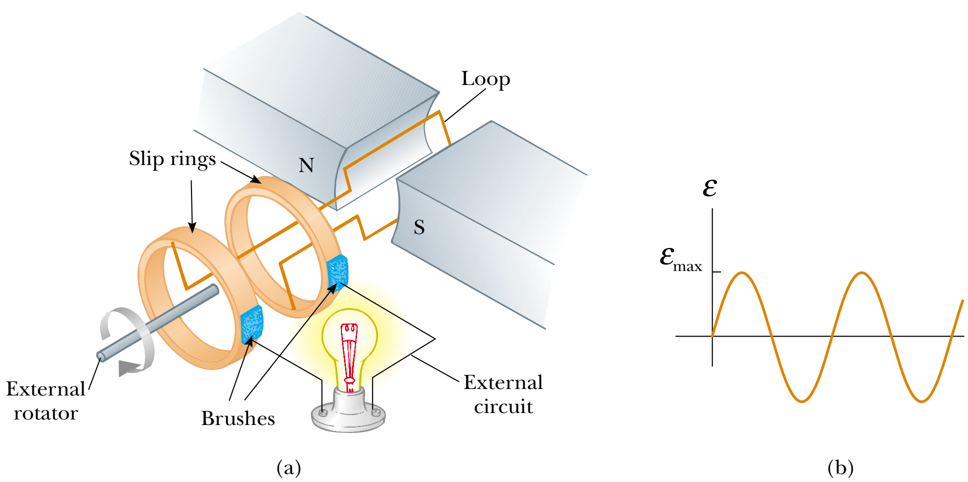


Figure 31.19 (a) Schematic diagram of an ac generator. An emf is induced in a loop that rotates in a magnetic field. (b) The alternating emf induced in the loop plotted as a function of time.

Suppose that, instead of a single turn, the loop has N turns (a more practical situation), all of the same area A, and rotates in a magnetic field with a constant angular speed ω. If θ is the angle between the magnetic field and the normal to the plane of the loop, as shown in Figure 31.20, then the magnetic flux through the loop at any time t is

$$\Phi_B = BA\cos\theta = BA\cos\omega t$$

where we have used the relationship $\theta = \omega t$ between angular displacement and angular speed (see Eq. 10.3). (We have set the clock so that $t = 0$ when $\theta = 0$.) Hence, the induced emf in the coil is

$$\mathcal{E} = -N\frac{d\Phi_B}{dt} = -NAB\frac{d}{dt}(\cos\omega t) = NAB\omega\sin\omega t \qquad \textbf{(31.10)}$$

This result shows that the emf varies sinusoidally with time, as was plotted in Figure 31.19b. From Equation 31.10 we see that the maximum emf has the value

$$\mathcal{E}_{\text{max}} = NAB\omega \qquad \textbf{(31.11)}$$

which occurs when $\omega t = 90°$ or $270°$. In other words, $\mathcal{E} = \mathcal{E}_{\text{max}}$ when the magnetic field is in the plane of the coil and the time rate of change of flux is a maximum. Furthermore, the emf is zero when $\omega t = 0$ or $180°$, that is, when **B** is perpendicular to the plane of the coil and the time rate of change of flux is zero.

The frequency for commercial generators in the United States and Canada is 60 Hz, whereas in some European countries it is 50 Hz. (Recall that $\omega = 2\pi f$, where f is the frequency in hertz.)

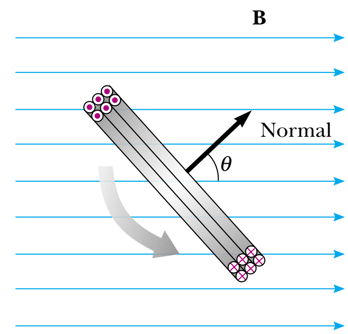


Figure 31.20 A loop enclosing an area A and containing N turns, rotating with constant angular speed ω in a magnetic field. The emf induced in the loop varies sinusoidally in time.

EXAMPLE 31.9 emf Induced in a Generator

An ac generator consists of 8 turns of wire, each of area $A = 0.090\,0\ \text{m}^2$, and the total resistance of the wire is 12.0 Ω. The loop rotates in a 0.500-T magnetic field at a constant frequency of 60.0 Hz. (a) Find the maximum induced emf.

Solution First, we note that $\omega = 2\pi f = 2\pi(60.0\ \text{Hz}) = 377\ \text{s}^{-1}$. Thus, Equation 31.11 gives

$$\mathcal{E}_{\text{max}} = NAB\omega = 8(0.090\,0\ \text{m}^2)(0.500\ \text{T})(377\ \text{s}^{-1}) = \boxed{136\ \text{V}}$$

(b) What is the maximum induced current when the output terminals are connected to a low-resistance conductor?

Solution From Equation 27.8 and the results to part (a), we have

$$I_{\text{max}} = \frac{\mathcal{E}_{\text{max}}}{R} = \frac{136\ \text{V}}{12.0\ \Omega} = \boxed{11.3\ \text{A}}$$

Exercise Determine how the induced emf and induced current vary with time.

Answer $\mathcal{E} = \mathcal{E}_{\text{max}}\sin\omega t = (136\ \text{V})\sin 377t$; $I = I_{\text{max}}\sin\omega t = (11.3\ \text{A})\sin 377t$.

The **direct current** (dc) **generator** is illustrated in Figure 31.21a. Such generators are used, for instance, in older cars to charge the storage batteries used. The components are essentially the same as those of the ac generator except that the contacts to the rotating loop are made using a split ring called a *commutator.*

In this configuration, the output voltage always has the same polarity and pulsates with time, as shown in Figure 31.21b. We can understand the reason for this by noting that the contacts to the split ring reverse their roles every half cycle. At the same time, the polarity of the induced emf reverses; hence, the polarity of the

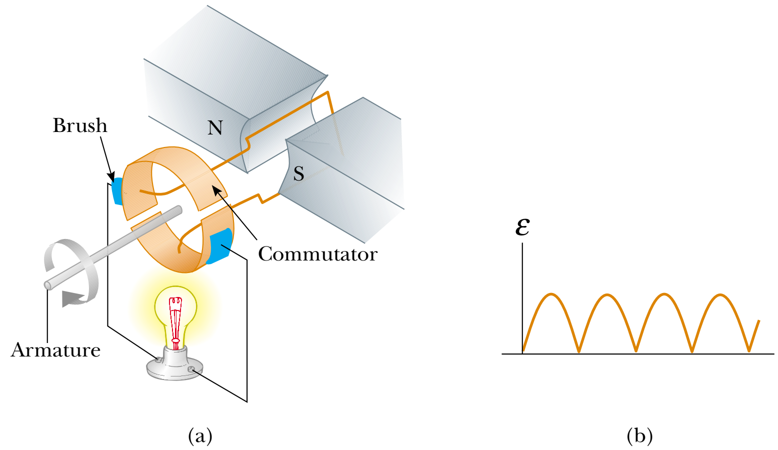


Figure 31.21 (a) Schematic diagram of a dc generator. (b) The magnitude of the emf varies in time but the polarity never changes.

split ring (which is the same as the polarity of the output voltage) remains the same.

A pulsating dc current is not suitable for most applications. To obtain a more steady dc current, commercial dc generators use many coils and commutators distributed so that the sinusoidal pulses from the various coils are out of phase. When these pulses are superimposed, the dc output is almost free of fluctuations.

Motors are devices that convert electrical energy to mechanical energy. Essentially, a motor is a generator operating in reverse. Instead of generating a current by rotating a loop, a current is supplied to the loop by a battery, and the torque acting on the current-carrying loop causes it to rotate.

Useful mechanical work can be done by attaching the rotating armature to some external device. However, as the loop rotates in a magnetic field, the changing magnetic flux induces an emf in the loop; this induced emf always acts to reduce the current in the loop. If this were not the case, Lenz's law would be violated. The back emf increases in magnitude as the rotational speed of the armature increases. (The phrase *back emf* is used to indicate an emf that tends to reduce the supplied current.) Because the voltage available to supply current equals the difference between the supply voltage and the back emf, the current in the rotating coil is limited by the back emf.

When a motor is turned on, there is initially no back emf; thus, the current is very large because it is limited only by the resistance of the coils. As the coils begin to rotate, the induced back emf opposes the applied voltage, and the current in the coils is reduced. If the mechanical load increases, the motor slows down; this causes the back emf to decrease. This reduction in the back emf increases the current in the coils and therefore also increases the power needed from the external voltage source. For this reason, the power requirements for starting a motor and for running it are greater for heavy loads than for light ones. If the motor is allowed to run under no mechanical load, the back emf reduces the current to a value just large enough to overcome energy losses due to internal energy and friction. If a very heavy load jams the motor so that it cannot rotate, the lack of a back emf can lead to dangerously high current in the motor's wire. If the problem is not corrected, a fire could result.

EXAMPLE 31.10 The Induced Current in a Motor

Assume that a motor in which the coils have a total resistance of 10 Ω is supplied by a voltage of 120 V. When the motor is running at its maximum speed, the back emf is 70 V. Find the current in the coils (a) when the motor is turned on and (b) when it has reached maximum speed.

Solution (a) When the motor is turned on, the back emf is zero (because the coils are motionless). Thus, the current in the coils is a maximum and equal to

$$I = \frac{\mathcal{E}}{R} = \frac{120\text{ V}}{10\ \Omega} = \boxed{12\text{ A}}$$

(b) At the maximum speed, the back emf has its maximum value. Thus, the effective supply voltage is that of the external source minus the back emf. Hence, the current is reduced to

$$I = \frac{\mathcal{E} - \mathcal{E}_{\text{back}}}{R} = \frac{120\text{ V} - 70\text{ V}}{10\ \Omega} = \frac{50\text{ V}}{10\ \Omega} = \boxed{5.0\text{ A}}$$

Exercise If the current in the motor is 8.0 A at some instant, what is the back emf at this time?

Answer 40 V.

Optional Section

31.6 EDDY CURRENTS

As we have seen, an emf and a current are induced in a circuit by a changing magnetic flux. In the same manner, circulating currents called **eddy currents** are induced in bulk pieces of metal moving through a magnetic field. This can easily be demonstrated by allowing a flat copper or aluminum plate attached at the end of a rigid bar to swing back and forth through a magnetic field (Fig. 31.22). As the plate enters the field, the changing magnetic flux induces an emf in the plate, which in turn causes the free electrons in the plate to move, producing the swirling eddy currents. According to Lenz's law, the direction of the eddy currents must oppose the change that causes them. For this reason, the eddy currents must produce effective magnetic poles on the plate, which are repelled by the poles of the magnet; this gives rise to a repulsive force that opposes the motion of the plate. (If the opposite were true, the plate would accelerate and its energy would

QuickLab

Hang a strong magnet from two strings so that it swings back and forth in a plane. Start it oscillating and determine approximately how much time passes before it stops swinging. Start it oscillating again and quickly bring the flat surface of an aluminum cooking sheet up to within a millimeter of the plane of oscillation, taking care not to touch the magnet. How long does it take the oscillating magnet to stop now?

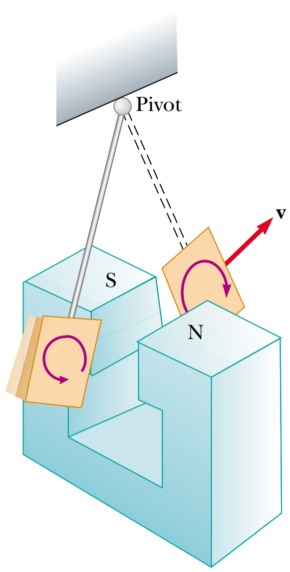


Figure 31.22 Formation of eddy currents in a conducting plate moving through a magnetic field. As the plate enters or leaves the field, the changing magnetic flux induces an emf, which causes eddy currents in the plate.

increase after each swing, in violation of the law of conservation of energy.) As you may have noticed while carrying out the QuickLab on page 997, you can "feel" the retarding force by pulling a copper or aluminum sheet through the field of a strong magnet.

As indicated in Figure 31.23, with **B** directed into the page, the induced eddy current is counterclockwise as the swinging plate enters the field at position 1. This is because the external magnetic flux into the page through the plate is increasing, and hence by Lenz's law the induced current must provide a magnetic flux out of the page. The opposite is true as the plate leaves the field at position 2, where the current is clockwise. Because the induced eddy current always produces a magnetic retarding force $\mathbf{F}_B$ when the plate enters or leaves the field, the swinging plate eventually comes to rest.

If slots are cut in the plate, as shown in Figure 31.24, the eddy currents and the corresponding retarding force are greatly reduced. We can understand this by realizing that the cuts in the plate prevent the formation of any large current loops.

The braking systems on many subway and rapid-transit cars make use of electromagnetic induction and eddy currents. An electromagnet attached to the train is positioned near the steel rails. (An electromagnet is essentially a solenoid with an iron core.) The braking action occurs when a large current is passed through the electromagnet. The relative motion of the magnet and rails induces eddy currents in the rails, and the direction of these currents produces a drag force on the moving train. The loss in mechanical energy of the train is transformed to internal energy in the rails and wheels. Because the eddy currents decrease steadily in magnitude as the train slows down, the braking effect is quite smooth. Eddy-current brakes are also used in some mechanical balances and in various machines. Some power tools use eddy currents to stop rapidly spinning blades once the device is turned off.

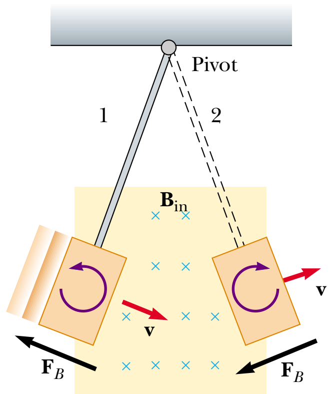


Figure 31.23 As the conducting plate enters the field (position 1), the eddy currents are counterclockwise. As the plate leaves the field (position 2), the currents are clockwise. In either case, the force on the plate is opposite the velocity, and eventually the plate comes to rest.

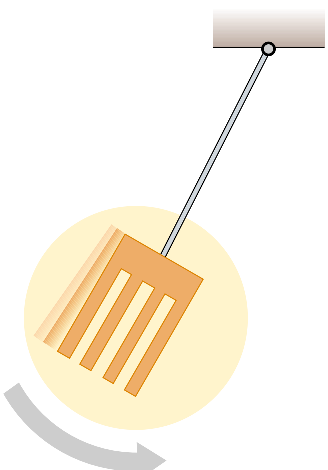


Figure 31.24 When slots are cut in the conducting plate, the eddy currents are reduced and the plate swings more freely through the magnetic field.

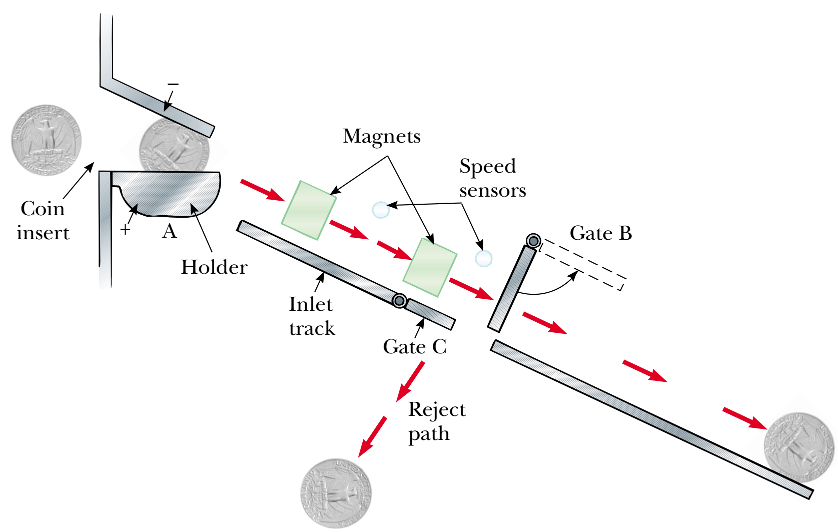


Figure 31.25 As the coin enters the vending machine, a potential difference is applied across the coin at A, and its resistance is measured. If the resistance is acceptable, the holder drops down, releasing the coin and allowing it to roll along the inlet track. Two magnets induce eddy currents in the coin, and magnetic forces control its speed. If the speed sensors indicate that the coin has the correct speed, gate B swings up to allow the coin to be accepted. If the coin is not moving at the correct speed, gate C opens to allow the coin to follow the reject path.

Eddy currents are often undesirable because they represent a transformation of mechanical energy to internal energy. To reduce this energy loss, moving conducting parts are often laminated—that is, they are built up in thin layers separated by a nonconducting material such as lacquer or a metal oxide. This layered structure increases the resistance of the possible paths of the eddy currents and effectively confines the currents to individual layers. Such a laminated structure is used in transformer cores and motors to minimize eddy currents and thereby increase the efficiency of these devices.

Even a task as simple as buying a candy bar from a vending machine involves eddy currents, as shown in Figure 31.25. After entering the slot, a coin is stopped momentarily while its electrical resistance is checked. If its resistance falls within an acceptable range, the coin is allowed to continue down a ramp and through a magnetic field. As it moves through the field, eddy currents are produced in the coin, and magnetic forces slow it down slightly. How much it is slowed down depends on its metallic composition. Sensors measure the coin's speed after it moves past the magnets, and this speed is compared with expected values. If the coin is legal and passes these tests, a gate is opened and the coin is accepted; otherwise, a second gate moves it into the reject path.

31.7 MAXWELL'S WONDERFUL EQUATIONS

12.10 We conclude this chapter by presenting four equations that are regarded as the basis of all electrical and magnetic phenomena. These equations, developed by James Clerk Maxwell, are as fundamental to electromagnetic phenomena as Newton's laws are to mechanical phenomena. In fact, the theory that Maxwell developed was more far-reaching than even he imagined because it turned out to be in agreement with the special theory of relativity, as Einstein showed in 1905.

Maxwell's equations represent the laws of electricity and magnetism that we have already discussed, but they have additional important consequences. In Chapter 34 we shall show that these equations predict the existence of electromagnetic waves (traveling patterns of electric and magnetic fields), which travel with a speed $c = 1/\sqrt{\mu_0\epsilon_0} = 3.00 \times 10^8$ m/s, the speed of light. Furthermore, the theory shows that such waves are radiated by accelerating charges.

For simplicity, we present **Maxwell's equations** as applied to free space, that is, in the absence of any dielectric or magnetic material. The four equations are

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0} \tag{31.12}$$

Gauss's law

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \tag{31.13}$$

Gauss's law in magnetism

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \tag{31.14}$$

Faraday's law

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \epsilon_0\mu_0 \frac{d\Phi_E}{dt} \tag{31.15}$$

Ampère–Maxwell law

Equation 31.12 is Gauss's law: **The total electric flux through any closed surface equals the net charge inside that surface divided by ϵ_0.** This law relates an electric field to the charge distribution that creates it.

Equation 31.13, which can be considered Gauss's law in magnetism, states that **the net magnetic flux through a closed surface is zero.** That is, the number of magnetic field lines that enter a closed volume must equal the number that leave that volume. This implies that magnetic field lines cannot begin or end at any point. If they did, it would mean that isolated magnetic monopoles existed at those points. The fact that isolated magnetic monopoles have not been observed in nature can be taken as a confirmation of Equation 31.13.

Equation 31.14 is Faraday's law of induction, which describes the creation of an electric field by a changing magnetic flux. This law states that **the emf, which is the line integral of the electric field around any closed path, equals the rate of change of magnetic flux through any surface area bounded by that path.** One consequence of Faraday's law is the current induced in a conducting loop placed in a time-varying magnetic field.

Equation 31.15, usually called the Ampère–Maxwell law, is the generalized form of Ampère's law, which describes the creation of a magnetic field by an electric field and electric currents: **The line integral of the magnetic field around any closed path is the sum of μ_0 times the net current through that path and $\epsilon_0\mu_0$ times the rate of change of electric flux through any surface bounded by that path.**

Once the electric and magnetic fields are known at some point in space, the force acting on a particle of charge q can be calculated from the expression

Lorentz force law

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B} \tag{31.16}$$

This relationship is called the **Lorentz force law.** (We saw this relationship earlier as Equation 29.16.) Maxwell's equations, together with this force law, completely describe all classical electromagnetic interactions.

It is interesting to note the symmetry of Maxwell's equations. Equations 31.12 and 31.13 are symmetric, apart from the absence of the term for magnetic monopoles in Equation 31.13. Furthermore, Equations 31.14 and 31.15 are symmetric in that the line integrals of $\mathbf{E}$ and $\mathbf{B}$ around a closed path are related to the rate of change of magnetic flux and electric flux, respectively. "Maxwell's wonderful equations," as they were called by John R. Pierce,[3] are of fundamental importance not only to electromagnetism but to all of science. Heinrich Hertz once wrote, "One cannot escape the feeling that these mathematical formulas have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than we put into them."

SUMMARY

Faraday's law of induction states that the emf induced in a circuit is directly proportional to the time rate of change of magnetic flux through the circuit:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} \tag{31.1}$$

where $\Phi_B = \int \mathbf{B} \cdot d\mathbf{A}$ is the magnetic flux.

[3] John R. Pierce, *Electrons and Waves,* New York, Doubleday Science Study Series, 1964. Chapter 6 of this interesting book is recommended as supplemental reading.

When a conducting bar of length ℓ moves at a velocity $\mathbf{v}$ through a magnetic field $\mathbf{B}$, where $\mathbf{B}$ is perpendicular to the bar and to $\mathbf{v}$, the **motional emf** induced in the bar is

$$\mathcal{E} = -B\ell v \tag{31.5}$$

Lenz's law states that the induced current and induced emf in a conductor are in such a direction as to oppose the change that produced them.

A general form of **Faraday's law of induction** is

$$\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \tag{31.9}$$

where $\mathbf{E}$ is the nonconservative electric field that is produced by the changing magnetic flux.

When used with the Lorentz force law, $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$, **Maxwell's equations** describe all electromagnetic phenomena:

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0} \tag{31.12}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \tag{31.13}$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \tag{31.14}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt} \tag{31.15}$$

The Ampère–Maxwell law (Eq. 31.15) describes how a magnetic field can be produced by both a conduction current and a changing electric flux.

QUESTIONS

1. A loop of wire is placed in a uniform magnetic field. For what orientation of the loop is the magnetic flux a maximum? For what orientation is the flux zero? Draw pictures of these two situations.
2. As the conducting bar shown in Figure Q31.2 moves to the right, an electric field directed downward is set up in the bar. Explain why the electric field would be upward if the bar were to move to the left.
3. As the bar shown in Figure Q31.2 moves in a direction perpendicular to the field, is an applied force required to keep it moving with constant speed? Explain.
4. The bar shown in Figure Q31.4 moves on rails to the right with a velocity $\mathbf{v}$, and the uniform, constant magnetic field is directed out of the page. Why is the induced current clockwise? If the bar were moving to the left, what would be the direction of the induced current?
5. Explain why an applied force is necessary to keep the bar shown in Figure Q31.4 moving with a constant speed.
6. A large circular loop of wire lies in the horizontal plane. A bar magnet is dropped through the loop. If the axis of the magnet remains horizontal as it falls, describe the emf induced in the loop. How is the situation altered if the axis of the magnet remains vertical as it falls?

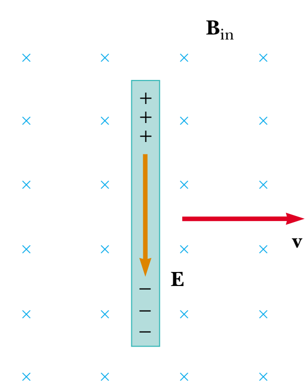


Figure Q31.2 (Questions 2 and 3).

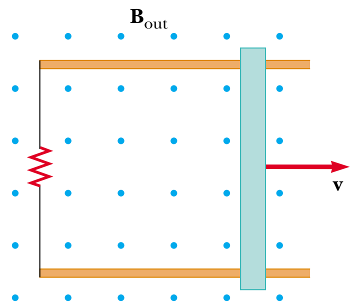


Figure Q31.4 (Questions 4 and 5).

7. When a small magnet is moved toward a solenoid, an emf is induced in the coil. However, if the magnet is moved around inside a toroid, no emf is induced. Explain.
8. Will dropping a magnet down a long copper tube produce a current in the walls of the tube? Explain.
9. How is electrical energy produced in dams (that is, how is the energy of motion of the water converted to alternating current electricity)?
10. In a beam–balance scale, an aluminum plate is sometimes used to slow the oscillations of the beam near equilibrium. The plate is mounted at the end of the beam and moves between the poles of a small horseshoe magnet attached to the frame. Why are the oscillations strongly damped near equilibrium?
11. What happens when the rotational speed of a generator coil is increased?
12. Could a current be induced in a coil by the rotation of a magnet inside the coil? If so, how?
13. When the switch shown in Figure Q31.13a is closed, a current is set up in the coil, and the metal ring springs upward (Fig. Q31.13b). Explain this behavior.

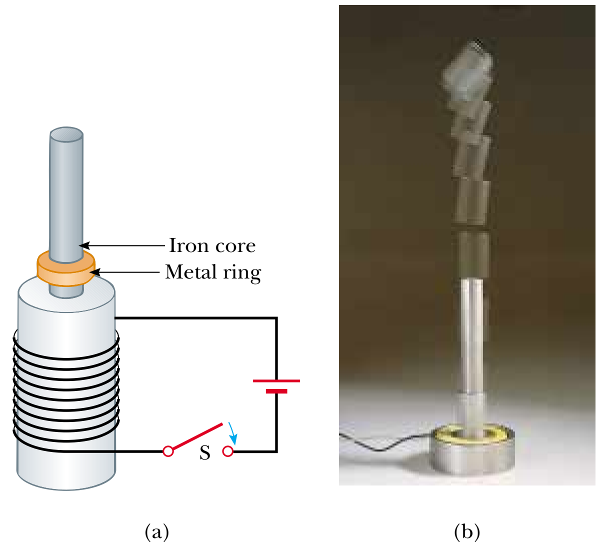


Figure Q31.13 (Questions 13 and 14). *(Photo courtesy of Central Scientific Company)*

14. Assume that the battery shown in Figure Q31.13a is replaced by an alternating current source and that the switch is held closed. If held down, the metal ring on top of the solenoid becomes hot. Why?
15. Do Maxwell's equations allow for the existence of magnetic monopoles? Explain.

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
☐ = paired numerical/symbolic problems

Section 31.1 Faraday's Law of Induction
Section 31.2 Motional emf
Section 31.3 Lenz's Law

1. A 50-turn rectangular coil of dimensions 5.00 cm × 10.0 cm is allowed to fall from a position where $B = 0$ to a new position where $B = 0.500$ T and is directed perpendicular to the plane of the coil. Calculate the magnitude of the average emf induced in the coil if the displacement occurs in 0.250 s.
2. A flat loop of wire consisting of a single turn of cross-sectional area 8.00 cm^2 is perpendicular to a magnetic field that increases uniformly in magnitude from 0.500 T to 2.50 T in 1.00 s. What is the resulting induced current if the loop has a resistance of 2.00 Ω?
3. A 25-turn circular coil of wire has a diameter of 1.00 m. It is placed with its axis along the direction of the Earth's magnetic field of 50.0 μT, and then in 0.200 s it is flipped 180°. An average emf of what magnitude is generated in the coil?
4. A rectangular loop of area A is placed in a region where the magnetic field is perpendicular to the plane of the loop. The magnitude of the field is allowed to vary in time according to the expression $B = B_{max}e^{-t/\tau}$, where B_{max} and τ are constants. The field has the constant value B_{max} for $t < 0$. (a) Use Faraday's law to show that the emf induced in the loop is given by

$$\mathcal{E} = (AB_{max}/\tau)e^{-t/\tau}$$

(b) Obtain a numerical value for $\mathcal{E}$ at $t = 4.00$ s when

$A = 0.160\ m^2$, $B_{max} = 0.350$ T, and $\tau = 2.00$ s. (c) For the values of A, B_{max}, and τ given in part (b), what is the maximum value of $\mathcal{E}$?

WEB **5.** A strong electromagnet produces a uniform field of 1.60 T over a cross-sectional area of 0.200 m^2. A coil having 200 turns and a total resistance of 20.0 Ω is placed around the electromagnet. The current in the electromagnet is then smoothly decreased until it reaches zero in 20.0 ms. What is the current induced in the coil?

6. A magnetic field of 0.200 T exists within a solenoid of 500 turns and a diameter of 10.0 cm. How rapidly (that is, within what period of time) must the field be reduced to zero if the average induced emf within the coil during this time interval is to be 10.0 kV?

WEB **7.** An aluminum ring with a radius of 5.00 cm and a resistance of 3.00×10^{-4} Ω is placed on top of a long air-core solenoid with 1 000 turns per meter and a radius of 3.00 cm, as shown in Figure P31.7. Assume that the axial component of the field produced by the solenoid over the area of the end of the solenoid is one-half as strong as at the center of the solenoid. Assume that the solenoid produces negligible field outside its cross-sectional area. (a) If the current in the solenoid is increasing at a rate of 270 A/s, what is the induced current in the ring? (b) At the center of the ring, what is the magnetic field produced by the induced current in the ring? (c) What is the direction of this field?

8. An aluminum ring of radius r_1 and resistance R is placed on top of a long air-core solenoid with n turns per meter and smaller radius r_2, as shown in Figure P31.7. Assume that the axial component of the field produced by the solenoid over the area of the end of the solenoid is one-half as strong as at the center of the solenoid. Assume that the solenoid produces negligible field outside its cross-sectional area. (a) If the current in the solenoid is increasing at a rate of $\Delta I/\Delta t$, what is the induced current in the ring? (b) At the center of the ring, what is the magnetic field produced by the induced current in the ring? (c) What is the direction of this field?

9. A loop of wire in the shape of a rectangle of width w and length L and a long, straight wire carrying a current I lie on a tabletop as shown in Figure P31.9. (a) Determine the magnetic flux through the loop due to the current I. (b) Suppose that the current is changing with time according to $I = a + bt$, where a and b are constants. Determine the induced emf in the loop if $b = 10.0$ A/s, $h = 1.00$ cm, $w = 10.0$ cm, and $L = 100$ cm. What is the direction of the induced current in the rectangle?

10. A coil of 15 turns and radius 10.0 cm surrounds a long solenoid of radius 2.00 cm and 1.00×10^3 turns per meter (Fig. P31.10). If the current in the solenoid changes as $I = (5.00\text{ A}) \sin(120t)$, find the induced emf in the 15-turn coil as a function of time.

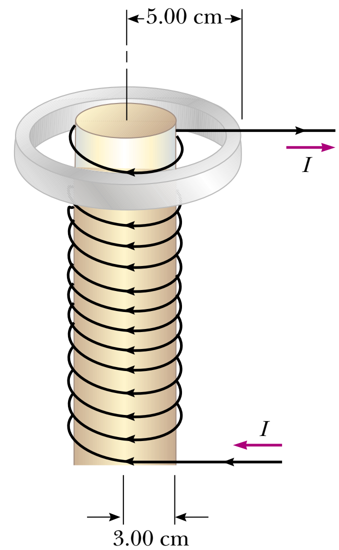


Figure P31.7 Problems 7 and 8.

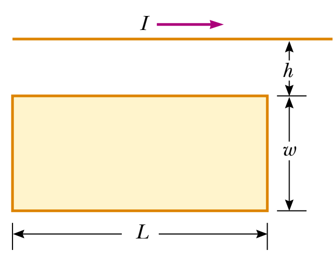


Figure P31.9 Problems 9 and 73.

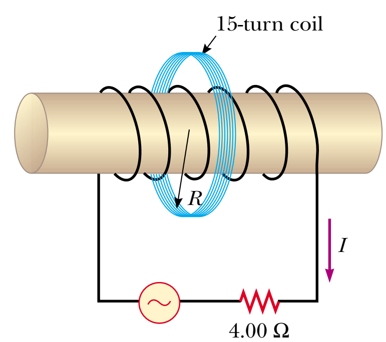


Figure P31.10

11. Find the current through section PQ of length a = 65.0 cm shown in Figure P31.11. The circuit is located in a magnetic field whose magnitude varies with time according to the expression $B = (1.00 \times 10^{-3}\text{ T/s})t$. Assume that the resistance per length of the wire is 0.100 Ω/m.

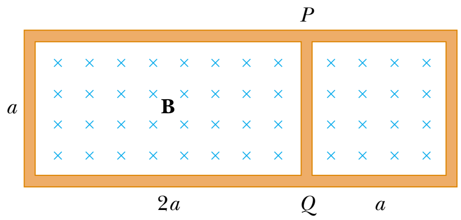


Figure P31.11

12. A 30-turn circular coil of radius 4.00 cm and resistance 1.00 Ω is placed in a magnetic field directed perpendicular to the plane of the coil. The magnitude of the magnetic field varies in time according to the expression $B = 0.010\,0t + 0.040\,0t^2$, where t is in seconds and B is in tesla. Calculate the induced emf in the coil at $t = 5.00$ s.

13. A long solenoid has 400 turns per meter and carries a current $I = (30.0\text{ A})(1 - e^{-1.60t})$. Inside the solenoid and coaxial with it is a coil that has a radius of 6.00 cm and consists of a total of 250 turns of fine wire (Fig. P31.13). What emf is induced in the coil by the changing current?

14. A long solenoid has n turns per meter and carries a current $I = I_{max}(1 - e^{-\alpha t})$. Inside the solenoid and coaxial with it is a coil that has a radius R and consists of a total of N turns of fine wire (see Fig. P31.13). What emf is induced in the coil by the changing current?

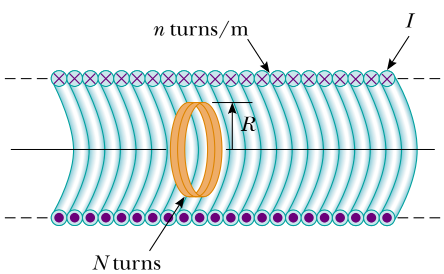


Figure P31.13 Problems 13 and 14.

15. A coil formed by wrapping 50 turns of wire in the shape of a square is positioned in a magnetic field so that the normal to the plane of the coil makes an angle of 30.0° with the direction of the field. When the magnetic field is increased uniformly from 200 μT to 600 μT in 0.400 s, an emf of magnitude 80.0 mV is induced in the coil. What is the total length of the wire?

16. A closed loop of wire is given the shape of a circle with a radius of 0.500 m. It lies in a plane perpendicular to a uniform magnetic field of magnitude 0.400 T. If in 0.100 s the wire loop is reshaped into a square but remains in the same plane, what is the magnitude of the average induced emf in the wire during this time?

17. A toroid having a rectangular cross-section (a = 2.00 cm by b = 3.00 cm) and inner radius R = 4.00 cm consists of 500 turns of wire that carries a current $I = I_{max} \sin \omega t$, with I_{max} = 50.0 A and a frequency $f = \omega/2\pi$ = 60.0 Hz. A coil that consists of 20 turns of wire links with the toroid, as shown in Figure P31.17. Determine the emf induced in the coil as a function of time.

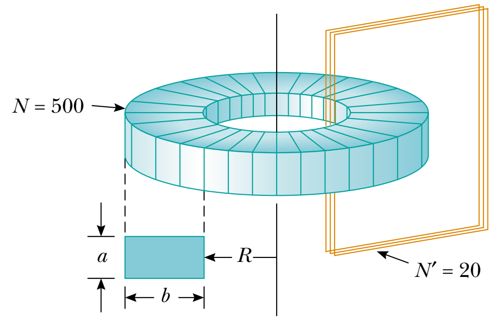


Figure P31.17

18. A single-turn, circular loop of radius R is coaxial with a long solenoid of radius r and length ℓ and having N turns (Fig. P31.18). The variable resistor is changed so that the solenoid current decreases linearly from I_1 to I_2 in an interval Δt. Find the induced emf in the loop.

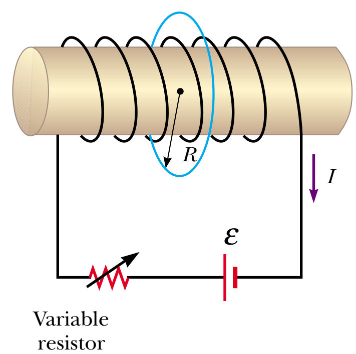


Figure P31.18

19. A circular coil enclosing an area of 100 cm² is made of 200 turns of copper wire, as shown in Figure P31.19. Ini-

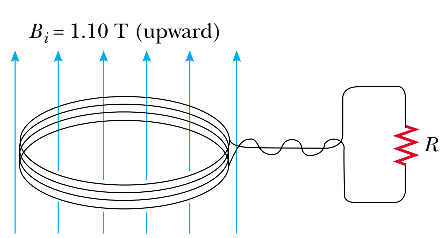


Figure P31.19

tially, a 1.10-T uniform magnetic field points in a perpendicular direction upward through the plane of the coil. The direction of the field then reverses. During the time the field is changing its direction, how much charge flows through the coil if $R = 5.00\ \Omega$?

20. Consider the arrangement shown in Figure P31.20. Assume that $R = 6.00\ \Omega$, $\ell = 1.20$ m, and a uniform 2.50-T magnetic field is directed into the page. At what speed should the bar be moved to produce a current of 0.500 A in the resistor?

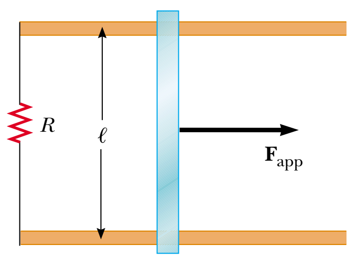


Figure P31.20 Problems 20, 21, and 22.

21. Figure P31.20 shows a top view of a bar that can slide without friction. The resistor is 6.00 Ω and a 2.50-T magnetic field is directed perpendicularly downward, into the paper. Let $\ell = 1.20$ m. (a) Calculate the applied force required to move the bar to the right at a constant speed of 2.00 m/s. (b) At what rate is energy delivered to the resistor?

22. A conducting rod of length ℓ moves on two horizontal, frictionless rails, as shown in Figure P31.20. If a constant force of 1.00 N moves the bar at 2.00 m/s through a magnetic field **B** that is directed into the page, (a) what is the current through an 8.00-Ω resistor R? (b) What is the rate at which energy is delivered to the resistor? (c) What is the mechanical power delivered by the force $\mathbf{F}_{app}$?

23. A Boeing-747 jet with a wing span of 60.0 m is flying horizontally at a speed of 300 m/s over Phoenix, Arizona, at a location where the Earth's magnetic field is 50.0 μT at 58.0° below the horizontal. What voltage is generated between the wingtips?

24. The square loop in Figure P31.24 is made of wires with total series resistance 10.0 Ω. It is placed in a uniform 0.100-T magnetic field directed perpendicular into the plane of the paper. The loop, which is hinged at each corner, is pulled as shown until the separation between points A and B is 3.00 m. If this process takes 0.100 s, what is the average current generated in the loop? What is the direction of the current?

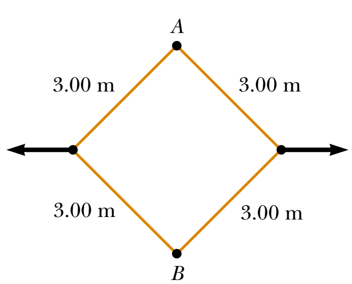


Figure P31.24

25. A helicopter has blades with a length of 3.00 m extending outward from a central hub and rotating at 2.00 rev/s. If the vertical component of the Earth's magnetic field is 50.0 μT, what is the emf induced between the blade tip and the center hub?

26. Use Lenz's law to answer the following questions concerning the direction of induced currents: (a) What is the direction of the induced current in resistor R shown in Figure P31.26a when the bar magnet is moved to the left? (b) What is the direction of the current induced in the resistor R right after the switch S in Figure P31.26b is closed? (c) What is the direction of the induced current in R when the current I in Figure P31.26c decreases rapidly to zero? (d) A copper bar is moved to the right while its axis is maintained in a direction perpendicular to a magnetic field, as shown in Figure P31.26d. If the top of the bar becomes positive relative to the bottom, what is the direction of the magnetic field?

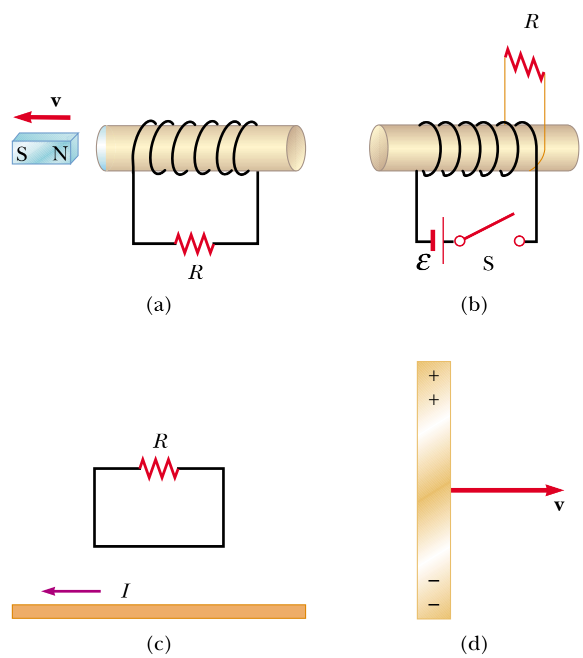


Figure P31.26

27. A rectangular coil with resistance R has N turns, each of length ℓ and width w as shown in Figure P31.27. The coil moves into a uniform magnetic field **B** with a velocity **v**. What are the magnitude and direction of the resultant force on the coil (a) as it enters the magnetic field, (b) as it moves within the field, and (c) as it leaves the field?

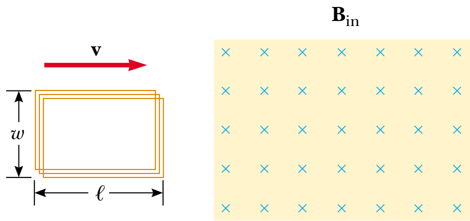


Figure P31.27

28. In 1832 Faraday proposed that the apparatus shown in Figure P31.28 could be used to generate electric current from the water flowing in the Thames River.[4] Two conducting plates of lengths a and widths b are placed facing each other on opposite sides of the river, a distance w apart, and are immersed entirely. The flow velocity of the river is $\mathbf{v}$ and the vertical component of the Earth's magnetic field is B. (a) Show that the current in the load resistor R is

$$I = \frac{abvB}{\rho + abR/w}$$

where ρ is the electrical resistivity of the water. (b) Calculate the short-circuit current ($R = 0$) if $a = 100$ m, $b = 5.00$ m, $v = 3.00$ m/s, $B = 50.0\ \mu$T, and $\rho = 100\ \Omega \cdot$ m.

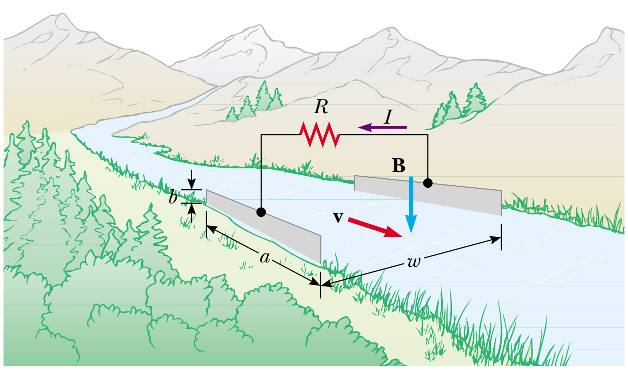


Figure P31.28

29. In Figure P31.29, the bar magnet is moved toward the loop. Is $V_a - V_b$ positive, negative, or zero? Explain.

30. A metal bar spins at a constant rate in the magnetic field of the Earth as in Figure 31.10. The rotation occurs in a region where the component of the Earth's magnetic field perpendicular to the plane of rotation is 3.30×10^{-5} T. If the bar is 1.00 m in length and its angular speed is $5.00\ \pi$ rad/s, what potential difference is developed between its ends?

[4] The idea for this problem and Figure P31.28 is from Oleg D. Jefimenko, *Electricity and Magnetism: An Introduction to the Theory of Electric and Magnetic Fields.* Star City, WV, Electret Scientific Co., 1989.

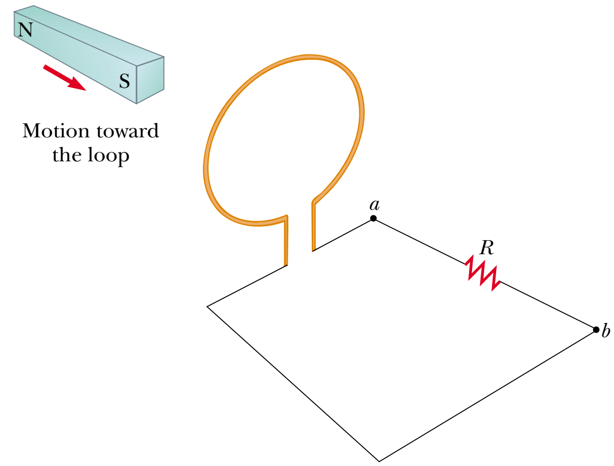


Figure P31.29

31. Two parallel rails with negligible resistance are 10.0 cm apart and are connected by a 5.00-Ω resistor. The circuit also contains two metal rods having resistances of 10.0 Ω and 15.0 Ω sliding along the rails (Fig. P31.31). The rods are pulled away from the resistor at constant speeds 4.00 m/s and 2.00 m/s, respectively. A uniform magnetic field of magnitude 0.010 0 T is applied perpendicular to the plane of the rails. Determine the current in the 5.00-Ω resistor.

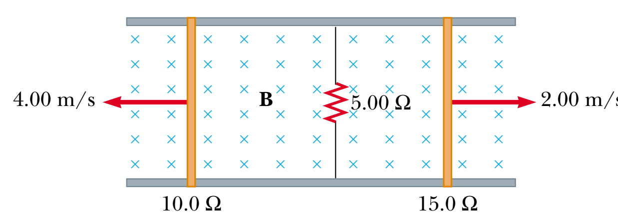


Figure P31.31

Section 31.4 Induced emf and Electric Fields

32. For the situation described in Figure P31.32, the magnetic field changes with time according to the expression $B = (2.00t^3 - 4.00t^2 + 0.800)$ T, and $r_2 = 2R = 5.00$ cm. (a) Calculate the magnitude and direction of

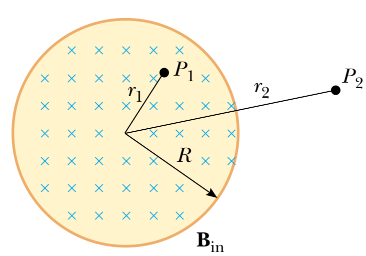


Figure P31.32 Problems 32 and 33.

the force exerted on an electron located at point P_2 when $t = 2.00$ s. (b) At what time is this force equal to zero?

33. A magnetic field directed into the page changes with time according to $B = (0.030\,0t^2 + 1.40)$ T, where t is in seconds. The field has a circular cross-section of radius $R = 2.50$ cm (see Fig. P31.32). What are the magnitude and direction of the electric field at point P_1 when $t = 3.00$ s and $r_1 = 0.020\,0$ m?

34. A solenoid has a radius of 2.00 cm and 1 000 turns per meter. Over a certain time interval the current varies with time according to the expression $I = 3e^{0.2t}$, where I is in amperes and t is in seconds. Calculate the electric field 5.00 cm from the axis of the solenoid at $t = 10.0$ s.

35. A long solenoid with 1 000 turns per meter and radius 2.00 cm carries an oscillating current $I = (5.00\text{ A})\sin(100\pi t)$. (a) What is the electric field induced at a radius $r = 1.00$ cm from the axis of the solenoid? (b) What is the direction of this electric field when the current is increasing counterclockwise in the coil?

(Optional)

Section 31.5 Generators and Motors

36. In a 250-turn automobile alternator, the magnetic flux in each turn is $\Phi_B = (2.50 \times 10^{-4}\text{ T}\cdot\text{m}^2)\cos(\omega t)$, where ω is the angular speed of the alternator. The alternator is geared to rotate three times for each engine revolution. When the engine is running at an angular speed of 1 000 rev/min, determine (a) the induced emf in the alternator as a function of time and (b) the maximum emf in the alternator.

WEB 37. A coil of area 0.100 m^2 is rotating at 60.0 rev/s with the axis of rotation perpendicular to a 0.200-T magnetic field. (a) If there are 1 000 turns on the coil, what is the maximum voltage induced in it? (b) What is the orientation of the coil with respect to the magnetic field when the maximum induced voltage occurs?

38. A square coil (20.0 cm × 20.0 cm) that consists of 100 turns of wire rotates about a vertical axis at 1 500 rev/min, as indicated in Figure P31.38. The horizontal component of the Earth's magnetic field at the location of the coil is 2.00×10^{-5} T. Calculate the maximum emf induced in the coil by this field.

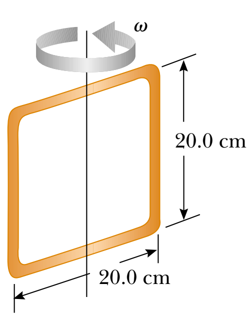


Figure P31.38

39. A long solenoid, with its axis along the x axis, consists of 200 turns per meter of wire that carries a steady current of 15.0 A. A coil is formed by wrapping 30 turns of thin wire around a circular frame that has a radius of 8.00 cm. The coil is placed inside the solenoid and mounted on an axis that is a diameter of the coil and coincides with the y axis. The coil is then rotated with an angular speed of 4.00π rad/s. (The plane of the coil is in the yz plane at $t = 0$.) Determine the emf developed in the coil as a function of time.

40. A bar magnet is spun at constant angular speed ω around an axis, as shown in Figure P31.40. A flat rectangular conducting loop surrounds the magnet, and at $t = 0$, the magnet is oriented as shown. Make a qualitative graph of the induced current in the loop as a function of time, plotting counterclockwise currents as positive and clockwise currents as negative.

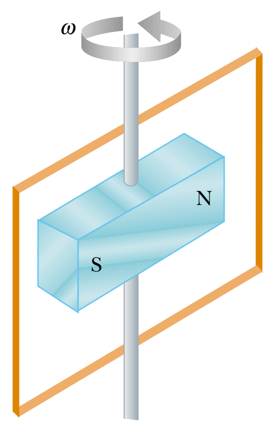


Figure P31.40

41. (a) What is the maximum torque delivered by an electric motor if it has 80 turns of wire wrapped on a rectangular coil of dimensions 2.50 cm by 4.00 cm? Assume that the motor uses 10.0 A of current and that a uniform 0.800-T magnetic field exists within the motor. (b) If the motor rotates at 3 600 rev/min, what is the peak power produced by the motor?

42. A semicircular conductor of radius $R = 0.250$ m is rotated about the axis AC at a constant rate of 120 rev/min (Fig. P31.42). A uniform magnetic field in all of the lower half of the figure is directed out of the plane of rotation and has a magnitude of 1.30 T. (a) Calculate the maximum value of the emf induced in the conductor. (b) What is the value of the average induced emf for each complete rotation? (c) How would the answers to parts (a) and (b) change if **B** were allowed to extend a distance R above the axis of rotation? Sketch the emf versus time (d) when the field is as drawn in Figure P31.42 and (e) when the field is extended as described in part (c).

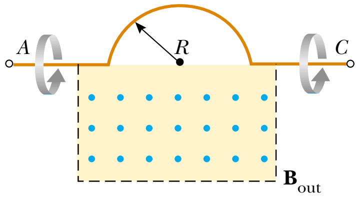


Figure P31.42

43. The rotating loop in an ac generator is a square 10.0 cm on a side. It is rotated at 60.0 Hz in a uniform field of 0.800 T. Calculate (a) the flux through the loop as a function of time, (b) the emf induced in the loop, (c) the current induced in the loop for a loop resistance of 1.00 Ω, (d) the power in the resistance of the loop, and (e) the torque that must be exerted to rotate the loop.

(Optional)
Section 31.6 Eddy Currents

44. A 0.150-kg wire in the shape of a closed rectangle 1.00 m wide and 1.50 m long has a total resistance of 0.750 Ω. The rectangle is allowed to fall through a magnetic field directed perpendicular to the direction of motion of the rectangle (Fig. P31.44). The rectangle accelerates downward as it approaches a terminal speed of 2.00 m/s, with its top not yet in the region of the field. Calculate the magnitude of **B**.

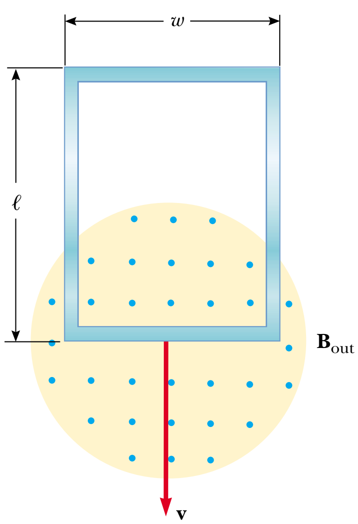


Figure P31.44 Problems 44 and 45.

WEB 45. A conducting rectangular loop of mass M, resistance R, and dimensions w by ℓ falls from rest into a magnetic field **B** as in Figure P31.44. The loop approaches terminal speed v_t. (a) Show that

$$v_t = \frac{MgR}{B^2w^2}$$

(b) Why is v_t proportional to R? (c) Why is it inversely proportional to B^2?

46. Figure P31.46 represents an electromagnetic brake that utilizes eddy currents. An electromagnet hangs from a railroad car near one rail. To stop the car, a large steady current is sent through the coils of the electromagnet. The moving electromagnet induces eddy currents in the rails, whose fields oppose the change in the field of the electromagnet. The magnetic fields of the eddy currents exert force on the current in the electromagnet, thereby slowing the car. The direction of the car's motion and the direction of the current in the electromagnet are shown correctly in the picture. Determine which of the eddy currents shown on the rails is correct. Explain your answer.

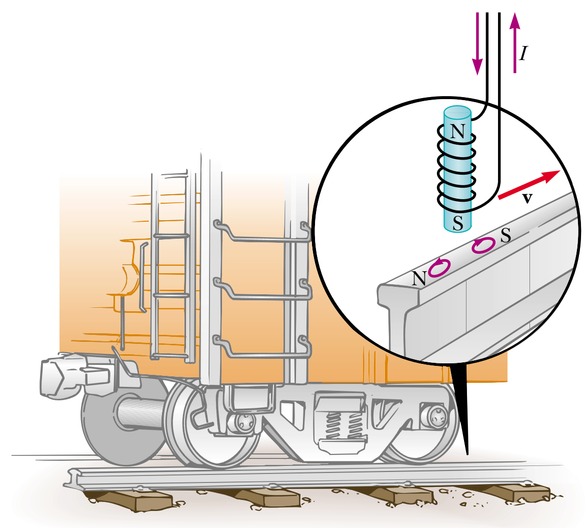


Figure P31.46

Section 31.7 Maxwell's Wonderful Equations

47. A proton moves through a uniform electric field $\mathbf{E} = 50.0\mathbf{j}$ V/m and a uniform magnetic field $\mathbf{B} = (0.200\mathbf{i} + 0.300\mathbf{j} + 0.400\mathbf{k})$ T. Determine the acceleration of the proton when it has a velocity $\mathbf{v} = 200\mathbf{i}$ m/s.

48. An electron moves through a uniform electric field $\mathbf{E} = (2.50\mathbf{i} + 5.00\mathbf{j})$ V/m and a uniform magnetic field $\mathbf{B} = 0.400\mathbf{k}$ T. Determine the acceleration of the electron when it has a velocity $\mathbf{v} = 10.0\mathbf{i}$ m/s.

ADDITIONAL PROBLEMS

49. A steel guitar string vibrates (see Fig. 31.5). The component of the magnetic field perpendicular to the area of

a pickup coil nearby is given by

$$B = 50.0 \text{ mT} + (3.20 \text{ mT}) \sin(2\pi 523\, t/\text{s})$$

The circular pickup coil has 30 turns and radius 2.70 mm. Find the emf induced in the coil as a function of time.

50. Figure P31.50 is a graph of the induced emf versus time for a coil of N turns rotating with angular velocity ω in a uniform magnetic field directed perpendicular to the axis of rotation of the coil. Copy this graph (on a larger scale), and on the same set of axes show the graph of emf versus t (a) if the number of turns in the coil is doubled, (b) if instead the angular velocity is doubled, and (c) if the angular velocity is doubled while the number of turns in the coil is halved.

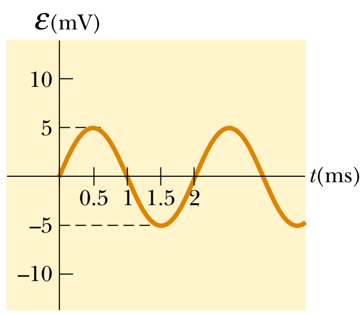


Figure P31.50

51. A technician wearing a brass bracelet enclosing an area of 0.005 00 m^2 places her hand in a solenoid whose magnetic field is 5.00 T directed perpendicular to the plane of the bracelet. The electrical resistance around the circumference of the bracelet is 0.020 0 Ω. An unexpected power failure causes the field to drop to 1.50 T in a time of 20.0 ms. Find (a) the current induced in the bracelet and (b) the power delivered to the resistance of the bracelet. (*Note:* As this problem implies, you should not wear any metallic objects when working in regions of strong magnetic fields.)

52. Two infinitely long solenoids (seen in cross-section) thread a circuit as shown in Figure P31.52. The magnitude of **B** inside each is the same and is increasing at the rate of 100 T/s. What is the current in each resistor?

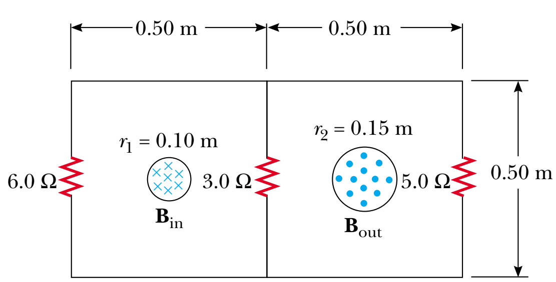


Figure P31.52

53. A conducting rod of length ℓ = 35.0 cm is free to slide on two parallel conducting bars, as shown in Figure P31.53. Two resistors R_1 = 2.00 Ω and R_2 = 5.00 Ω are connected across the ends of the bars to form a loop. A constant magnetic field B = 2.50 T is directed perpendicular into the page. An external agent pulls the rod to the left with a constant speed of v = 8.00 m/s. Find (a) the currents in both resistors, (b) the total power delivered to the resistance of the circuit, and (c) the magnitude of the applied force that is needed to move the rod with this constant velocity.

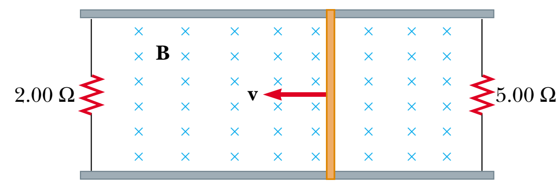


Figure P31.53

54. Suppose you wrap wire onto the core from a roll of cellophane tape to make a coil. Describe how you can use a bar magnet to produce an induced voltage in the coil. What is the order of magnitude of the emf you generate? State the quantities you take as data and their values.

55. A bar of mass m, length d, and resistance R slides without friction on parallel rails, as shown in Figure P31.55. A battery that maintains a constant emf $\mathcal{E}$ is connected between the rails, and a constant magnetic field **B** is directed perpendicular to the plane of the page. If the bar starts from rest, show that at time t it moves with a speed

$$v = \frac{\mathcal{E}}{Bd}\left(1 - e^{-B^2d^2t/mR}\right)$$

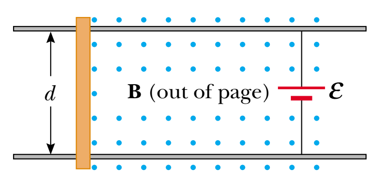


Figure P31.55

56. An automobile has a vertical radio antenna 1.20 m long. The automobile travels at 65.0 km/h on a horizontal road where the Earth's magnetic field is 50.0 μT directed toward the north and downward at an angle of 65.0° below the horizontal. (a) Specify the direction that the automobile should move to generate the maxi-

mum motional emf in the antenna, with the top of the antenna positive relative to the bottom. (b) Calculate the magnitude of this induced emf.

57. The plane of a square loop of wire with edge length $a = 0.200$ m is perpendicular to the Earth's magnetic field at a point where $B = 15.0\ \mu$T, as shown in Figure P31.57. The total resistance of the loop and the wires connecting it to the galvanometer is 0.500 Ω. If the loop is suddenly collapsed by horizontal forces as shown, what total charge passes through the galvanometer?

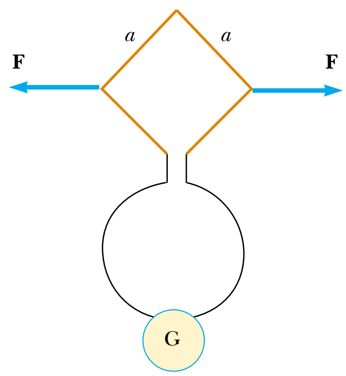


Figure P31.57

58. Magnetic field values are often determined by using a device known as a *search coil.* This technique depends on the measurement of the total charge passing through a coil in a time interval during which the magnetic flux linking the windings changes either because of the motion of the coil or because of a change in the value of B. (a) Show that as the flux through the coil changes from Φ_1 to Φ_2, the charge transferred through the coil will be given by $Q = N(\Phi_2 - \Phi_1)/R$, where R is the resistance of the coil and associated circuitry (galvanometer) and N is the number of turns. (b) As a specific example, calculate B when a 100-turn coil of resistance 200 Ω and cross-sectional area 40.0 cm^2 produces the following results. A total charge of 5.00×10^{-4} C passes through the coil when it is rotated in a uniform field from a position where the plane of the coil is perpendicular to the field to a position where the coil's plane is parallel to the field.

59. In Figure P31.59, the rolling axle, 1.50 m long, is pushed along horizontal rails at a constant speed $v = 3.00$ m/s. A resistor $R = 0.400\ \Omega$ is connected to the rails at points a and b, directly opposite each other. (The wheels make good electrical contact with the rails, and so the axle, rails, and R form a closed-loop circuit. The only significant resistance in the circuit is R.) There is a uniform magnetic field $B = 0.080\ 0$ T vertically downward. (a) Find the induced current I in the resistor. (b) What horizontal force F is required to keep the

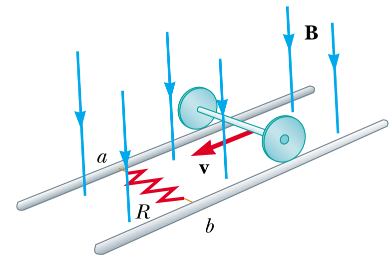


Figure P31.59

axle rolling at constant speed? (c) Which end of the resistor, a or b, is at the higher electric potential? (d) After the axle rolls past the resistor, does the current in R reverse direction? Explain your answer.

60. A conducting rod moves with a constant velocity **v** perpendicular to a long, straight wire carrying a current I as shown in Figure P31.60. Show that the magnitude of the emf generated between the ends of the rod is

$$|\boldsymbol{\mathcal{E}}| = \frac{\mu_0 v I}{2\pi r}\,\ell$$

In this case, note that the emf decreases with increasing r, as you might expect.

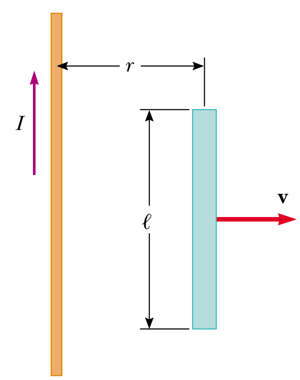


Figure P31.60

61. A circular loop of wire of radius r is in a uniform magnetic field, with the plane of the loop perpendicular to the direction of the field (Fig. P31.61). The magnetic field varies with time according to $B(t) = a + bt$, where a and b are constants. (a) Calculate the magnetic flux through the loop at $t = 0$. (b) Calculate the emf induced in the loop. (c) If the resistance of the loop is R, what is the induced current? (d) At what rate is electrical energy being delivered to the resistance of the loop?

62. In Figure P31.62, a uniform magnetic field decreases at a constant rate $dB/dt = -K$, where K is a positive constant. A circular loop of wire of radius a containing a re-

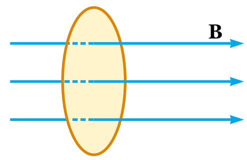


Figure P31.61

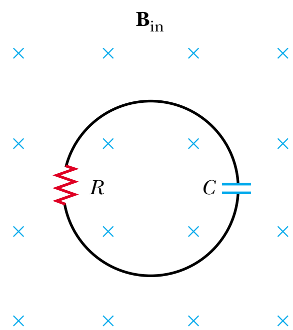


Figure P31.62

sistance R and a capacitance C is placed with its plane normal to the field. (a) Find the charge Q on the capacitor when it is fully charged. (b) Which plate is at the higher potential? (c) Discuss the force that causes the separation of charges.

63. A rectangular coil of 60 turns, dimensions 0.100 m by 0.200 m and total resistance 10.0 Ω, rotates with angular speed 30.0 rad/s about the y axis in a region where a 1.00-T magnetic field is directed along the x axis. The rotation is initiated so that the plane of the coil is perpendicular to the direction of **B** at $t = 0$. Calculate (a) the maximum induced emf in the coil, (b) the maximum rate of change of magnetic flux through the coil, (c) the induced emf at $t = 0.050\,0$ s, and (d) the torque exerted on the coil by the magnetic field at the instant when the emf is a maximum.

64. A small circular washer of radius 0.500 cm is held directly below a long, straight wire carrying a current of 10.0 A. The washer is located 0.500 m above the top of the table (Fig. P31.64). (a) If the washer is dropped from rest, what is the magnitude of the average induced emf in the washer from the time it is released to the moment it hits the tabletop? Assume that the magnetic field is nearly constant over the area of the washer and equal to the magnetic field at the center of the washer. (b) What is the direction of the induced current in the washer?

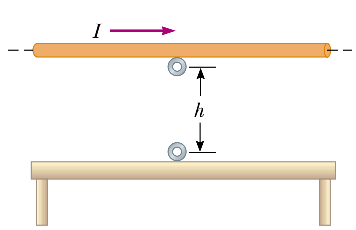


Figure P31.64

65. To monitor the breathing of a hospital patient, a thin belt is wrapped around the patient's chest. The belt is a 200-turn coil. When the patient inhales, the area encircled by the coil increases by 39.0 cm^2. The magnitude of the Earth's magnetic field is 50.0 μT and makes an angle of 28.0° with the plane of the coil. If a patient takes 1.80 s to inhale, find the average induced emf in the coil during this time.

66. A conducting rod of length ℓ moves with velocity **v** parallel to a long wire carrying a steady current I. The axis of the rod is maintained perpendicular to the wire with the near end a distance r away, as shown in Figure P31.66. Show that the magnitude of the emf induced in the rod is

$$|\mathcal{E}| = \frac{\mu_0 I}{2\pi} v \ln\left(1 + \frac{\ell}{r}\right)$$

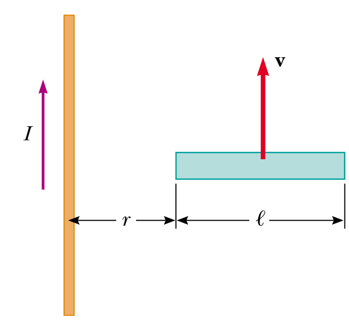


Figure P31.66

67. A rectangular loop of dimensions ℓ and w moves with a constant velocity **v** away from a long wire that carries a current I in the plane of the loop (Fig. P31.67). The to-

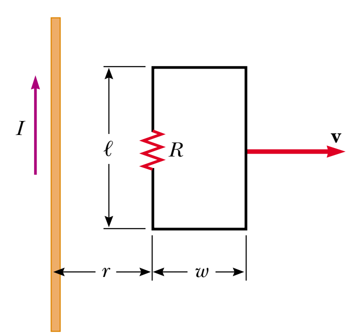


Figure P31.67

tal resistance of the loop is R. Derive an expression that gives the current in the loop at the instant the near side is a distance r from the wire.

68. A horizontal wire is free to slide on the vertical rails of a conducting frame, as shown in Figure P31.68. The wire has mass m and length ℓ, and the resistance of the circuit is R. If a uniform magnetic field is directed perpendicular to the frame, what is the terminal speed of the wire as it falls under the force of gravity?

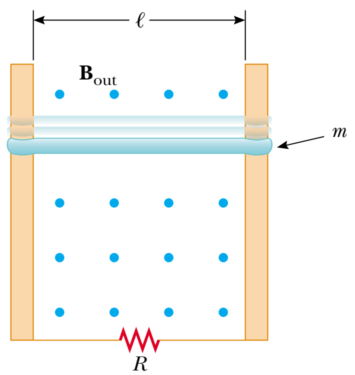


Figure P31.68

69. The magnetic flux threading a metal ring varies with time t according to $\Phi_B = 3(at^3 - bt^2)$ T·m², with $a = 2.00$ s^{-3} and $b = 6.00$ s^{-2}. The resistance of the ring is 3.00 Ω. Determine the maximum current induced in the ring during the interval from $t = 0$ to $t = 2.00$ s.

70. Review Problem. The bar of mass m shown in Figure P31.70 is pulled horizontally across parallel rails by a massless string that passes over an ideal pulley and is attached to a suspended mass M. The uniform magnetic field has a magnitude B, and the distance between the rails is ℓ. The rails are connected at one end by a load resistor R. Derive an expression that gives the horizontal speed of the bar as a function of time, assuming that the suspended mass is released with the bar at rest at $t = 0$. Assume no friction between rails and bar.

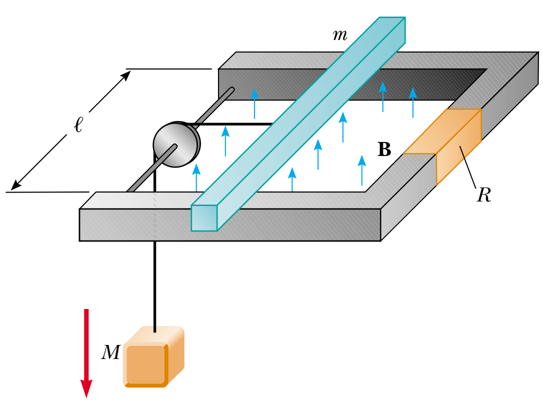


Figure P31.70

71. A solenoid wound with 2 000 turns/m is supplied with current that varies in time according to $I = 4\sin(120\pi t)$, where I is in A and t is in s. A small coaxial circular coil of 40 turns and radius $r = 5.00$ cm is located inside the solenoid near its center. (a) Derive an expression that describes the manner in which the emf in the small coil varies in time. (b) At what average rate is energy transformed into internal energy in the small coil if the windings have a total resistance of 8.00 Ω?

72. A wire 30.0 cm long is held parallel to and 80.0 cm above a long wire carrying 200 A and resting on the floor (Fig. P31.72). The 30.0-cm wire is released and falls, remaining parallel with the current-carrying wire as it falls. Assume that the falling wire accelerates at 9.80 m/s² and derive an equation for the emf induced in it. Express your result as a function of the time t after the wire is dropped. What is the induced emf 0.300 s after the wire is released?

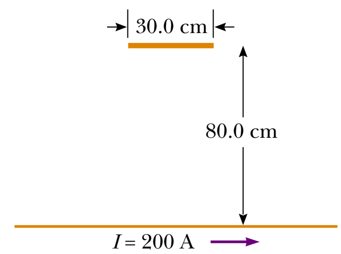


Figure P31.72

WEB **73.** A long, straight wire carries a current $I = I_{max}\sin(\omega t + \phi)$ and lies in the plane of a rectangular coil of N turns of wire, as shown in Figure P31.9. The quantities I_{max}, ω, and ϕ are all constants. Determine the emf induced in the coil by the magnetic field created by the current in the straight wire. Assume $I_{max} = 50.0$ A, $\omega = 200\pi$ s^{-1}, $N = 100$, $h = w = 5.00$ cm, and $L = 20.0$ cm.

74. A dime is suspended from a thread and hung between the poles of a strong horseshoe magnet as shown in Figure P31.74. The dime rotates at constant angular speed ω about a vertical axis. Letting θ represent the angle between the direction of **B** and the normal to the face of the dime, sketch a graph of the torque due to induced currents as a function of θ for $0 < \theta < 2\pi$.

75. The wire shown in Figure P31.75 is bent in the shape of a tent, with $\theta = 60.0°$ and $L = 1.50$ m, and is placed in a uniform magnetic field of magnitude 0.300 T perpendicular to the tabletop. The wire is rigid but hinged at points a and b. If the "tent" is flattened out on the table in 0.100 s, what is the average induced emf in the wire during this time?

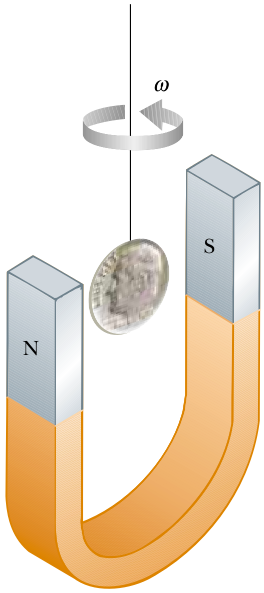


Figure P31.74

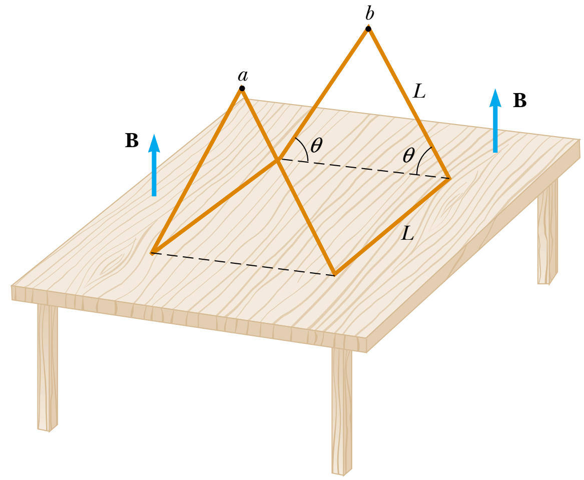


Figure P31.75

Answers to Quick Quizzes

31.1 Because the magnetic field now points in the opposite direction, you must replace θ with $\theta + \pi$. Because $\cos(\theta + \pi) = -\cos\theta$, the sign of the induced emf is reversed.

31.2 The one on the west side of the plane. As we saw in Section 30.9, the Earth's magnetic field has a downward component in the northern hemisphere. As the plane flies north, the right-hand rule illustrated in Figure 29.4 indicates that positive charge experiences a force directed toward the west. Thus, the left wingtip becomes positively charged and the right wingtip negatively charged.

31.3 Inserting. Because the south pole of the magnet is nearest the solenoid, the field lines created by the magnet point upward in Figure 31.14. Because the current induced in the solenoid is clockwise when viewed from above, the magnetic field lines produced by this current point downward in Figure 31.14. If the magnet were being withdrawn, it would create a decreasing upward flux. The induced current would counteract this decrease by producing its own upward flux. This would require a counterclockwise current in the solenoid, contrary to what is observed.

PUZZLER

The marks in the pavement are part of a sensor that controls the traffic lights at this intersection. What are these marks, and how do they detect when a car is waiting at the light? *(© David R. Frazier)*

chapter

32 Inductance

Chapter Outline

In Chapter 31, we saw that emfs and currents are induced in a circuit when the magnetic flux through the area enclosed by the circuit changes with time. This electromagnetic induction has some practical consequences, which we describe in this chapter. First, we describe an effect known as *self-induction,* in which a time-varying current in a circuit produces in the circuit an induced emf that opposes the emf that initially set up the time-varying current. Self-induction is the basis of the *inductor,* an electrical element that has an important role in circuits that use time-varying currents. We discuss the energy stored in the magnetic field of an inductor and the energy density associated with the magnetic field.

Next, we study how an emf is induced in a circuit as a result of a changing magnetic flux produced by a second circuit; this is the basic principle of *mutual induction.* Finally, we examine the characteristics of circuits that contain inductors, resistors, and capacitors in various combinations.

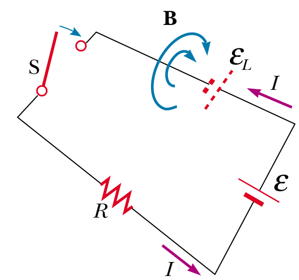


Figure 32.1 After the switch is thrown closed, the current produces a magnetic flux through the area enclosed by the loop. As the current increases toward its equilibrium value, this magnetic flux changes in time and induces an emf in the loop. The battery symbol drawn with dashed lines represents the self-induced emf.

32.1 SELF-INDUCTANCE

In this chapter, we need to distinguish carefully between emfs and currents that are caused by batteries or other sources and those that are induced by changing magnetic fields. We use the adjective *source* (as in the terms *source emf* and *source current*) to describe the parameters associated with a physical source, and we use the adjective *induced* to describe those emfs and currents caused by a changing magnetic field.

Consider a circuit consisting of a switch, a resistor, and a source of emf, as shown in Figure 32.1. When the switch is thrown to its closed position, the source current does not immediately jump from zero to its maximum value $\mathcal{E}/R$. Faraday's law of electromagnetic induction (Eq. 31.1) can be used to describe this effect as follows: As the source current increases with time, the magnetic flux through the circuit loop due to this current also increases with time. This increasing flux creates an induced emf in the circuit. The direction of the induced emf is such that it would cause an induced current in the loop (if a current were not already flowing in the loop), which would establish a magnetic field that would oppose the change in the source magnetic field. Thus, the direction of the induced emf is opposite the direction of the source emf; this results in a gradual rather than instantaneous increase in the source current to its final equilibrium value. This effect is called *self-induction* because the changing flux through the circuit and the resultant induced emf arise from the circuit itself. The emf $\mathcal{E}_L$ set up in this case is called a **self-induced emf.** It is also often called a **back emf.**

As a second example of self-induction, consider Figure 32.2, which shows a coil wound on a cylindrical iron core. (A practical device would have several hun-

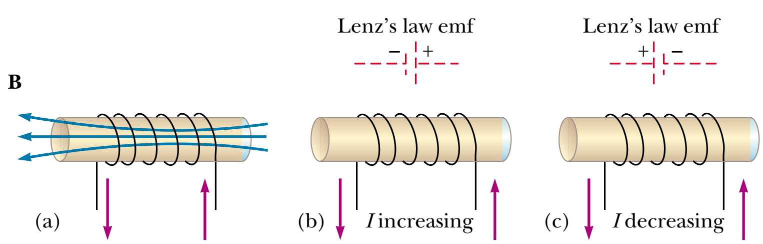


Figure 32.2 (a) A current in the coil produces a magnetic field directed to the left. (b) If the current increases, the increasing magnetic flux creates an induced emf having the polarity shown by the dashed battery. (c) The polarity of the induced emf reverses if the current decreases.

Joseph Henry (1797–1878)
Henry, an American physicist, became the first director of the Smithsonian Institution and first president of the Academy of Natural Science. He improved the design of the electromagnet and constructed one of the first motors. He also discovered the phenomenon of self-induction but failed to publish his findings. The unit of inductance, the henry, is named in his honor. *(North Wind Picture Archives)*

dred turns.) Assume that the source current in the coil either increases or decreases with time. When the source current is in the direction shown, a magnetic field directed from right to left is set up inside the coil, as seen in Figure 32.2a. As the source current changes with time, the magnetic flux through the coil also changes and induces an emf in the coil. From Lenz's law, the polarity of this induced emf must be such that it opposes the change in the magnetic field from the source current. If the source current is increasing, the polarity of the induced emf is as pictured in Figure 32.2b, and if the source current is decreasing, the polarity of the induced emf is as shown in Figure 32.2c.

To obtain a quantitative description of self-induction, we recall from Faraday's law that the induced emf is equal to the negative time rate of change of the magnetic flux. The magnetic flux is proportional to the magnetic field due to the source current, which in turn is proportional to the source current in the circuit. Therefore, **a self-induced emf $\mathcal{E}_L$ is always proportional to the time rate of change of the source current.** For a closely spaced coil of N turns (a toroid or an ideal solenoid) carrying a source current I, we find that

Self-induced emf

$$\mathcal{E}_L = -N\frac{d\Phi_B}{dt} = -L\frac{dI}{dt} \qquad \textbf{(32.1)}$$

where L is a proportionality constant—called the **inductance** of the coil—that depends on the geometry of the circuit and other physical characteristics. From this expression, we see that the inductance of a coil containing N turns is

Inductance of an N-turn coil

$$L = \frac{N\Phi_B}{I} \qquad \textbf{(32.2)}$$

where it is assumed that the same flux passes through each turn. Later, we shall use this equation to calculate the inductance of some special circuit geometries.

From Equation 32.1, we can also write the inductance as the ratio

Inductance

$$L = -\frac{\mathcal{E}_L}{dI/dt} \qquad \textbf{(32.3)}$$

Just as resistance is a measure of the opposition to current ($R = \Delta V/I$), inductance is a measure of the opposition to a *change* in current.

The SI unit of inductance is the **henry** (H), which, as we can see from Equation 32.3, is 1 volt-second per ampere:

$$1\ \text{H} = 1\ \frac{\text{V}\cdot\text{s}}{\text{A}}$$

That the inductance of a device depends on its geometry is analogous to the capacitance of a capacitor depending on the geometry of its plates, as we found in Chapter 26. Inductance calculations can be quite difficult to perform for complicated geometries; however, the following examples involve simple situations for which inductances are easily evaluated.

EXAMPLE 32.1 Inductance of a Solenoid

Find the inductance of a uniformly wound solenoid having N turns and length ℓ. Assume that ℓ is much longer than the radius of the windings and that the core of the solenoid is air.

Solution We can assume that the interior magnetic field due to the source current is uniform and given by Equation 30.17:

$$B = \mu_0 n I = \mu_0 \frac{N}{\ell} I$$

where $n = N/\ell$ is the number of turns per unit length. The magnetic flux through each turn is

$$\Phi_B = BA = \mu_0 \frac{NA}{\ell} I$$

where A is the cross-sectional area of the solenoid. Using this expression and Equation 32.2, we find that

$$L = \frac{N\Phi_B}{I} = \boxed{\frac{\mu_0 N^2 A}{\ell}} \tag{32.4}$$

This result shows that L depends on geometry and is proportional to the square of the number of turns. Because $N = n\ell$, we can also express the result in the form

$$L = \mu_0 \frac{(n\ell)^2}{\ell} A = \mu_0 n^2 A\ell = \mu_0 n^2 V \tag{32.5}$$

where $V = A\ell$ is the volume of the solenoid.

Exercise What would happen to the inductance if a ferromagnetic material were placed inside the solenoid? 13.6

Answer The inductance would increase. For a given current, the magnetic flux is now much greater because of the increase in the field originating from the magnetization of the ferromagnetic material. For example, if the material has a magnetic permeability of $500\mu_0$, the inductance would increase by a factor of 500.

The fact that various materials in the vicinity of a coil can substantially alter the coil's inductance is used to great advantage by traffic engineers. A flat, horizontal coil made of numerous loops of wire is placed in a shallow groove cut into the pavement of the lane approaching an intersection. (See the photograph at the beginning of this chapter.) These loops are attached to circuitry that measures inductance. When an automobile passes over the loops, the change in inductance caused by the large amount of iron passing over the loops is used to control the lights at the intersection.

EXAMPLE 32.2 Calculating Inductance and emf

(a) Calculate the inductance of an air-core solenoid containing 300 turns if the length of the solenoid is 25.0 cm and its cross-sectional area is 4.00 cm^2.

Solution Using Equation 32.4, we obtain

$$L = \frac{\mu_0 N^2 A}{\ell}$$

$$= (4\pi \times 10^{-7}\ \mathrm{T\cdot m/A}) \frac{(300)^2(4.00 \times 10^{-4}\ \mathrm{m^2})}{25.0 \times 10^{-2}\ \mathrm{m}}$$

$$= 1.81 \times 10^{-4}\ \mathrm{T\cdot m^2/A} = \boxed{0.181\ \mathrm{mH}}$$

(b) Calculate the self-induced emf in the solenoid if the current through it is decreasing at the rate of 50.0 A/s.

Solution Using Equation 32.1 and given that $dI/dt = -50.0$ A/s, we obtain

$$\mathcal{E}_L = -L\frac{dI}{dt} = -(1.81 \times 10^{-4}\ \mathrm{H})(-50.0\ \mathrm{A/s})$$

$$= \boxed{9.05\ \mathrm{mV}}$$

32.2 *RL* CIRCUITS

13.6 If a circuit contains a coil, such as a solenoid, the self-inductance of the coil prevents the current in the circuit from increasing or decreasing instantaneously. A circuit element that has a large self-inductance is called an **inductor** and has the circuit symbol . We always assume that the self-inductance of the remainder of a circuit is negligible compared with that of the inductor. Keep in mind, however, that even a circuit without a coil has some self-inductance that can affect the behavior of the circuit.

Because the inductance of the inductor results in a back emf, **an inductor in a circuit opposes changes in the current through that circuit.** If the battery voltage in the circuit is increased so that the current rises, the inductor opposes

this change, and the rise is not instantaneous. If the battery voltage is decreased, the presence of the inductor results in a slow drop in the current rather than an immediate drop. Thus, the inductor causes the circuit to be "sluggish" as it reacts to changes in the voltage.

Quick Quiz 32.1

A switch controls the current in a circuit that has a large inductance. Is a spark more likely to be produced at the switch when the switch is being closed or when it is being opened, or doesn't it matter?

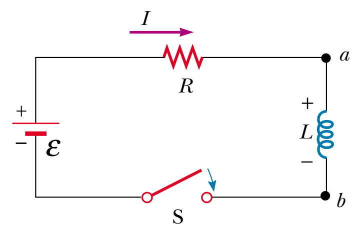


Figure 32.3 A series *RL* circuit. As the current increases toward its maximum value, an emf that opposes the increasing current is induced in the inductor.

Consider the circuit shown in Figure 32.3, in which the battery has negligible internal resistance. This is an ***RL* circuit** because the elements connected to the battery are a resistor and an inductor. Suppose that the switch S is thrown closed at $t = 0$. The current in the circuit begins to increase, and a back emf that opposes the increasing current is induced in the inductor. The back emf is, from Equation 32.1,

$$\mathcal{E}_L = -L\frac{dI}{dt}$$

Because the current is increasing, dI/dt is positive; thus, $\mathcal{E}_L$ is negative. This negative value reflects the decrease in electric potential that occurs in going from a to b across the inductor, as indicated by the positive and negative signs in Figure 32.3.

With this in mind, we can apply Kirchhoff's loop rule to this circuit, traversing the circuit in the clockwise direction:

$$\mathcal{E} - IR - L\frac{dI}{dt} = 0 \qquad \textbf{(32.6)}$$

where IR is the voltage drop across the resistor. (We developed Kirchhoff's rules for circuits with steady currents, but we can apply them to a circuit in which the current is changing if we imagine them to represent the circuit at one *instant* of time.) We must now look for a solution to this differential equation, which is similar to that for the *RC* circuit (see Section 28.4).

A mathematical solution of Equation 32.6 represents the current in the circuit as a function of time. To find this solution, we change variables for convenience, letting $x = \dfrac{\mathcal{E}}{R} - I$, so that $dx = -dI$. With these substitutions, we can write Equation 32.6 as

$$x + \frac{L}{R}\frac{dx}{dt} = 0$$

$$\frac{dx}{x} = -\frac{R}{L}\,dt$$

Integrating this last expression, we have

$$\ln\frac{x}{x_0} = -\frac{R}{L}\,t$$

where we take the integrating constant to be $-\ln x_0$ and x_0 is the value of x at time $t = 0$. Taking the antilogarithm of this result, we obtain

$$x = x_0 e^{-Rt/L}$$

Because $I = 0$ at $t = 0$, we note from the definition of x that $x_0 = \mathcal{E}/R$. Hence, this last expression is equivalent to

$$\frac{\mathcal{E}}{R} - I = \frac{\mathcal{E}}{R} e^{-Rt/L}$$

$$I = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L})$$

This expression shows the effect of the inductor. The current does not increase instantly to its final equilibrium value when the switch is closed but instead increases according to an exponential function. If we remove the inductance in the circuit, which we can do by letting L approach zero, the exponential term becomes zero and we see that there is no time dependence of the current in this case—the current increases instantaneously to its final equilibrium value in the absence of the inductance.

We can also write this expression as

$$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) \tag{32.7}$$

where the constant τ is the **time constant** of the RL circuit:

Time constant of an RL circuit

$$\tau = L/R \tag{32.8}$$

Physically, τ is the time it takes the current in the circuit to reach $(1 - e^{-1}) = 0.63$ of its final value $\mathcal{E}/R$. The time constant is a useful parameter for comparing the time responses of various circuits.

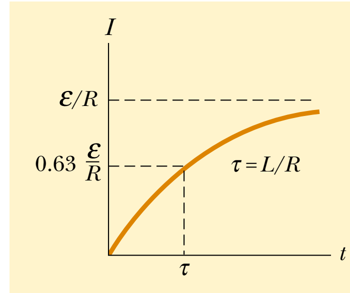


Figure 32.4 Plot of the current versus time for the RL circuit shown in Figure 32.3. The switch is thrown closed at $t = 0$, and the current increases toward its maximum value $\mathcal{E}/R$. The time constant τ is the time it takes I to reach 63% of its maximum value.

Figure 32.4 shows a graph of the current versus time in the RL circuit. Note that the equilibrium value of the current, which occurs as t approaches infinity, is $\mathcal{E}/R$. We can see this by setting dI/dt equal to zero in Equation 32.6 and solving for the current I. (At equilibrium, the change in the current is zero.) Thus, we see that the current initially increases very rapidly and then gradually approaches the equilibrium value $\mathcal{E}/R$ as t approaches infinity.

Let us also investigate the time rate of change of the current in the circuit. Taking the first time derivative of Equation 32.7, we have

$$\frac{dI}{dt} = \frac{\mathcal{E}}{L} e^{-t/\tau} \tag{32.9}$$

From this result, we see that the time rate of change of the current is a maximum (equal to $\mathcal{E}/L$) at $t = 0$ and falls off exponentially to zero as t approaches infinity (Fig. 32.5).

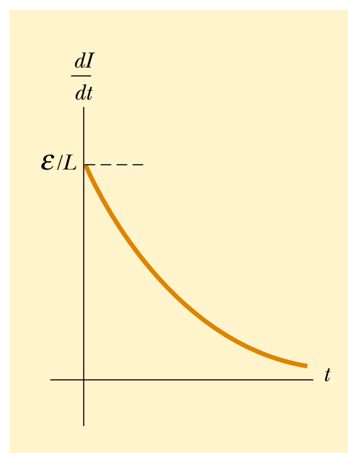


Figure 32.5 Plot of dI/dt versus time for the RL circuit shown in Figure 32.3. The time rate of change of current is a maximum at $t = 0$, which is the instant at which the switch is thrown closed. The rate decreases exponentially with time as I increases toward its maximum value.

Now let us consider the RL circuit shown in Figure 32.6. The circuit contains two switches that operate such that when one is closed, the other is opened. Suppose that S_1 has been closed for a length of time sufficient to allow the current to reach its equilibrium value $\mathcal{E}/R$. In this situation, the circuit is described completely by the outer loop in Figure 32.6. If S_2 is closed at the instant at which S_1 is opened, the circuit changes so that it is described completely by just the upper loop in Figure 32.6. The lower loop no longer influences the behavior of the circuit. Thus, we have a circuit with no battery ($\mathcal{E} = 0$). If we apply Kirchhoff's loop rule to the upper loop at the instant the switches are thrown, we obtain

$$IR + L\frac{dI}{dt} = 0$$

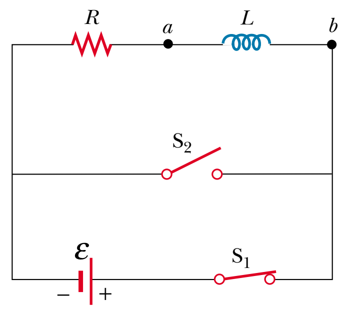


Figure 32.6 An RL circuit containing two switches. When S_1 is closed and S_2 open as shown, the battery is in the circuit. At the instant S_2 is closed, S_1 is opened, and the battery is no longer part of the circuit.

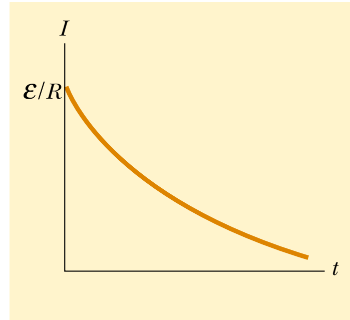


Figure 32.7 Current versus time for the upper loop of the circuit shown in Figure 32.6. For $t < 0$, S_1 is closed and S_2 is open. At $t = 0$, S_2 is closed as S_1 is opened, and the current has its maximum value $\mathcal{E}/R$.

It is left as a problem (Problem 18) to show that the solution of this differential equation is

$$I = \frac{\mathcal{E}}{R} e^{-t/\tau} = I_0 e^{-t/\tau} \tag{32.10}$$

where $\mathcal{E}$ is the emf of the battery and $I_0 = \mathcal{E}/R$ is the current at $t = 0$, the instant at which S_2 is closed as S_1 is opened.

If no inductor were present in the circuit, the current would immediately decrease to zero if the battery were removed. When the inductor is present, it acts to oppose the decrease in the current and to maintain the current. A graph of the current in the circuit versus time (Fig. 32.7) shows that the current is continuously decreasing with time. Note that the slope dI/dt is always negative and has its maximum value at $t = 0$. The negative slope signifies that $\mathcal{E}_L = -L\,(dI/dt)$ is now positive; that is, point a in Figure 32.6 is at a lower electric potential than point b.

Quick Quiz 32.2

Two circuits like the one shown in Figure 32.6 are identical except for the value of L. In circuit A the inductance of the inductor is L_A, and in circuit B it is L_B. Switch S_1 is thrown closed at $t = 0$, while switch S_2 remains open. At $t = 10$ s, switch S_1 is opened and switch S_2 is closed. The resulting time rates of change for the two currents are as graphed in Figure 32.8. If we assume that the time constant of each circuit is much less than 10 s, which of the following is true? (a) $L_A > L_B$; (b) $L_A < L_B$; (c) not enough information to tell.

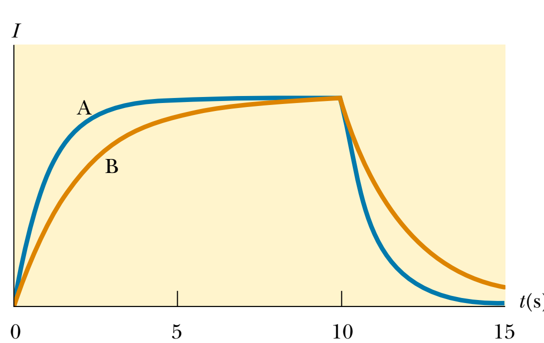


Figure 32.8

EXAMPLE 32.3 Time Constant of an *RL* Circuit

The switch in Figure 32.9a is thrown closed at $t = 0$. (a) Find the time constant of the circuit.

Solution The time constant is given by Equation 32.8:

$$\tau = \frac{L}{R} = \frac{30.0 \times 10^{-3}\ \text{H}}{6.00\ \Omega} = \boxed{5.00\ \text{ms}}$$

(b) Calculate the current in the circuit at $t = 2.00$ ms.

Solution Using Equation 32.7 for the current as a function of time (with t and τ in milliseconds), we find that at $t = 2.00$ ms

$$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) = \frac{12.0\ \text{V}}{6.00\ \Omega}(1 - e^{-0.400}) = \boxed{0.659\ \text{A}}$$

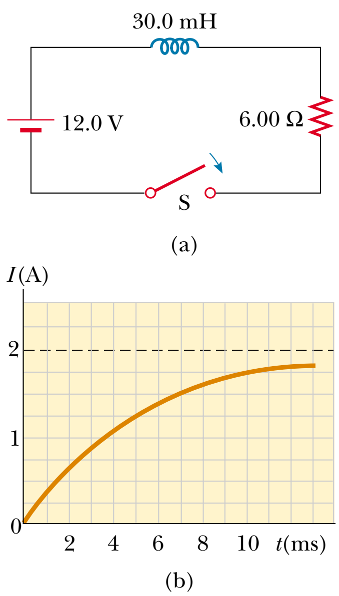


Figure 32.9 (a) The switch in this *RL* circuit is thrown closed at $t = 0$. (b) A graph of the current versus time for the circuit in part (a).

A plot of Equation 32.7 for this circuit is given in Figure 32.9b.

(c) Compare the potential difference across the resistor with that across the inductor.

Solution At the instant the switch is closed, there is no current and thus no potential difference across the resistor. At this instant, the battery voltage appears entirely across the inductor in the form of a back emf of 12.0 V as the inductor tries to maintain the zero-current condition. (The left end of the inductor is at a higher electric potential than the right end.) As time passes, the emf across the inductor decreases and the current through the resistor (and hence the potential difference across it) increases. The sum of the two potential differences at all times is 12.0 V, as shown in Figure 32.10.

Exercise Calculate the current in the circuit and the voltage across the resistor after a time interval equal to one time constant has elapsed.

Answer 1.26 A, 7.56 V.

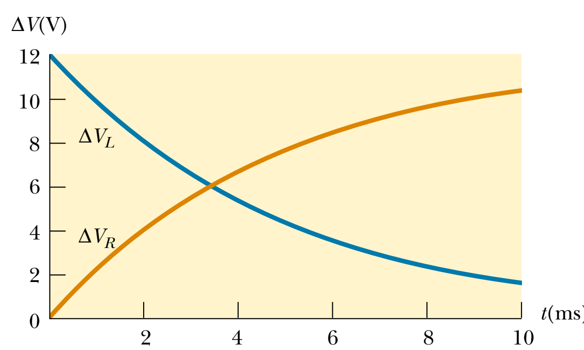


Figure 32.10 The sum of the potential differences across the resistor and inductor in Figure 32.9a is 12.0 V (the battery emf) at all times.

32.3 ENERGY IN A MAGNETIC FIELD

13.6 Because the emf induced in an inductor prevents a battery from establishing an instantaneous current, the battery must do work against the inductor to create a current. Part of the energy supplied by the battery appears as internal energy in the resistor, while the remaining energy is stored in the magnetic field of the inductor. If we multiply each term in Equation 32.6 by I and rearrange the expression, we have

$$I\mathcal{E} = I^2R + LI\frac{dI}{dt} \qquad \textbf{(32.11)}$$

This expression indicates that the rate at which energy is supplied by the battery ($I\mathcal{E}$) equals the sum of the rate at which energy is delivered to the resistor, I^2R, and the rate at which energy is stored in the inductor, $LI(dI/dt)$. Thus, Equation 32.11 is simply an expression of energy conservation. If we let U denote the energy stored in the inductor at any time, then we can write the rate dU/dt at which energy is stored as

$$\frac{dU}{dt} = LI\frac{dI}{dt}$$

To find the total energy stored in the inductor, we can rewrite this expression as $dU = LI\,dI$ and integrate:

$$U = \int dU = \int_0^I LI\,dI = L\int_0^I I\,dI$$

Energy stored in an inductor

$$U = \tfrac{1}{2}LI^2 \qquad \textbf{(32.12)}$$

where L is constant and has been removed from the integral. This expression represents the energy stored in the magnetic field of the inductor when the current is I. Note that this equation is similar in form to Equation 26.11 for the energy stored in the electric field of a capacitor, $U = Q^2/2C$. In either case, we see that energy is required to establish a field.

We can also determine the energy density of a magnetic field. For simplicity, consider a solenoid whose inductance is given by Equation 32.5:

$$L = \mu_0 n^2 A\ell$$

The magnetic field of a solenoid is given by Equation 30.17:

$$B = \mu_0 nI$$

Substituting the expression for L and $I = B/\mu_0 n$ into Equation 32.12 gives

$$U = \tfrac{1}{2}LI^2 = \tfrac{1}{2}\mu_0 n^2 A\ell\left(\frac{B}{\mu_0 n}\right)^2 = \frac{B^2}{2\mu_0}A\ell \qquad \textbf{(32.13)}$$

Because $A\ell$ is the volume of the solenoid, the energy stored per unit volume in the magnetic field surrounding the inductor is

Magnetic energy density

$$u_B = \frac{U}{A\ell} = \frac{B^2}{2\mu_0} \qquad \textbf{(32.14)}$$

Although this expression was derived for the special case of a solenoid, it is valid for any region of space in which a magnetic field exists. Note that Equation 32.14 is similar in form to Equation 26.13 for the energy per unit volume stored in an electric field, $u_E = \tfrac{1}{2}\epsilon_0 E^2$. In both cases, the energy density is proportional to the square of the magnitude of the field.

EXAMPLE 32.4 What Happens to the Energy in the Inductor?

Consider once again the RL circuit shown in Figure 32.6, in which switch S_2 is closed at the instant S_1 is opened (at $t = 0$). Recall that the current in the upper loop decays exponentially with time according to the expression $I = I_0 e^{-t/\tau}$, where $I_0 = \mathcal{E}/R$ is the initial current in the circuit and $\tau = L/R$ is the time constant. Show that all the energy initially stored in the magnetic field of the inductor appears as internal energy in the resistor as the current decays to zero.

Solution The rate dU/dt at which energy is delivered to the resistor (which is the power) is equal to I^2R, where I is the instantaneous current:

$$\frac{dU}{dt} = I^2R = (I_0e^{-Rt/L})^2R = I_0{}^2Re^{-2Rt/L}$$

To find the total energy delivered to the resistor, we solve for dU and integrate this expression over the limits $t = 0$ to $t \rightarrow \infty$ (the upper limit is infinity because it takes an infinite amount of time for the current to reach zero):

$$(1) \qquad U = \int_0^\infty I_0{}^2Re^{-2Rt/L}dt = I_0{}^2R\int_0^\infty e^{-2Rt/L}dt$$

The value of the definite integral is $L/2R$ (this is left for the student to show in the exercise at the end of this example), and so U becomes

$$U = I_0{}^2R\left(\frac{L}{2R}\right) = \frac{1}{2}LI_0{}^2$$

Note that this is equal to the initial energy stored in the magnetic field of the inductor, given by Equation 32.13, as we set out to prove.

Exercise Show that the integral on the right-hand side of Equation (1) has the value $L/2R$.

EXAMPLE 32.5 The Coaxial Cable

Coaxial cables are often used to connect electrical devices, such as your stereo system and a loudspeaker. Model a long coaxial cable as consisting of two thin concentric cylindrical conducting shells of radii a and b and length ℓ, as shown in Figure 32.11. The conducting shells carry the same current I in opposite directions. Imagine that the inner conductor carries current to a device and that the outer one acts as a return path carrying the current back to the source. (a) Calculate the self-inductance L of this cable.

Solution To obtain L, we must know the magnetic flux through any cross-section in the region between the two shells, such as the light blue rectangle in Figure 32.11. Ampère's law (see Section 30.3) tells us that the magnetic field in the region between the shells is $B = \mu_0I/2\pi r$, where r is measured from the common center of the shells. The magnetic field is zero outside the outer shell ($r > b$) because the net current through the area enclosed by a circular path surrounding the cable is zero, and hence from Ampère's law, $\oint \mathbf{B} \cdot d\mathbf{s} = 0$. The magnetic field is zero inside the inner shell because the shell is hollow and no current is present within a radius $r < a$.

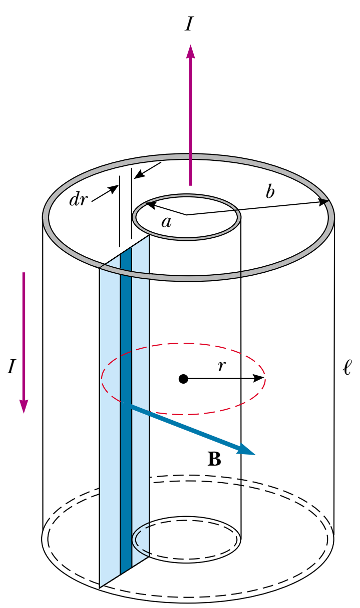


Figure 32.11 Section of a long coaxial cable. The inner and outer conductors carry equal currents in opposite directions.

The magnetic field is perpendicular to the light blue rectangle of length ℓ and width $b - a$, the cross-section of interest. Because the magnetic field varies with radial position across this rectangle, we must use calculus to find the total magnetic flux. Dividing this rectangle into strips of width dr, such as the dark blue strip in Figure 32.11, we see that the area of each strip is $\ell\, dr$ and that the flux through each strip is $B\, dA = B\ell\, dr$. Hence, we find the total flux through the entire cross-section by integrating:

$$\Phi_B = \int B\, dA = \int_a^b \frac{\mu_0I}{2\pi r}\,\ell\, dr = \frac{\mu_0I\ell}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0I\ell}{2\pi}\ln\left(\frac{b}{a}\right)$$

Using this result, we find that the self-inductance of the cable is

$$L = \frac{\Phi_B}{I} = \boxed{\frac{\mu_0\ell}{2\pi}\ln\left(\frac{b}{a}\right)}$$

(b) Calculate the total energy stored in the magnetic field of the cable.

Solution Using Equation 32.12 and the results to part (a) gives

$$U = \tfrac{1}{2}LI^2 = \boxed{\frac{\mu_0\ell I^2}{4\pi}\ln\left(\frac{b}{a}\right)}$$

32.4 MUTUAL INDUCTANCE

Very often, the magnetic flux through the area enclosed by a circuit varies with time because of time-varying currents in nearby circuits. This condition induces an emf through a process known as *mutual induction,* so called because it depends on the interaction of two circuits.

Consider the two closely wound coils of wire shown in cross-sectional view in Figure 32.12. The current I_1 in coil 1, which has N_1 turns, creates magnetic field lines, some of which pass through coil 2, which has N_2 turns. The magnetic flux caused by the current in coil 1 and passing through coil 2 is represented by Φ_{12}. In analogy to Equation 32.2, we define the **mutual inductance** M_{12} of coil 2 with respect to coil 1:

Definition of mutual inductance

$$M_{12} \equiv \frac{N_2 \Phi_{12}}{I_1} \tag{32.15}$$

Quick Quiz 32.3

Referring to Figure 32.12, tell what happens to M_{12} (a) if coil 1 is brought closer to coil 2 and (b) if coil 1 is rotated so that it lies in the plane of the page.

Quick Quiz 32.3 demonstrates that mutual inductance depends on the geometry of both circuits and on their orientation with respect to each other. As the circuit separation distance increases, the mutual inductance decreases because the flux linking the circuits decreases.

If the current I_1 varies with time, we see from Faraday's law and Equation 32.15 that the emf induced by coil 1 in coil 2 is

$$\mathcal{E}_2 = -N_2 \frac{d\Phi_{12}}{dt} = -N_2 \frac{d}{dt}\left(\frac{M_{12} I_1}{N_2}\right) = -M_{12} \frac{dI_1}{dt} \tag{32.16}$$

In the preceding discussion, we assumed that the source current is in coil 1. We can also imagine a source current I_2 in coil 2. The preceding discussion can be repeated to show that there is a mutual inductance M_{21}. If the current I_2 varies with time, the emf induced by coil 2 in coil 1 is

$$\mathcal{E}_1 = -M_{21} \frac{dI_2}{dt} \tag{32.17}$$

In mutual induction, the emf induced in one coil is always proportional to the rate at which the current in the other coil is changing. Although the

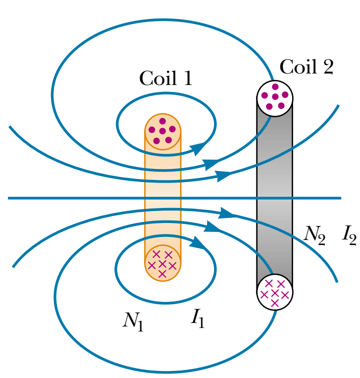


Figure 32.12 A cross-sectional view of two adjacent coils. A current in coil 1 sets up a magnetic flux, part of which passes through coil 2.

proportionality constants M_{12} and M_{21} appear to have different values, it can be shown that they are equal. Thus, with $M_{12} = M_{21} = M$, Equations 32.16 and 32.17 become

$$\mathcal{E}_2 = -M\frac{dI_1}{dt} \quad \text{and} \quad \mathcal{E}_1 = -M\frac{dI_2}{dt}$$

These two equations are similar in form to Equation 32.1 for the self-induced emf $\mathcal{E} = -L(dI/dt)$. The unit of mutual inductance is the henry.

QuickLab

Tune in a relatively weak station on a radio. Now slowly rotate the radio about a vertical axis through its center. What happens to the reception? Can you explain this in terms of the mutual induction of the station's broadcast antenna and your radio's antenna?

Quick Quiz 32.4

(a) Can you have mutual inductance without self-inductance? (b) How about self-inductance without mutual inductance?

EXAMPLE 32.6 "Wireless" Battery Charger

An electric toothbrush has a base designed to hold the toothbrush handle when not in use. As shown in Figure 32.13a, the handle has a cylindrical hole that fits loosely over a matching cylinder on the base. When the handle is placed on the base, a changing current in a solenoid inside the base cylinder induces a current in a coil inside the handle. This induced current charges the battery in the handle.

We can model the base as a solenoid of length ℓ with N_B turns (Fig. 32.13b), carrying a source current I, and having a cross-sectional area A. The handle coil contains N_H turns. Find the mutual inductance of the system.

Solution Because the base solenoid carries a source current I, the magnetic field in its interior is

$$B = \frac{\mu_0 N_B I}{\ell}$$

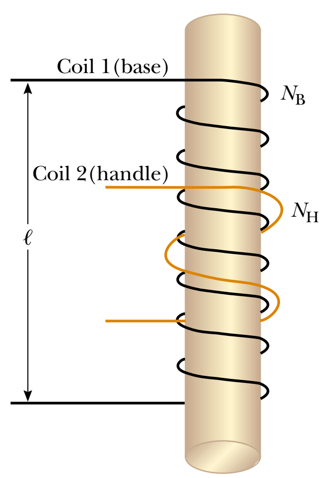
Because the magnetic flux Φ_{BH} through the handle's coil caused by the magnetic field of the base coil is BA, the mutual inductance is

$$M = \frac{N_H \Phi_{BH}}{I} = \frac{N_H BA}{I} = \boxed{\mu_0 \frac{N_H N_B A}{\ell}}$$

Exercise Calculate the mutual inductance of two solenoids with $N_B = 1\,500$ turns, $A = 1.0 \times 10^{-4}\ \text{m}^2$, $\ell = 0.02$ m, and $N_H = 800$ turns.

Answer 7.5 mH.

(a)



(b)

Figure 32.13 (a) This electric toothbrush uses the mutual induction of solenoids as part of its battery-charging system. (b) A coil of N_H turns wrapped around the center of a solenoid of N_B turns.

32.5 OSCILLATIONS IN AN *LC* CIRCUIT

13.7 When a capacitor is connected to an inductor as illustrated in Figure 32.14, the combination is an ***LC* circuit.** If the capacitor is initially charged and the switch is then closed, both the current in the circuit and the charge on the capacitor oscillate between maximum positive and negative values. If the resistance of the circuit is zero, no energy is transformed to internal energy. In the following analysis, we neglect the resistance in the circuit. We also assume an idealized situation in which energy is not radiated away from the circuit. We shall discuss this radiation in Chapter 34, but we neglect it for now. With these idealizations—zero resistance and no radiation—the oscillations in the circuit persist indefinitely.

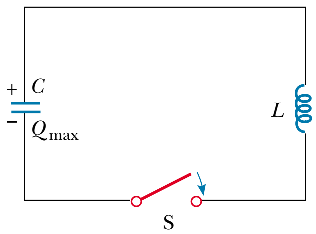


Figure 32.14 A simple LC circuit. The capacitor has an initial charge Q_{max}, and the switch is thrown closed at $t = 0$.

Assume that the capacitor has an initial charge Q_{max} (the maximum charge) and that the switch is thrown closed at $t = 0$. Let us look at what happens from an energy viewpoint.

When the capacitor is fully charged, the energy U in the circuit is stored in the electric field of the capacitor and is equal to $Q_{max}^2/2C$ (Eq. 26.11). At this time, the current in the circuit is zero, and thus no energy is stored in the inductor. After the switch is thrown closed, the rate at which charges leave or enter the capacitor plates (which is also the rate at which the charge on the capacitor changes) is equal to the current in the circuit. As the capacitor begins to discharge after the switch is closed, the energy stored in its electric field decreases. The discharge of the capacitor represents a current in the circuit, and hence some energy is now stored in the magnetic field of the inductor. Thus, energy is transferred from the electric field of the capacitor to the magnetic field of the inductor. When the capacitor is fully discharged, it stores no energy. At this time, the current reaches its maximum value, and all of the energy is stored in the inductor. The current continues in the same direction, decreasing in magnitude, with the capacitor eventually becoming fully charged again but with the polarity of its plates now opposite the initial polarity. This is followed by another discharge until the circuit returns to its original state of maximum charge Q_{max} and the plate polarity shown in Figure 32.14. The energy continues to oscillate between inductor and capacitor.

The oscillations of the LC circuit are an electromagnetic analog to the mechanical oscillations of a block–spring system, which we studied in Chapter 13. Much of what we discussed is applicable to LC oscillations. For example, we investigated the effect of driving a mechanical oscillator with an external force, which leads to the phenomenon of *resonance*. We observe the same phenomenon in the LC circuit. For example, a radio tuner has an LC circuit with a natural frequency, which we determine as follows: When the circuit is driven by the electromagnetic oscillations of a radio signal detected by the antenna, the tuner circuit responds with a large amplitude of electrical oscillation only for the station frequency that matches the natural frequency. Thus, only the signal from one station is passed on to the amplifier, even though signals from all stations are driving the circuit at the same time. When you turn the knob on the radio tuner to change the station, you are changing the natural frequency of the circuit so that it will exhibit a resonance response to a different driving frequency.

A graphical description of the energy transfer between the inductor and the capacitor in an LC circuit is shown in Figure 32.15. The right side of the figure shows the analogous energy transfer in the oscillating block–spring system studied in Chapter 13. In each case, the situation is shown at intervals of one-fourth the period of oscillation T. The potential energy $\frac{1}{2}kx^2$ stored in a stretched spring is analogous to the electric potential energy $Q_{max}^2/2C$ stored in the capacitor. The kinetic energy $\frac{1}{2}mv^2$ of the moving block is analogous to the magnetic energy $\frac{1}{2}LI^2$

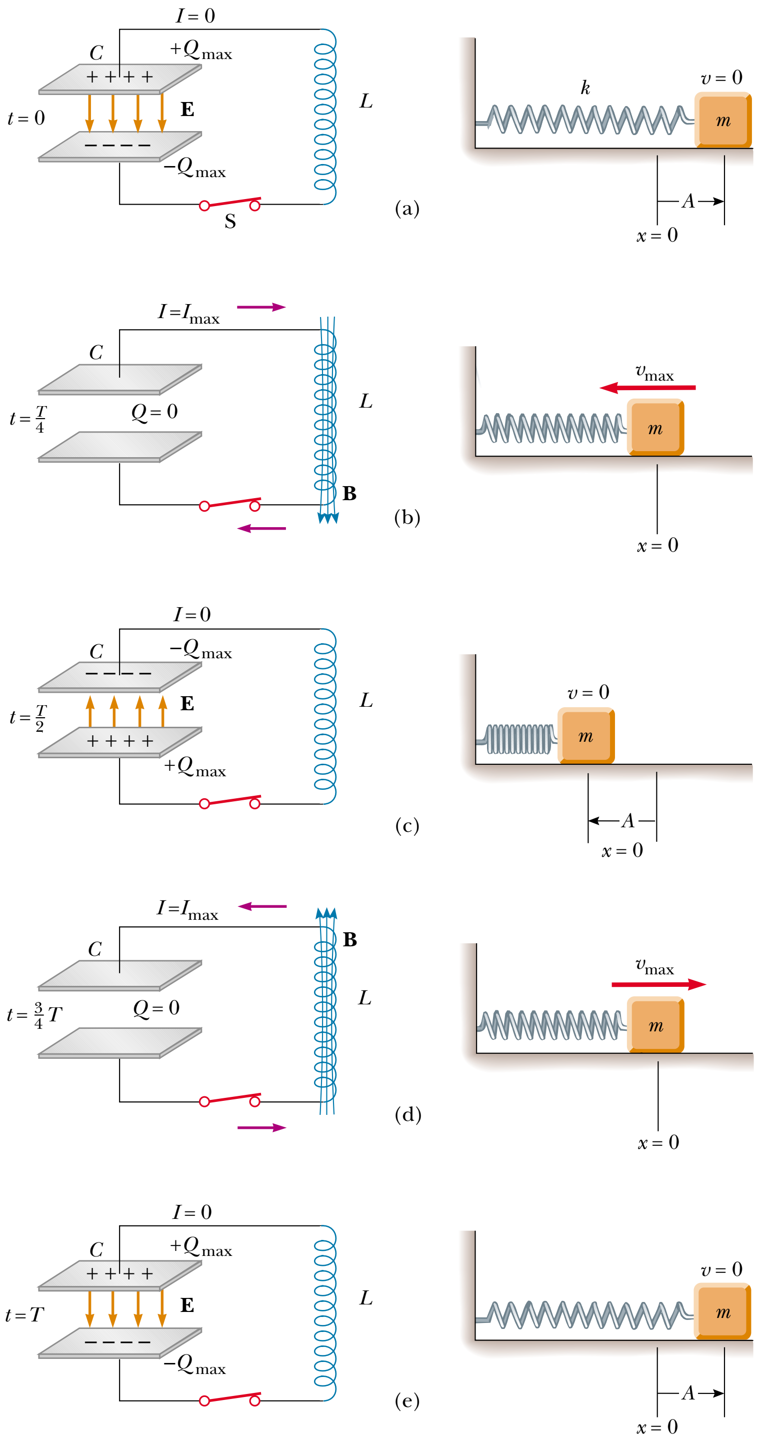


Figure 32.15 Energy transfer in a resistanceless, non-radiating LC circuit. The capacitor has a charge Q_{max} at $t = 0$, the instant at which the switch is thrown closed. The mechanical analog of this circuit is a block–spring system.

stored in the inductor, which requires the presence of moving charges. In Figure 32.15a, all of the energy is stored as electric potential energy in the capacitor at $t = 0$. In Figure 32.15b, which is one fourth of a period later, all of the energy is stored as magnetic energy $\frac{1}{2}LI_{\max}^2$ in the inductor, where $I_{\max}$ is the maximum current in the circuit. In Figure 32.15c, the energy in the LC circuit is stored completely in the capacitor, with the polarity of the plates now opposite what it was in Figure 32.15a. In parts d and e the system returns to the initial configuration over the second half of the cycle. At times other than those shown in the figure, part of the energy is stored in the electric field of the capacitor and part is stored in the magnetic field of the inductor. In the analogous mechanical oscillation, part of the energy is potential energy in the spring and part is kinetic energy of the block.

Let us consider some arbitrary time t after the switch is closed, so that the capacitor has a charge $Q < Q_{\max}$ and the current is $I < I_{\max}$. At this time, both elements store energy, but the sum of the two energies must equal the total initial energy U stored in the fully charged capacitor at $t = 0$:

Total energy stored in an LC circuit

$$U = U_C + U_L = \frac{Q^2}{2C} + \frac{1}{2}LI^2 \tag{32.18}$$

Because we have assumed the circuit resistance to be zero, no energy is transformed to internal energy, and hence *the total energy must remain constant in time.* This means that $dU/dt = 0$. Therefore, by differentiating Equation 32.18 with respect to time while noting that Q and I vary with time, we obtain

The total energy in an ideal LC circuit remains constant; $dU/dt = 0$

$$\frac{dU}{dt} = \frac{d}{dt}\left(\frac{Q^2}{2C} + \frac{1}{2}LI^2\right) = \frac{Q}{C}\frac{dQ}{dt} + LI\frac{dI}{dt} = 0 \tag{32.19}$$

We can reduce this to a differential equation in one variable by remembering that the current in the circuit is equal to the rate at which the charge on the capacitor changes: $I = dQ/dt$. From this, it follows that $dI/dt = d^2Q/dt^2$. Substitution of these relationships into Equation 32.19 gives

$$\frac{Q}{C} + L\frac{d^2Q}{dt^2} = 0$$

$$\frac{d^2Q}{dt^2} = -\frac{1}{LC}Q \tag{32.20}$$

We can solve for Q by noting that this expression is of the same form as the analogous Equations 13.16 and 13.17 for a block–spring system:

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x = -\omega^2 x$$

where k is the spring constant, m is the mass of the block, and $\omega = \sqrt{k/m}$. The solution of this equation has the general form

$$x = A\cos(\omega t + \phi)$$

where ω is the angular frequency of the simple harmonic motion, A is the amplitude of motion (the maximum value of x), and ϕ is the phase constant; the values of A and ϕ depend on the initial conditions. Because it is of the same form as the differential equation of the simple harmonic oscillator, we see that Equation 32.20 has the solution

Charge versus time for an ideal LC circuit

$$Q = Q_{\max}\cos(\omega t + \phi) \tag{32.21}$$

where Q_{max} is the maximum charge of the capacitor and the angular frequency ω is

$$\omega = \frac{1}{\sqrt{LC}} \tag{32.22}$$

Angular frequency of oscillation

Note that the angular frequency of the oscillations depends solely on the inductance and capacitance of the circuit. This is the *natural frequency* of oscillation of the LC circuit.

Because Q varies sinusoidally, the current in the circuit also varies sinusoidally. We can easily show this by differentiating Equation 32.21 with respect to time:

$$I = \frac{dQ}{dt} = -\omega Q_{max} \sin (\omega t + \phi) \tag{32.23}$$

Current versus time for an ideal LC current

To determine the value of the phase angle ϕ, we examine the initial conditions, which in our situation require that at $t = 0$, $I = 0$ and $Q = Q_{max}$. Setting $I = 0$ at $t = 0$ in Equation 32.23, we have

$$0 = -\omega Q_{max} \sin \phi$$

which shows that $\phi = 0$. This value for ϕ also is consistent with Equation 32.21 and with the condition that $Q = Q_{max}$ at $t = 0$. Therefore, in our case, the expressions for Q and I are

$$Q = Q_{max} \cos \omega t \tag{32.24}$$

$$I = -\omega Q_{max} \sin \omega t = -I_{max} \sin \omega t \tag{32.25}$$

Graphs of Q versus t and I versus t are shown in Figure 32.16. Note that the charge on the capacitor oscillates between the extreme values Q_{max} and $-Q_{max}$, and that the current oscillates between I_{max} and $-I_{max}$. Furthermore, the current is 90° out of phase with the charge. That is, when the charge is a maximum, the current is zero, and when the charge is zero, the current has its maximum value.

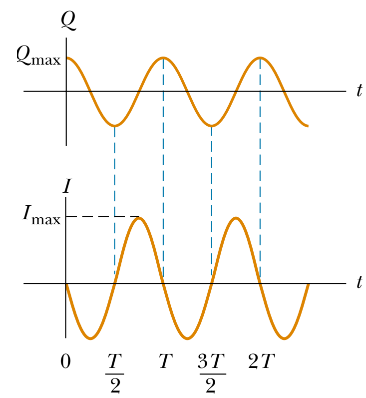


Figure 32.16 Graphs of charge versus time and current versus time for a resistanceless, nonradiating LC circuit. Note that Q and I are 90° out of phase with each other.

Quick Quiz 32.5

What is the relationship between the amplitudes of the two curves in Figure 32.16?

Let us return to the energy discussion of the LC circuit. Substituting Equations 32.24 and 32.25 in Equation 32.18, we find that the total energy is

$$U = U_C + U_L = \frac{Q_{max}^2}{2C} \cos^2 \omega t + \frac{LI_{max}^2}{2} \sin^2 \omega t \tag{32.26}$$

This expression contains all of the features described qualitatively at the beginning of this section. It shows that the energy of the LC circuit continuously oscillates between energy stored in the electric field of the capacitor and energy stored in the magnetic field of the inductor. When the energy stored in the capacitor has its maximum value $Q_{max}^2/2C$, the energy stored in the inductor is zero. When the energy stored in the inductor has its maximum value $\frac{1}{2}LI_{max}^2$, the energy stored in the capacitor is zero.

Plots of the time variations of U_C and U_L are shown in Figure 32.17. The sum $U_C + U_L$ is a constant and equal to the total energy $Q_{max}^2/2C$ or $LI_{max}^2/2$. Analytical verification of this is straightforward. The amplitudes of the two graphs in Figure 32.17 must be equal because the maximum energy stored in the capacitor

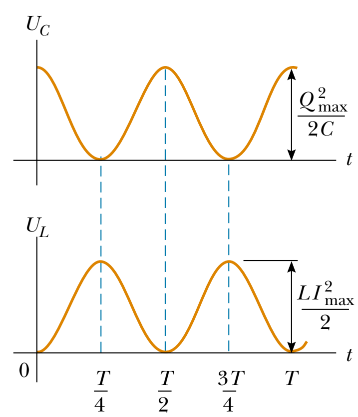


Figure 32.17 Plots of U_C versus t and U_L versus t for a resistanceless, nonradiating LC circuit. The sum of the two curves is a constant and equal to the total energy stored in the circuit.

(when $I = 0$) must equal the maximum energy stored in the inductor (when $Q = 0$). This is mathematically expressed as

$$\frac{Q_{max}^2}{2C} = \frac{LI_{max}^2}{2}$$

Using this expression in Equation 32.26 for the total energy gives

$$U = \frac{Q_{max}^2}{2C}(\cos^2 \omega t + \sin^2 \omega t) = \frac{Q_{max}^2}{2C} \quad \textbf{(32.27)}$$

because $\cos^2 \omega t + \sin^2 \omega t = 1$.

In our idealized situation, the oscillations in the circuit persist indefinitely; however, we remember that the total energy U of the circuit remains constant only if energy transfers and transformations are neglected. In actual circuits, there is always some resistance, and hence energy is transformed to internal energy. We mentioned at the beginning of this section that we are also ignoring radiation from the circuit. In reality, radiation is inevitable in this type of circuit, and the total energy in the circuit continuously decreases as a result of this process.

EXAMPLE 32.7 An Oscillatory *LC* Circuit

In Figure 32.18, the capacitor is initially charged when switch S_1 is open and S_2 is closed. Switch S_1 is then thrown closed at the same instant that S_2 is opened, so that the capacitor is connected directly across the inductor. (a) Find the frequency of oscillation of the circuit.

Solution Using Equation 32.22 gives for the frequency

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{LC}}$$

$$= \frac{1}{2\pi[(2.81 \times 10^{-3}\ \text{H})(9.00 \times 10^{-12}\ \text{F})]^{1/2}}$$

$$= \boxed{1.00 \times 10^6\ \text{Hz}}$$

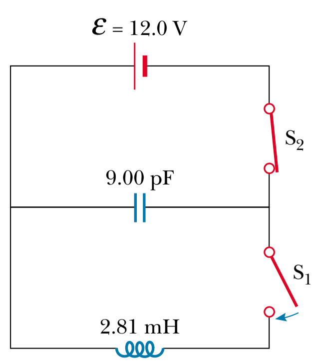


Figure 32.18 First the capacitor is fully charged with the switch S_1 open and S_2 closed. Then, S_1 is thrown closed at the same time that S_2 is thrown open.

(b) What are the maximum values of charge on the capacitor and current in the circuit?

Solution The initial charge on the capacitor equals the maximum charge, and because $C = Q/\mathcal{E}$, we have

$$Q_{max} = C\mathcal{E} = (9.00 \times 10^{-12}\ \text{F})(12.0\ \text{V}) = \boxed{1.08 \times 10^{-10}\ \text{C}}$$

From Equation 32.25, we can see how the maximum current is related to the maximum charge:

$$I_{max} = \omega Q_{max} = 2\pi f Q_{max}$$

$$= (2\pi \times 10^6\ \text{s}^{-1})(1.08 \times 10^{-10}\ \text{C})$$

$$= \boxed{6.79 \times 10^{-4}\ \text{A}}$$

(c) Determine the charge and current as functions of time.

Solution Equations 32.24 and 32.25 give the following expressions for the time variation of Q and I:

$$Q = Q_{max} \cos \omega t$$

$$= \boxed{(1.08 \times 10^{-10}\ \text{C}) \cos[(2\pi \times 10^6\ \text{rad/s})t]}$$

$$I = -I_{max} \sin \omega t$$

$$= \boxed{(-6.79 \times 10^{-4}\ \text{A}) \sin[(2\pi \times 10^6\ \text{rad/s})t]}$$

Exercise What is the total energy stored in the circuit?

Answer 6.48×10^{-10} J.

Optional Section

32.6 THE *RLC* CIRCUIT

13.7 We now turn our attention to a more realistic circuit consisting of an inductor, a capacitor, and a resistor connected in series, as shown in Figure 32.19. We let the resistance of the resistor represent all of the resistance in the circuit. We assume that the capacitor has an initial charge Q_{max} before the switch is closed. Once the switch is thrown closed and a current is established, the total energy stored in the capacitor and inductor at any time is given, as before, by Equation 32.18. However, the total energy is no longer constant, as it was in the LC circuit, because the resistor causes transformation to internal energy. Because the rate of energy transformation to internal energy within a resistor is I^2R, we have

$$\frac{dU}{dt} = -I^2R$$

where the negative sign signifies that the energy U of the circuit is decreasing in time. Substituting this result into Equation 32.19 gives

$$LI\frac{dI}{dt} + \frac{Q}{C}\frac{dQ}{dt} = -I^2R \quad \textbf{(32.28)}$$

To convert this equation into a form that allows us to compare the electrical oscillations with their mechanical analog, we first use the fact that $I = dQ/dt$ and move all terms to the left-hand side to obtain

$$LI\frac{d^2Q}{dt^2} + \frac{Q}{C}I + I^2R = 0$$

Now we divide through by I:

$$L\frac{d^2Q}{dt^2} + \frac{Q}{C} + IR = 0$$

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \quad \textbf{(32.29)}$$

The RLC circuit is analogous to the damped harmonic oscillator discussed in Section 13.6 and illustrated in Figure 32.20. The equation of motion for this mechanical system is, from Equation 13.32,

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0 \quad \textbf{(32.30)}$$

Comparing Equations 32.29 and 32.30, we see that Q corresponds to the position x of the block at any instant, L to the mass m of the block, R to the damping coefficient b, and C to $1/k$, where k is the force constant of the spring. These and other relationships are listed in Table 32.1.

Because the analytical solution of Equation 32.29 is cumbersome, we give only a qualitative description of the circuit behavior. In the simplest case, when $R = 0$, Equation 32.29 reduces to that of a simple LC circuit, as expected, and the charge and the current oscillate sinusoidally in time. This is equivalent to removal of all damping in the mechanical oscillator.

When R is small, a situation analogous to light damping in the mechanical oscillator, the solution of Equation 32.29 is

$$Q = Q_{max}e^{-Rt/2L}\cos\omega_d t \quad \textbf{(32.31)}$$

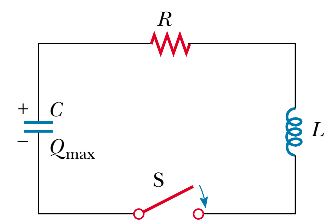


Figure 32.19 A series RLC circuit. The capacitor has a charge Q_{max} at $t = 0$, the instant at which the switch is thrown closed.

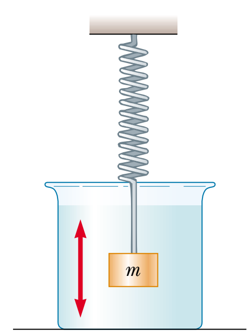


Figure 32.20 A block–spring system moving in a viscous medium with damped harmonic motion is analogous to an RLC circuit.

TABLE 32.1 Analogies Between Electrical and Mechanical Systems

Electric Circuit		One-Dimensional Mechanical System
Charge	$Q \leftrightarrow x$	Displacement
Current	$I \leftrightarrow v_x$	Velocity
Potential difference	$\Delta V \leftrightarrow F_x$	Force
Resistance	$R \leftrightarrow b$	Viscous damping coefficient
Capacitance	$C \leftrightarrow 1/k$	(k = spring constant)
Inductance	$L \leftrightarrow m$	Mass
Current = time derivative of charge	$I = \frac{dQ}{dt} \leftrightarrow v_x = \frac{dx}{dt}$	Velocity = time derivative of position
Rate of change of current = second time derivative of charge	$\frac{dI}{dt} = \frac{d^2Q}{dt^2} \leftrightarrow a_x = \frac{dv_x}{dt} = \frac{d^2x}{dt^2}$	Acceleration = second time derivative of position
Energy in inductor	$U_L = \frac{1}{2}LI^2 \leftrightarrow K = \frac{1}{2}mv^2$	Kinetic energy of moving mass
Energy in capacitor	$U_C = \frac{1}{2}\frac{Q^2}{C} \leftrightarrow U = \frac{1}{2}kx^2$	Potential energy stored in a spring
Rate of energy loss due to resistance	$I^2R \leftrightarrow bv^2$	Rate of energy loss due to friction
RLC circuit	$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0 \leftrightarrow m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$	Damped mass on a spring

where

$$\omega_d = \left[\frac{1}{LC} - \left(\frac{R}{2L}\right)^2\right]^{1/2} \tag{32.32}$$

is the angular frequency at which the circuit oscillates. That is, the value of the charge on the capacitor undergoes a damped harmonic oscillation in analogy with a mass–spring system moving in a viscous medium. From Equation 32.32, we see that, when $R \ll \sqrt{4L/C}$ (so that the second term in the brackets is much smaller than the first), the frequency ω_d of the damped oscillator is close to that of the undamped oscillator, $1/\sqrt{LC}$. Because $I = dQ/dt$, it follows that the current also undergoes damped harmonic oscillation. A plot of the charge versus time for the damped oscillator is shown in Figure 32.21a. Note that the maximum value of Q decreases after each oscillation, just as the amplitude of a damped block–spring system decreases in time.

Quick Quiz 32.6

Figure 32.21a has two dashed blue lines that form an "envelope" around the curve. What is the equation for the upper dashed line?

When we consider larger values of R, we find that the oscillations damp out more rapidly; in fact, there exists a critical resistance value $R_c = \sqrt{4L/C}$ above which no oscillations occur. A system with $R = R_c$ is said to be *critically damped*. When R exceeds R_c, the system is said to be *overdamped* (Fig. 32.22).

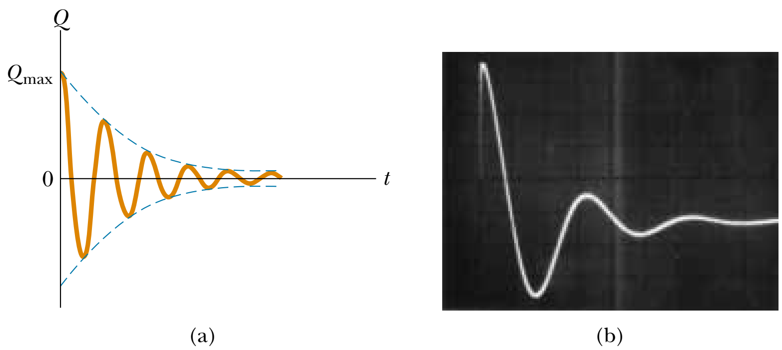


Figure 32.21 (a) Charge versus time for a damped *RLC* circuit. The charge decays in this way when $R \ll \sqrt{4L/C}$. The Q-versus-t curve represents a plot of Equation 32.31. (b) Oscilloscope pattern showing the decay in the oscillations of an *RLC* circuit. The parameters used were $R = 75\ \Omega$, $L = 10$ mH, and $C = 0.19\ \mu$F.

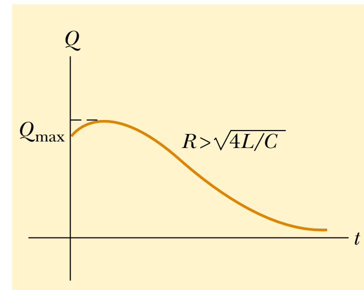


Figure 32.22 Plot of Q versus t for an overdamped *RLC* circuit, which occurs for values of $R > \sqrt{4L/C}$.

SUMMARY

When the current in a coil changes with time, an emf is induced in the coil according to Faraday's law. The **self-induced emf** is

$$\mathcal{E}_L = -L\frac{dI}{dt} \tag{32.1}$$

where L is the **inductance** of the coil. Inductance is a measure of how much opposition an electrical device offers to a change in current passing through the device. Inductance has the SI unit of **henry** (H), where $1\ \text{H} = 1\ \text{V}\cdot\text{s/A}$.

The inductance of any coil is

$$L = \frac{N\Phi_B}{I} \tag{32.2}$$

where Φ_B is the magnetic flux through the coil and N is the total number of turns. The inductance of a device depends on its geometry. For example, the inductance of an air-core solenoid is

$$L = \frac{\mu_0 N^2 A}{\ell} \tag{32.4}$$

where A is the cross-sectional area, and ℓ is the length of the solenoid.

If a resistor and inductor are connected in series to a battery of emf $\mathcal{E}$, and if a switch in the circuit is thrown closed at $t = 0$, then the current in the circuit varies in time according to the expression

$$I = \frac{\mathcal{E}}{R}(1 - e^{-t/\tau}) \tag{32.7}$$

where $\tau = L/R$ is the time constant of the *RL* circuit. That is, the current increases to an equilibrium value of $\mathcal{E}/R$ after a time that is long compared with τ. If the battery in the circuit is replaced by a resistanceless wire, the current decays exponentially with time according to the expression

$$I = \frac{\mathcal{E}}{R}e^{-t/\tau} \tag{32.10}$$

where $\mathcal{E}/R$ is the initial current in the circuit.

The energy stored in the magnetic field of an inductor carrying a current I is

$$U = \tfrac{1}{2}LI^2 \tag{32.12}$$

This energy is the magnetic counterpart to the energy stored in the electric field of a charged capacitor.

The energy density at a point where the magnetic field is B is

$$u_B = \frac{B^2}{2\mu_0} \tag{32.14}$$

The mutual inductance of a system of two coils is given by

$$M_{12} = \frac{N_2\Phi_{12}}{I_1} = M_{21} = \frac{N_1\Phi_{21}}{I_2} = M \tag{32.15}$$

This mutual inductance allows us to relate the induced emf in a coil to the changing source current in a nearby coil using the relationships

$$\mathcal{E}_2 = -M_{12}\frac{dI_1}{dt} \quad \text{and} \quad \mathcal{E}_1 = -M_{21}\frac{dI_2}{dt} \tag{32.16, 32.17}$$

In an LC circuit that has zero resistance and does not radiate electromagnetically (an idealization), the values of the charge on the capacitor and the current in the circuit vary in time according to the expressions

$$Q = Q_{\max}\cos(\omega t + \phi) \tag{32.21}$$

$$I = \frac{dQ}{dt} = -\omega Q_{\max}\sin(\omega t + \phi) \tag{32.23}$$

where $Q_{\max}$ is the maximum charge on the capacitor, ϕ is a phase constant, and ω is the angular frequency of oscillation:

$$\omega = \frac{1}{\sqrt{LC}} \tag{32.22}$$

The energy in an LC circuit continuously transfers between energy stored in the capacitor and energy stored in the inductor. The total energy of the LC circuit at any time t is

$$U = U_C + U_L = \frac{Q_{\max}^2}{2C}\cos^2\omega t + \frac{LI_{\max}^2}{2}\sin^2\omega t \tag{32.26}$$

At $t = 0$, all of the energy is stored in the electric field of the capacitor ($U = Q_{\max}^2/2C$). Eventually, all of this energy is transferred to the inductor ($U = LI_{\max}^2/2$). However, the total energy remains constant because energy transformations are neglected in the ideal LC circuit.

QUESTIONS

1. Why is the induced emf that appears in an inductor called a "counter" or "back" emf?
2. The current in a circuit containing a coil, resistor, and battery reaches a constant value. Does the coil have an inductance? Does the coil affect the value of the current?
3. What parameters affect the inductance of a coil? Does the inductance of a coil depend on the current in the coil?
4. How can a long piece of wire be wound on a spool so that the wire has a negligible self-inductance?
5. A long, fine wire is wound as a solenoid with a self-inductance L. If it is connected across the terminals of a battery, how does the maximum current depend on L?
6. For the series RL circuit shown in Figure Q32.6, can the back emf ever be greater than the battery emf? Explain.

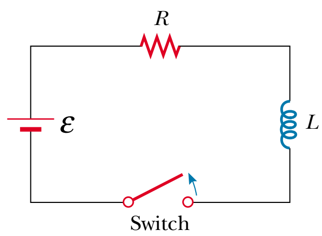


Figure Q32.6

7. Consider this thesis: "Joseph Henry, America's first professional physicist, changed the view of the Universe during a school vacation at the Albany Academy in 1830. Before that time, one could think of the Universe as consisting of just one thing: matter. In Henry's experiment, after a battery is removed from a coil, the energy that keeps the current flowing for a while does not belong to any piece of matter. This energy belongs to the magnetic field surrounding the coil. With Henry's discovery of self-induction, Nature forced us to admit that the Universe consists of fields as well as matter." What in your view constitutes the Universe? Argue for your answer.
8. Discuss the similarities and differences between the energy stored in the electric field of a charged capacitor and the energy stored in the magnetic field of a current-carrying coil.
9. What is the inductance of two inductors connected in series? Does it matter if they are solenoids or toroids?
10. The centers of two circular loops are separated by a fixed distance. For what relative orientation of the loops is their mutual inductance a maximum? a minimum? Explain.
11. Two solenoids are connected in series so that each carries the same current at any instant. Is mutual induction present? Explain.
12. In the LC circuit shown in Figure 32.15, the charge on the capacitor is sometimes zero, even though current is in the circuit. How is this possible?
13. If the resistance of the wires in an LC circuit were not zero, would the oscillations persist? Explain.
14. How can you tell whether an RLC circuit is overdamped or underdamped?
15. What is the significance of critical damping in an RLC circuit?
16. Can an object exert a force on itself? When a coil induces an emf in itself, does it exert a force on itself?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
= paired numerical/symbolic problems

Section 32.1 Self-Inductance

1. A coil has an inductance of 3.00 mH, and the current through it changes from 0.200 A to 1.50 A in a time of 0.200 s. Find the magnitude of the average induced emf in the coil during this time.
2. A coiled telephone cord forms a spiral with 70 turns, a diameter of 1.30 cm, and an unstretched length of 60.0 cm. Determine the self-inductance of one conductor in the unstretched cord.
3. A 2.00-H inductor carries a steady current of 0.500 A. When the switch in the circuit is thrown open, the current is effectively zero in 10.0 ms. What is the average induced emf in the inductor during this time?
4. A small air-core solenoid has a length of 4.00 cm and a radius of 0.250 cm. If the inductance is to be 0.060 0 mH, how many turns per centimeter are required?
5. Calculate the magnetic flux through the area enclosed by a 300-turn, 7.20-mH coil when the current in the coil is 10.0 mA.
6. The current in a solenoid is increasing at a rate of 10.0 A/s. The cross-sectional area of the solenoid is π cm^2, and there are 300 turns on its 15.0-cm length. What is the induced emf opposing the increasing current?

WEB 7. A 10.0-mH inductor carries a current $I = I_{max} \sin \omega t$, with $I_{max} = 5.00$ A and $\omega/2\pi = 60.0$ Hz. What is the back emf as a function of time?
8. An emf of 24.0 mV is induced in a 500-turn coil at an instant when the current is 4.00 A and is changing at the rate of 10.0 A/s. What is the magnetic flux through each turn of the coil?
9. An inductor in the form of a solenoid contains 420 turns, is 16.0 cm in length, and has a cross-sectional area of 3.00 cm^2. What uniform rate of decrease of current through the inductor induces an emf of 175 μV?
10. An inductor in the form of a solenoid contains N turns, has length ℓ, and has cross-sectional area A. What uniform rate of decrease of current through the inductor induces an emf $\mathcal{E}$?
11. The current in a 90.0-mH inductor changes with time as $I = t^2 - 6.00t$ (in SI units). Find the magnitude of the induced emf at (a) $t = 1.00$ s and (b) $t = 4.00$ s. (c) At what time is the emf zero?
12. A 40.0-mA current is carried by a uniformly wound air-core solenoid with 450 turns, a 15.0-mm diameter, and 12.0-cm length. Compute (a) the magnetic field inside the solenoid, (b) the magnetic flux through each turn,

and (c) the inductance of the solenoid. (d) Which of these quantities depends on the current?

13. A solenoid has 120 turns uniformly wrapped around a wooden core, which has a diameter of 10.0 mm and a length of 9.00 cm. (a) Calculate the inductance of the solenoid. (b) The wooden core is replaced with a soft iron rod that has the same dimensions but a magnetic permeability $\mu_m = 800\mu_0$. What is the new inductance?

14. A toroid has a major radius R and a minor radius r, and it is tightly wound with N turns of wire, as shown in Figure P32.14. If $R \gg r$, the magnetic field within the region of the torus, of cross-sectional area $A = \pi r^2$, is essentially that of a long solenoid that has been bent into a large circle of radius R. Using the uniform field of a long solenoid, show that the self-inductance of such a toroid is approximately

$$L \cong \mu_0 N^2 A / 2\pi R$$

(An exact expression for the inductance of a toroid with a rectangular cross-section is derived in Problem 64.)

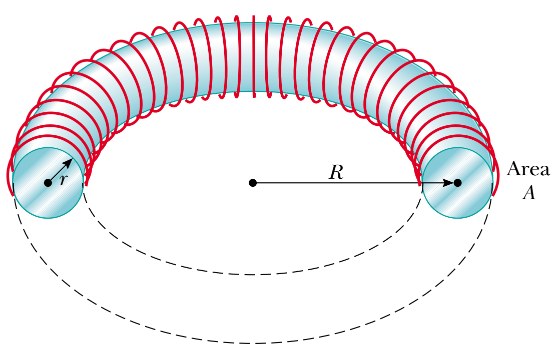


Figure P32.14

15. An emf self-induced in a solenoid of inductance L changes in time as $\mathcal{E} = \mathcal{E}_0 e^{-kt}$. Find the total charge that passes through the solenoid, if the charge is finite.

Section 32.2 *RL* Circuits

16. Calculate the resistance in an RL circuit in which $L = 2.50$ H and the current increases to 90.0% of its final value in 3.00 s.

17. A 12.0-V battery is connected into a series circuit containing a 10.0-Ω resistor and a 2.00-H inductor. How long will it take the current to reach (a) 50.0% and (b) 90.0% of its final value?

18. Show that $I = I_0 e^{-t/\tau}$ is a solution of the differential equation

$$IR + L\frac{dI}{dt} = 0$$

where $\tau = L/R$ and I_0 is the current at $t = 0$.

19. Consider the circuit in Figure P32.19, taking $\mathcal{E} = 6.00$ V, $L = 8.00$ mH, and $R = 4.00\ \Omega$. (a) What is the inductive time constant of the circuit? (b) Calculate the current in the circuit 250 μs after the switch is closed. (c) What is the value of the final steady-state current? (d) How long does it take the current to reach 80.0% of its maximum value?

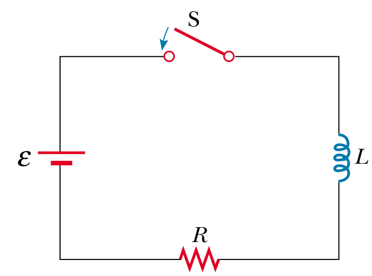


Figure P32.19 Problems 19, 20, 21, and 24.

20. In the circuit shown in Figure P32.19, let $L = 7.00$ H, $R = 9.00\ \Omega$, and $\mathcal{E} = 120$ V. What is the self-induced emf 0.200 s after the switch is closed?

WEB **21.** For the RL circuit shown in Figure P32.19, let $L = 3.00$ H, $R = 8.00\ \Omega$, and $\mathcal{E} = 36.0$ V. (a) Calculate the ratio of the potential difference across the resistor to that across the inductor when $I = 2.00$ A. (b) Calculate the voltage across the inductor when $I = 4.50$ A.

22. A 12.0-V battery is connected in series with a resistor and an inductor. The circuit has a time constant of 500 μs, and the maximum current is 200 mA. What is the value of the inductance?

23. An inductor that has an inductance of 15.0 H and a resistance of 30.0 Ω is connected across a 100-V battery. What is the rate of increase of the current (a) at $t = 0$ and (b) at $t = 1.50$ s?

24. When the switch in Figure P32.19 is thrown closed, the current takes 3.00 ms to reach 98.0% of its final value. If $R = 10.0\ \Omega$, what is the inductance?

25. The switch in Figure P32.25 is closed at time $t = 0$. Find the current in the inductor and the current through the switch as functions of time thereafter.

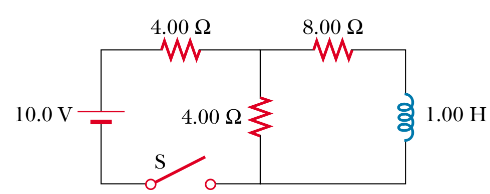


Figure P32.25

26. A series RL circuit with $L = 3.00$ H and a series RC circuit with $C = 3.00\ \mu$F have equal time constants. If the two circuits contain the same resistance R, (a) what is the value of R and (b) what is the time constant?

27. A current pulse is fed to the partial circuit shown in Figure P32.27. The current begins at zero, then becomes 10.0 A between $t = 0$ and $t = 200\ \mu s$, and then is zero once again. Determine the current in the inductor as a function of time.

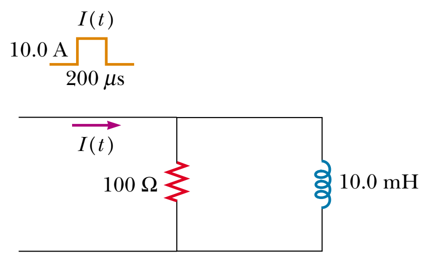


Figure P32.27

28. One application of an RL circuit is the generation of time-varying high voltage from a low-voltage source, as shown in Figure P32.28. (a) What is the current in the circuit a long time after the switch has been in position A? (b) Now the switch is thrown quickly from A to B. Compute the initial voltage across each resistor and the inductor. (c) How much time elapses before the voltage across the inductor drops to 12.0 V?

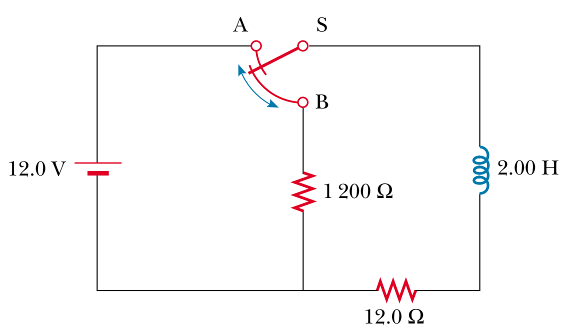


Figure P32.28

WEB **29.** A 140-mH inductor and a 4.90-Ω resistor are connected with a switch to a 6.00-V battery, as shown in Figure P32.29. (a) If the switch is thrown to the left (connecting the battery), how much time elapses before the current reaches 220 mA? (b) What is the current in the inductor 10.0 s after the switch is closed? (c) Now the switch is quickly thrown from A to B. How much time elapses before the current falls to 160 mA?

30. Consider two ideal inductors, L_1 and L_2, that have *zero* internal resistance and are far apart, so that their magnetic fields do not influence each other. (a) If these inductors are connected in series, show that they are equivalent to a single ideal inductor having $L_{eq} = L_1 + L_2$. (b) If these same two inductors are connected in parallel, show that they are equivalent to a single ideal inductor having $1/L_{eq} = 1/L_1 + 1/L_2$. (c) Now consider two inductors L_1 and L_2 that have *nonzero* internal resistances R_1 and R_2, respectively. Assume that they are still far apart so that their magnetic fields do not influence each other. If these inductors are connected in series, show that they are equivalent to a single inductor having $L_{eq} = L_1 + L_2$ and $R_{eq} = R_1 + R_2$. (d) If these same inductors are now connected in parallel, is it necessarily true that they are equivalent to a single ideal inductor having $1/L_{eq} = 1/L_1 + 1/L_2$ and $1/R_{eq} = 1/R_1 + 1/R_2$? Explain your answer.

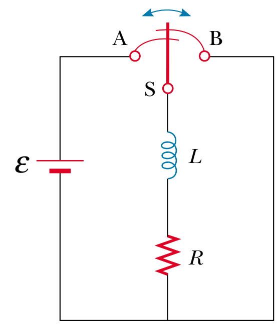


Figure P32.29

Section 32.3 Energy in a Magnetic Field

31. Calculate the energy associated with the magnetic field of a 200-turn solenoid in which a current of 1.75 A produces a flux of $3.70 \times 10^{-4}\ \text{T} \cdot \text{m}^2$ in each turn.

32. The magnetic field inside a superconducting solenoid is 4.50 T. The solenoid has an inner diameter of 6.20 cm and a length of 26.0 cm. Determine (a) the magnetic energy density in the field and (b) the energy stored in the magnetic field within the solenoid.

33. An air-core solenoid with 68 turns is 8.00 cm long and has a diameter of 1.20 cm. How much energy is stored in its magnetic field when it carries a current of 0.770 A?

34. At $t = 0$, an emf of 500 V is applied to a coil that has an inductance of 0.800 H and a resistance of 30.0 Ω. (a) Find the energy stored in the magnetic field when the current reaches half its maximum value. (b) After the emf is connected, how long does it take the current to reach this value?

WEB **35.** On a clear day there is a 100-V/m vertical electric field near the Earth's surface. At the same place, the Earth's magnetic field has a magnitude of 0.500×10^{-4} T. Compute the energy densities of the two fields.

36. An RL circuit in which $L = 4.00$ H and $R = 5.00\ \Omega$ is connected to a 22.0-V battery at $t = 0$. (a) What energy is stored in the inductor when the current is 0.500 A? (b) At what rate is energy being stored in the inductor when $I = 1.00$ A? (c) What power is being delivered to the circuit by the battery when $I = 0.500$ A?

37. A 10.0-V battery, a 5.00-Ω resistor, and a 10.0-H inductor are connected in series. After the current in the circuit

has reached its maximum value, calculate (a) the power being supplied by the battery, (b) the power being delivered to the resistor, (c) the power being delivered to the inductor, and (d) the energy stored in the magnetic field of the inductor.

38. A uniform electric field with a magnitude of 680 kV/m throughout a cylindrical volume results in a total energy of 3.40 μJ. What magnetic field over this same region stores the same total energy?

39. Assume that the magnitude of the magnetic field outside a sphere of radius R is $B = B_0(R/r)^2$, where B_0 is a constant. Determine the total energy stored in the magnetic field outside the sphere and evaluate your result for $B_0 = 5.00 \times 10^{-5}$ T and $R = 6.00 \times 10^6$ m, values appropriate for the Earth's magnetic field.

Section 32.4 Mutual Inductance

40. Two coils are close to each other. The first coil carries a time-varying current given by $I(t) = (5.00\text{ A})\, e^{-0.025\,0t} \sin(377t)$. At $t = 0.800$ s, the voltage measured across the second coil is -3.20 V. What is the mutual inductance of the coils?

41. Two coils, held in fixed positions, have a mutual inductance of 100 μH. What is the peak voltage in one when a sinusoidal current given by $I(t) = (10.0\text{ A}) \sin(1\,000t)$ flows in the other?

42. An emf of 96.0 mV is induced in the windings of a coil when the current in a nearby coil is increasing at the rate of 1.20 A/s. What is the mutual inductance of the two coils?

43. Two solenoids A and B, spaced close to each other and sharing the same cylindrical axis, have 400 and 700 turns, respectively. A current of 3.50 A in coil A produces an average flux of 300 μT · m^2 through each turn of A and a flux of 90.0 μT · m^2 through each turn of B. (a) Calculate the mutual inductance of the two solenoids. (b) What is the self-inductance of A? (c) What emf is induced in B when the current in A increases at the rate of 0.500 A/s?

44. A 70-turn solenoid is 5.00 cm long and 1.00 cm in diameter and carries a 2.00-A current. A single loop of wire, 3.00 cm in diameter, is held so that the plane of the loop is perpendicular to the long axis of the solenoid, as illustrated in Figure P31.18 (page 1004). What is the mutual inductance of the two if the plane of the loop passes through the solenoid 2.50 cm from one end?

45. Two single-turn circular loops of wire have radii R and r, with $R \gg r$. The loops lie in the same plane and are concentric. (a) Show that the mutual inductance of the pair is $M = \mu_0 \pi r^2/2R$. (*Hint:* Assume that the larger loop carries a current I and compute the resulting flux through the smaller loop.) (b) Evaluate M for $r = 2.00$ cm and $R = 20.0$ cm.

46. On a printed circuit board, a relatively long straight conductor and a conducting rectangular loop lie in the same plane, as shown in Figure P31.9 (page 1003). If $h = 0.400$ mm, $w = 1.30$ mm, and $L = 2.70$ mm, what is their mutual inductance?

47. Two inductors having self-inductances L_1 and L_2 are connected in parallel, as shown in Figure P32.47a. The mutual inductance between the two inductors is M. Determine the equivalent self-inductance L_{eq} for the system (Fig. P32.47b).

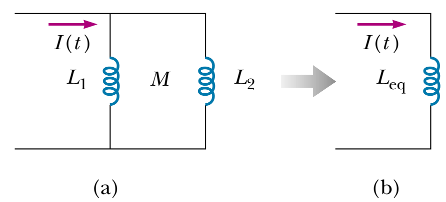


Figure P32.47

Section 32.5 Oscillations in an *LC* Circuit

48. A 1.00-μF capacitor is charged by a 40.0-V power supply. The fully-charged capacitor is then discharged through a 10.0-mH inductor. Find the maximum current in the resulting oscillations.

49. An *LC* circuit consists of a 20.0-mH inductor and a 0.500-μF capacitor. If the maximum instantaneous current is 0.100 A, what is the greatest potential difference across the capacitor?

50. In the circuit shown in Figure P32.50, $\mathcal{E} = 50.0$ V, $R = 250\ \Omega$, and $C = 0.500\ \mu$F. The switch S is closed for a long time, and no voltage is measured across the capacitor. After the switch is opened, the voltage across the capacitor reaches a maximum value of 150 V. What is the inductance L?

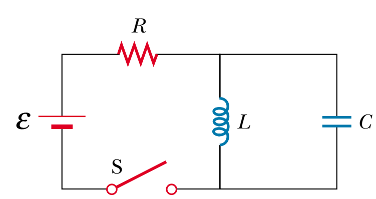


Figure P32.50

51. A fixed inductance $L = 1.05\ \mu$H is used in series with a variable capacitor in the tuning section of a radio. What capacitance tunes the circuit to the signal from a station broadcasting at 6.30 MHz?

52. Calculate the inductance of an *LC* circuit that oscillates at 120 Hz when the capacitance is 8.00 μF.

53. An *LC* circuit like the one shown in Figure 32.14 contains an 82.0-mH inductor and a 17.0-μF capacitor that initially carries a 180-μC charge. The switch is thrown closed at $t = 0$. (a) Find the frequency (in hertz) of the resulting oscillations. At $t = 1.00$ ms, find (b) the charge on the capacitor and (c) the current in the circuit.

54. The switch in Figure P32.54 is connected to point a for a long time. After the switch is thrown to point b, what are (a) the frequency of oscillation of the LC circuit, (b) the maximum charge that appears on the capacitor, (c) the maximum current in the inductor, and (d) the total energy the circuit possesses at $t = 3.00$ s?

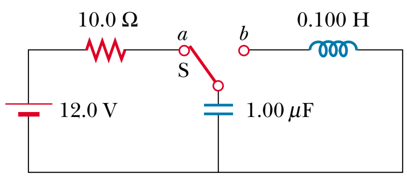


Figure P32.54

WEB 55. An LC circuit like that illustrated in Figure 32.14 consists of a 3.30-H inductor and an 840-pF capacitor, initially carrying a 105-μC charge. At $t = 0$ the switch is thrown closed. Compute the following quantities at $t = 2.00$ ms: (a) the energy stored in the capacitor; (b) the energy stored in the inductor; (c) the total energy in the circuit.

(Optional)

Section 32.6 The *RLC* Circuit

56. In Figure 32.19, let $R = 7.60\ \Omega$, $L = 2.20$ mH, and $C = 1.80\ \mu$F. (a) Calculate the frequency of the damped oscillation of the circuit. (b) What is the critical resistance?

57. Consider an LC circuit in which $L = 500$ mH and $C = 0.100\ \mu$F. (a) What is the resonant frequency ω_0? (b) If a resistance of 1.00 kΩ is introduced into this circuit, what is the frequency of the (damped) oscillations? (c) What is the percent difference between the two frequencies?

58. Show that Equation 32.29 in the text is Kirchhoff's loop rule as applied to Figure 32.19.

59. Electrical oscillations are initiated in a series circuit containing a capacitance C, inductance L, and resistance R. (a) If $R \ll \sqrt{4L/C}$ (weak damping), how much time elapses before the amplitude of the current oscillation falls off to 50.0% of its initial value? (b) How long does it take the energy to decrease to 50.0% of its initial value?

ADDITIONAL PROBLEMS

60. Initially, the capacitor in a series LC circuit is charged. A switch is closed, allowing the capacitor to discharge, and after time t the energy stored in the capacitor is one-fourth its initial value. Determine L if C is known.

61. A 1.00-mH inductor and a 1.00-μF capacitor are connected in series. The current in the circuit is described by $I = 20.0t$, where t is in seconds and I is in amperes. The capacitor initially has no charge. Determine (a) the voltage across the inductor as a function of time, (b) the voltage across the capacitor as a function of time, and (c) the time when the energy stored in the capacitor first exceeds that in the inductor.

62. An inductor having inductance L and a capacitor having capacitance C are connected in series. The current in the circuit increases linearly in time as described by $I = Kt$. The capacitor is initially uncharged. Determine (a) the voltage across the inductor as a function of time, (b) the voltage across the capacitor as a function of time, and (c) the time when the energy stored in the capacitor first exceeds that in the inductor.

63. A capacitor in a series LC circuit has an initial charge Q and is being discharged. Find, in terms of L and C, the flux through each of the N turns in the coil, when the charge on the capacitor is $Q/2$.

64. The toroid in Figure P32.64 consists of N turns and has a rectangular cross-section. Its inner and outer radii are a and b, respectively. (a) Show that

$$L = \frac{\mu_0 N^2 h}{2\pi} \ln \frac{b}{a}$$

(b) Using this result, compute the self-inductance of a 500-turn toroid for which $a = 10.0$ cm, $b = 12.0$ cm, and $h = 1.00$ cm. (c) In Problem 14, an approximate formula for the inductance of a toroid with $R \gg r$ was derived. To get a feel for the accuracy of that result, use the expression in Problem 14 to compute the approximate inductance of the toroid described in part (b). Compare the result with the answer to part (b).

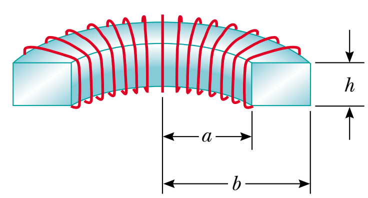


Figure P32.64

65. (a) A flat circular coil does not really produce a uniform magnetic field in the area it encloses, but estimate the self-inductance of a flat circular coil, with radius R and N turns, by supposing that the field at its center is uniform over its area. (b) A circuit on a laboratory table consists of a 1.5-V battery, a 270-Ω resistor, a switch, and three 30-cm-long cords connecting them. Suppose that the circuit is arranged to be circular. Think of it as a flat coil with one turn. Compute the order of magnitude of its self-inductance and (c) of the time constant describing how fast the current increases when you close the switch.

66. A soft iron rod ($\mu_m = 800\ \mu_0$) is used as the core of a solenoid. The rod has a diameter of 24.0 mm and is

10.0 cm long. A 10.0-m piece of 22-gauge copper wire (diameter = 0.644 mm) is wrapped around the rod in a single uniform layer, except for a 10.0-cm length at each end, which is to be used for connections. (a) How many turns of this wire can wrap around the rod? (*Hint:* The diameter of the wire adds to the diameter of the rod in determining the circumference of each turn. Also, the wire spirals diagonally along the surface of the rod.) (b) What is the resistance of this inductor? (c) What is its inductance?

67. A wire of nonmagnetic material with radius R carries current uniformly distributed over its cross-section. If the total current carried by the wire is I, show that the magnetic energy per unit length inside the wire is $\mu_0 I^2/16\pi$.

68. An 820-turn wire coil of resistance 24.0 Ω is placed around a 12 500-turn solenoid, 7.00 cm long, as shown in Figure P32.68. Both coil and solenoid have cross-sectional areas of 1.00×10^{-4} m^2. (a) How long does it take the solenoid current to reach 63.2 percent of its maximum value? Determine (b) the average back emf caused by the self-inductance of the solenoid during this interval, (c) the average rate of change in magnetic flux through the coil during this interval, and (d) the magnitude of the average induced current in the coil.

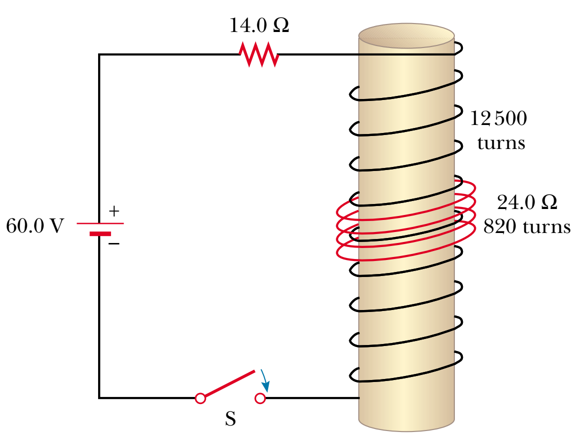


Figure P32.68

69. At $t = 0$, the switch in Figure P32.69 is thrown closed. Using Kirchhoff's laws for the instantaneous currents and voltages in this two-loop circuit, show that the current in the inductor is

$$I(t) = \frac{\mathcal{E}}{R_1}\left[1 - e^{-(R'/L)t}\right]$$

where $R' = R_1R_2/(R_1 + R_2)$.

70. In Figure P32.69, take $\mathcal{E} = 6.00$ V, $R_1 = 5.00\ \Omega$, and $R_2 = 1.00\ \Omega$. The inductor has negligible resistance. When the switch is thrown open after having been closed for a long time, the current in the inductor drops to 0.250 A in 0.150 s. What is the inductance of the inductor?

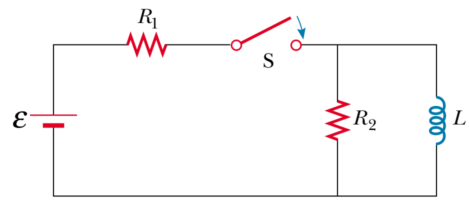


Figure P32.69 Problems 69 and 70.

71. In Figure P32.71, the switch is closed for $t < 0$, and steady-state conditions are established. The switch is thrown open at $t = 0$. (a) Find the initial voltage $\mathcal{E}_0$ across L just after $t = 0$. Which end of the coil is at the higher potential: a or b? (b) Make freehand graphs of the currents in R_1 and in R_2 as a function of time, treating the steady-state directions as positive. Show values before and after $t = 0$. (c) How long after $t = 0$ does the current in R_2 have the value 2.00 mA?

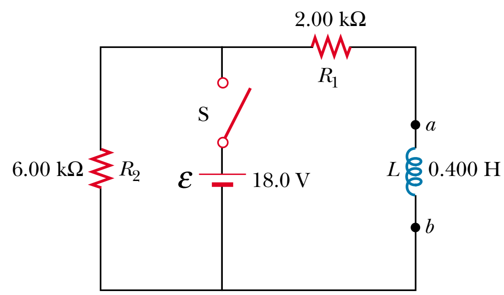


Figure P32.71

72. The switch in Figure P32.72 is thrown closed at $t = 0$. Before the switch is closed, the capacitor is uncharged, and all currents are zero. Determine the currents in L, C, and R and the potential differences across L, C, and R (a) the instant after the switch is closed and (b) long after it is closed.

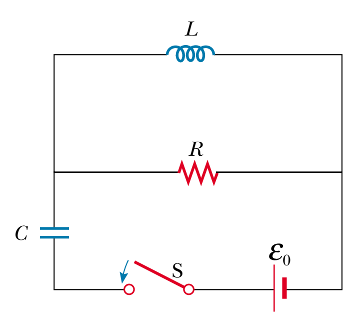


Figure P32.72

73. To prevent damage from arcing in an electric motor, a discharge resistor is sometimes placed in parallel with the armature. If the motor is suddenly unplugged while running, this resistor limits the voltage that appears across the armature coils. Consider a 12.0-V dc motor with an armature that has a resistance of 7.50 Ω and an inductance of 450 mH. Assume that the back emf in the armature coils is 10.0 V when the motor is running at normal speed. (The equivalent circuit for the armature is shown in Fig. P32.73.) Calculate the maximum resistance R that limits the voltage across the armature to 80.0 V when the motor is unplugged.

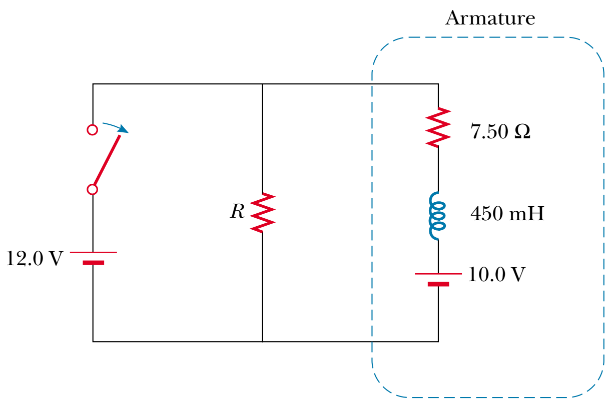


Figure P32.73

74. An air-core solenoid 0.500 m in length contains 1 000 turns and has a cross-sectional area of 1.00 cm^2. (a) If end effects are neglected, what is the self-inductance? (b) A secondary winding wrapped around the center of the solenoid has 100 turns. What is the mutual inductance? (c) The secondary winding carries a constant current of 1.00 A, and the solenoid is connected to a load of 1.00 kΩ. The constant current is suddenly stopped. How much charge flows through the load resistor?

75. The lead-in wires from a television antenna are often constructed in the form of two parallel wires (Fig. P32.75). (a) Why does this configuration of conductors have an inductance? (b) What constitutes the flux loop for this configuration? (c) Neglecting any magnetic flux inside the wires, show that the inductance of a length x of this type of lead-in is

$$L = \frac{\mu_0 x}{\pi} \ln\left(\frac{w - a}{a}\right)$$

where a is the radius of the wires and w is their center-to-center separation.

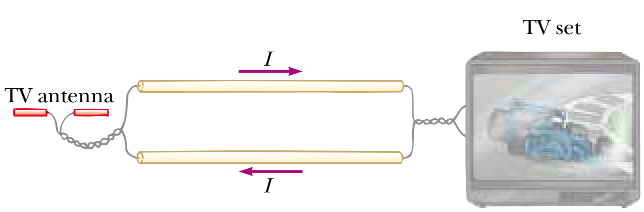


Figure P32.75

Note: Problems 76 through 79 require the application of ideas from this chapter and earlier chapters to some properties of superconductors, which were introduced in Section 27.5.

76. **Review Problem.** *The resistance of a superconductor.* In an experiment carried out by S. C. Collins between 1955 and 1958, a current was maintained in a superconducting lead ring for 2.50 yr with no observed loss. If the inductance of the ring was 3.14×10^{-8} H and the sensitivity of the experiment was 1 part in 10^9, what was the maximum resistance of the ring? (*Hint:* Treat this as a decaying current in an *RL* circuit, and recall that $e^{-x} \cong 1 - x$ for small x.)

77. **Review Problem.** A novel method of storing electrical energy has been proposed. A huge underground superconducting coil, 1.00 km in diameter, would be fabricated. It would carry a maximum current of 50.0 kA through each winding of a 150-turn Nb_3Sn solenoid. (a) If the inductance of this huge coil were 50.0 H, what would be the total energy stored? (b) What would be the compressive force per meter length acting between two adjacent windings 0.250 m apart?

78. **Review Problem.** *Superconducting Power Transmission.* The use of superconductors has been proposed for the manufacture of power transmission lines. A single coaxial cable (Fig. P32.78) could carry 1.00×10^3 MW (the output of a large power plant) at 200 kV, dc, over a distance of 1 000 km without loss. An inner wire with a radius of 2.00 cm, made from the superconductor Nb_3Sn, carries the current I in one direction. A surrounding superconducting cylinder, of radius 5.00 cm, would carry the return current I. In such a system, what is the magnetic field (a) at the surface of the inner conductor and (b) at the inner surface of the outer conductor? (c) How much energy would be stored in the space between the conductors in a 1 000-km superconducting line? (d) What is the pressure exerted on the outer conductor?

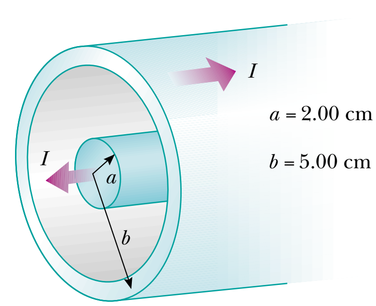


Figure P32.78

79. **Review Problem.** *The Meissner Effect.* Compare this problem with Problem 63 in Chapter 26 on the force attracting a perfect dielectric into a strong electric field. A fundamental property of a Type I superconducting material is *perfect diamagnetism,* or demonstration of the *Meissner effect,* illustrated in the photograph on page 855 and again in Figure 30.34, and described as follows: The superconducting material has $\mathbf{B} = 0$ everywhere inside it. If a sample of the material is placed into an externally produced magnetic field, or if it is cooled to become superconducting while it is in a magnetic field, electric currents appear on the surface of the sample. The currents have precisely the strength and orientation required to make the total magnetic field zero throughout the interior of the sample. The following problem will help you to understand the magnetic force that can then act on the superconducting sample.

Consider a vertical solenoid with a length of 120 cm and a diameter of 2.50 cm consisting of 1 400 turns of copper wire carrying a counterclockwise current of 2.00 A, as shown in Figure P32.79a. (a) Find the magnetic field in the vacuum inside the solenoid. (b) Find the energy density of the magnetic field, and note that the units J/m^3 of energy density are the same as the units $\text{N/m}^2 (= \text{Pa})$ of pressure. (c) A superconducting bar 2.20 cm in diameter is inserted partway into the solenoid. Its upper end is far outside the solenoid, where the magnetic field is small. The lower end of the bar is deep inside the solenoid. Identify the direction required for the current on the curved surface of the bar so that the total magnetic field is zero within the bar. The field created by the supercurrents is sketched in Figure P32.79b, and the total field is sketched in Figure P32.79c. (d) The field of the solenoid exerts a force on the current in the superconductor. Identify the direction of the force on the bar. (e) Calculate the magnitude of the force by multiplying the energy density of the solenoid field by the area of the bottom end of the superconducting bar.

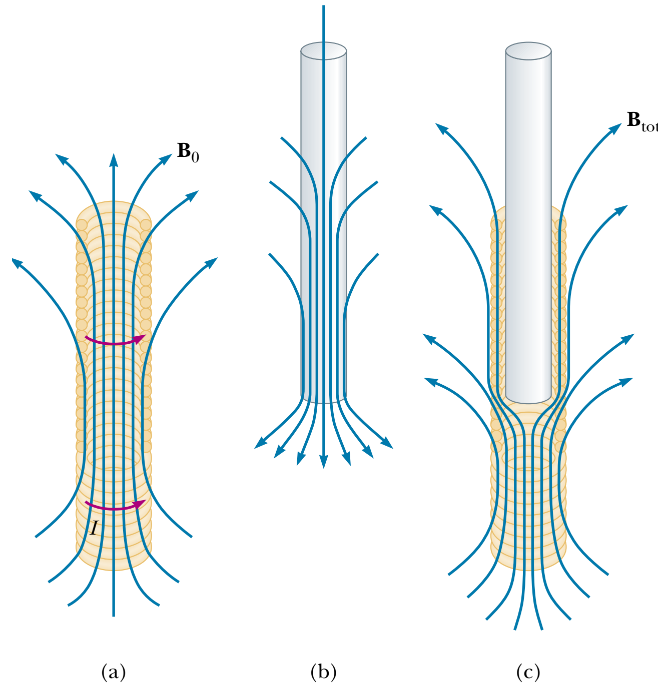


Figure P32.79

Answers to Quick Quizzes

32.1 When it is being opened. When the switch is initially open, there is no current in the circuit; when the switch is then closed, the inductor tends to maintain the no-current condition, and as a result there is very little chance of sparking. When the switch is initially closed, there is current in the circuit; when the switch is then opened, the current decreases. An induced emf is set up across the inductor, and this emf tends to maintain the original current. Sparking can occur as the current bridges the air gap between the poles of the switch.

32.2 (b). Figure 32.8 shows that circuit B has the greater time constant because in this circuit it takes longer for the current to reach its maximum value and then longer for this current to decrease to zero after switch S_2 is closed. Equation 32.8 indicates that, for equal resistances R_A and R_B, the condition $\tau_B > \tau_A$ means that $L_A < L_B$.

32.3 (a) M_{12} increases because the magnetic flux through coil 2 increases. (b) M_{12} decreases because rotation of coil 1 decreases its flux through coil 2.

32.4 (a) No. Mutual inductance requires a system of coils, and each coil has self-inductance. (b) Yes. A single coil has self-inductance but no mutual inductance because it does not interact with any other coils.

32.5 From Equation 32.25, $I_{\max} = \omega Q_{\max}$. Thus, the amplitude of the I-t graph is ω times the amplitude of the Q-t graph.

32.6 Equation 32.31 without the cosine factor. The dashed lines represent the positive and negative amplitudes (maximum values) for each oscillation period, and it is the $Q = Q_{\max} e^{-Rt/2L}$ part of Equation 32.31 that gives the value of the ever-decreasing amplitude.

PUZZLER

Small "black boxes" like this one are commonly used to supply power to electronic devices such as CD players and tape players. Whereas these devices need only about 12 V to operate, wall outlets provide an output of 120 V. What do the black boxes do, and how do they work? *(George Semple)*

chapter

33

Alternating-Current Circuits

Chapter Outline

In this chapter we describe alternating-current (ac) circuits. Every time we turn on a television set, a stereo, or any of a multitude of other electrical appliances, we are calling on alternating currents to provide the power to operate them. We begin our study by investigating the characteristics of simple series circuits that contain resistors, inductors, and capacitors and that are driven by a sinusoidal voltage. We shall find that the maximum alternating current in each element is proportional to the maximum alternating voltage across the element. We shall also find that when the applied voltage is sinusoidal, the current in each element is sinusoidal, too, but not necessarily in phase with the applied voltage. We conclude the chapter with two sections concerning transformers, power transmission, and *RC* filters.

33.1 AC SOURCES AND PHASORS

An ac circuit consists of circuit elements and a generator that provides the alternating current. As you recall from Section 31.5, the basic principle of the ac generator is a direct consequence of Faraday's law of induction. When a conducting loop is rotated in a magnetic field at constant angular frequency ω, a sinusoidal voltage (emf) is induced in the loop. This instantaneous voltage Δv is

$$\Delta v = \Delta V_{\text{max}} \sin \omega t$$

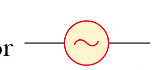
where ΔV_{max} is the maximum output voltage of the ac generator, or the **voltage amplitude.** From Equation 13.6, the angular frequency is

$$\omega = 2\pi f = \frac{2\pi}{T}$$

where f is the frequency of the generator (the voltage source) and T is the period. The generator determines the frequency of the current in any circuit connected to the generator. Because the output voltage of an ac generator varies sinusoidally with time, the voltage is positive during one half of the cycle and negative during the other half. Likewise, the current in any circuit driven by an ac generator is an alternating current that also varies sinusoidally with time. Commercial electric-power plants in the United States use a frequency of 60 Hz, which corresponds to an angular frequency of 377 rad/s.

The primary aim of this chapter can be summarized as follows: If an ac generator is connected to a series circuit containing resistors, inductors, and capacitors, we want to know the amplitude and time characteristics of the alternating current. To simplify our analysis of circuits containing two or more elements, we use graphical constructions called *phasor diagrams.* In these constructions, alternating (sinusoidal) quantities, such as current and voltage, are represented by rotating vectors called **phasors.** The length of the phasor represents the amplitude (maximum value) of the quantity, and the projection of the phasor onto the vertical axis represents the instantaneous value of the quantity. As we shall see, a phasor diagram greatly simplifies matters when we must combine several sinusoidally varying currents or voltages that have different phases.

33.2 RESISTORS IN AN AC CIRCUIT

Consider a simple ac circuit consisting of a resistor and an ac generator , as shown in Figure 33.1. At any instant, the algebraic sum of the voltages around a

closed loop in a circuit must be zero (Kirchhoff's loop rule). Therefore, $\Delta v - \Delta v_R = 0$, or[1]

$$\Delta v = \Delta v_R = \Delta V_{\max} \sin \omega t \tag{33.1}$$

where Δv_R is the **instantaneous voltage across the resistor.** Therefore, the instantaneous current in the resistor is

$$i_R = \frac{\Delta v_R}{R} = \frac{\Delta V_{\max}}{R} \sin \omega t = I_{\max} \sin \omega t \tag{33.2}$$

where $I_{\max}$ is the maximum current:

Maximum current in a resistor

$$I_{\max} = \frac{\Delta V_{\max}}{R}$$

From Equations 33.1 and 33.2, we see that the instantaneous voltage across the resistor is

$$\Delta v_R = I_{\max} R \sin \omega t \tag{33.3}$$

Let us discuss the current-versus-time curve shown in Figure 33.2a. At point *a*, the current has a maximum value in one direction, arbitrarily called the positive direction. Between points *a* and *b*, the current is decreasing in magnitude but is still in the positive direction. At *b*, the current is momentarily zero; it then begins to increase in the negative direction between points *b* and *c*. At *c*, the current has reached its maximum value in the negative direction.

The current in a resistor is in phase with the voltage

The current and voltage are in step with each other because they vary identically with time. Because i_R and Δv_R both vary as $\sin \omega t$ and reach their maximum values at the same time, as shown in Figure 33.2a, they are said to be **in phase.** Thus we can say that, for a sinusoidal applied voltage, the current in a resistor is always in phase with the voltage across the resistor.

A *phasor diagram* is used to represent current–voltage phase relationships. The lengths of the arrows correspond to $\Delta V_{\max}$ and $I_{\max}$. The projections of the phasor arrows onto the vertical axis give Δv_R and i_R values. As we showed in Section 13.5, if the phasor arrow is imagined to rotate steadily with angular speed ω, its vertical-axis component oscillates sinusoidally in time. In the case of the single-loop resistive circuit of Figure 33.1, the current and voltage phasors lie along the same line, as in Figure 33.2b, because i_R and Δv_R are in phase.

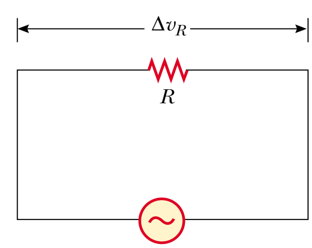


Figure 33.1 A circuit consisting of a resistor of resistance R connected to an ac generator, designated by the symbol

.

Note that **the average value of the current over one cycle is zero.** That is, the current is maintained in the positive direction for the same amount of time and at the same magnitude as it is maintained in the negative direction. However, the direction of the current has no effect on the behavior of the resistor. We can understand this by realizing that collisions between electrons and the fixed atoms of the resistor result in an increase in the temperature of the resistor. Although this temperature increase depends on the magnitude of the current, it is independent of the direction of the current.

We can make this discussion quantitative by recalling that the rate at which electrical energy is converted to internal energy in a resistor is the power $\mathcal{P} = i^2 R$, where i is the instantaneous current in the resistor. Because this rate is proportional to the square of the current, it makes no difference whether the current is direct or alternating—that is, whether the sign associated with the current is positive or negative. However, the temperature increase produced by an alternating

[1] The lowercase symbols v and i are used to indicate the instantaneous values of the voltage and the current.

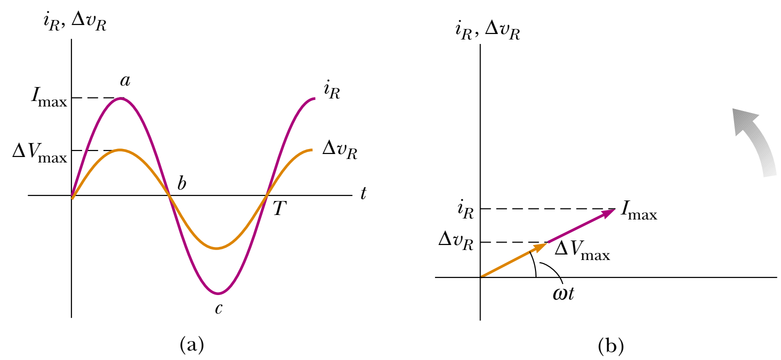


Figure 33.2 (a) Plots of the instantaneous current i_R and instantaneous voltage Δv_R across a resistor as functions of time. The current is in phase with the voltage, which means that the current is zero when the voltage is zero, maximum when the voltage is maximum, and minimum when the voltage is minimum. At time $t = T$, one cycle of the time-varying voltage and current has been completed. (b) Phasor diagram for the resistive circuit showing that the current is in phase with the voltage.

current having a maximum value I_{max} is not the same as that produced by a direct current equal to I_{max}. This is because the alternating current is at this maximum value for only an instant during each cycle (Fig. 33.3a). What is of importance in an ac circuit is an average value of current, referred to as the **rms current.** As we learned in Section 21.1, the notation *rms* stands for *root mean square,* which in this case means the square root of the mean (average) value of the square of the current: $I_{rms} = \sqrt{\overline{i^2}}$. Because i^2 varies as $\sin^2 \omega t$ and because the average value of i^2 is $\frac{1}{2}I_{max}^2$ (see Fig. 33.3b), the rms current is[2]

rms current

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} = 0.707 I_{max} \tag{33.4}$$

This equation states that an alternating current whose maximum value is 2.00 A delivers to a resistor the same power as a direct current that has a value of (0.707)(2.00 A) = 1.41 A. Thus, we can say that the average power delivered to a resistor that carries an alternating current is

Average power delivered to a resistor

$$\mathcal{P}_{av} = I_{rms}^2 R$$

[2] That the square root of the average value of i^2 is equal to $I_{max}/\sqrt{2}$ can be shown as follows: The current in the circuit varies with time according to the expression $i = I_{max} \sin \omega t$, so $i^2 = I_{max}^2 \sin^2 \omega t$. Therefore, we can find the average value of i^2 by calculating the average value of $\sin^2 \omega t$. A graph of $\cos^2 \omega t$ versus time is identical to a graph of $\sin^2 \omega t$ versus time, except that the points are shifted on the time axis. Thus, the time average of $\sin^2 \omega t$ is equal to the time average of $\cos^2 \omega t$ when taken over one or more complete cycles. That is,

$$(\sin^2 \omega t)_{av} = (\cos^2 \omega t)_{av}$$

Using this fact and the trigonometric identity $\sin^2 \theta + \cos^2 \theta = 1$, we obtain

$$(\sin^2 \omega t)_{av} + (\cos^2 \omega t)_{av} = 2(\sin^2 \omega t)_{av} = 1$$

$$(\sin^2 \omega t)_{av} = \tfrac{1}{2}$$

When we substitute this result in the expression $i^2 = I_{max}^2 \sin^2 \omega t$, we obtain $(i^2)_{av} = \overline{i^2} = I_{rms}^2 = I_{max}^2/2$, or $I_{rms} = I_{max}/\sqrt{2}$. The factor $1/\sqrt{2}$ is valid only for sinusoidally varying currents. Other waveforms, such as sawtooth variations, have different factors.

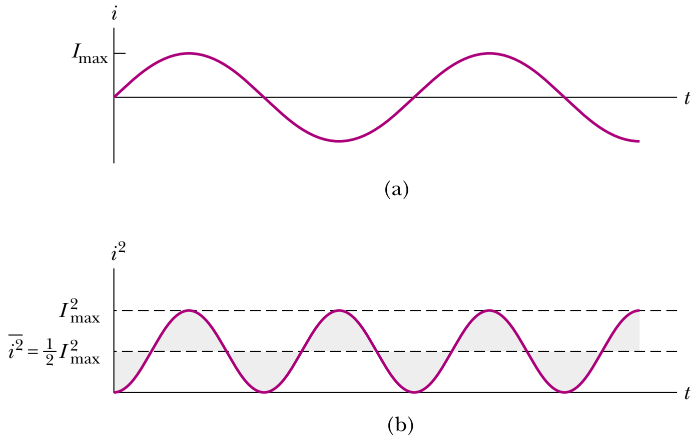


Figure 33.3 (a) Graph of the current in a resistor as a function of time. (b) Graph of the current squared in a resistor as a function of time. Notice that the gray shaded regions *under* the curve and *above* the dashed line for $I^2_{max}/2$ have the same area as the gray shaded regions *above* the curve and *below* the dashed line for $I^2_{max}/2$. Thus, the average value of i^2 is $I^2_{max}/2$.

Alternating voltage also is best discussed in terms of rms voltage, and the relationship is identical to that for current:

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = 0.707\,\Delta V_{max} \qquad \textbf{(33.5)}$$

rms voltage

When we speak of measuring a 120-V alternating voltage from an electrical outlet, we are referring to an rms voltage of 120 V. A quick calculation using Equation 33.5 shows that such an alternating voltage has a maximum value of about 170 V. One reason we use rms values when discussing alternating currents and voltages in this chapter is that ac ammeters and voltmeters are designed to read rms values. Furthermore, with rms values, many of the equations we use have the same form as their direct-current counterparts.

Quick Quiz 33.1

Which of the following statements might be true for a resistor connected to an ac generator? (a) $\mathcal{P}_{av} = 0$ and $i_{av} = 0$; (b) $\mathcal{P}_{av} = 0$ and $i_{av} > 0$; (c) $\mathcal{P}_{av} > 0$ and $i_{av} = 0$; (d) $\mathcal{P}_{av} > 0$ and $i_{av} > 0$.

EXAMPLE 33.1 What Is the rms Current?

The voltage output of a generator is given by $\Delta v = (200\text{ V})\sin \omega t$. Find the rms current in the circuit when this generator is connected to a 100-Ω resistor.

Solution Comparing this expression for voltage output with the general form $\Delta v = \Delta V_{max} \sin \omega t$, we see that $\Delta V_{max} = 200$ V. Thus, the rms voltage is

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = \frac{200\text{ V}}{\sqrt{2}} = 141\text{ V}$$

Therefore,

$$I_{rms} = \frac{\Delta V_{rms}}{R} = \frac{141\text{ V}}{100\ \Omega} = \boxed{1.41\text{ A}}$$

Exercise Find the maximum current in the circuit.

Answer 2.00 A.

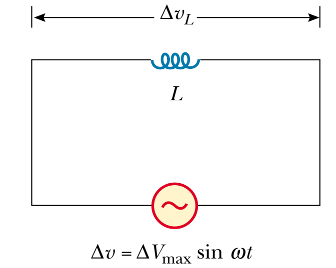


Figure 33.4 A circuit consisting of an inductor of inductance L connected to an ac generator.

33.3 INDUCTORS IN AN AC CIRCUIT

Now consider an ac circuit consisting only of an inductor connected to the terminals of an ac generator, as shown in Figure 33.4. If $\Delta v_L = \mathcal{E}_L = -L(di/dt)$ is the self-induced instantaneous voltage across the inductor (see Eq. 32.1), then Kirchhoff's loop rule applied to this circuit gives $\Delta v + \Delta v_L = 0$, or

$$\Delta v - L\frac{di}{dt} = 0$$

When we substitute $\Delta V_{\max} \sin \omega t$ for Δv and rearrange, we obtain

$$L\frac{di}{dt} = \Delta V_{\max} \sin \omega t \tag{33.6}$$

Solving this equation for di, we find that

$$di = \frac{\Delta V_{\max}}{L} \sin \omega t \, dt$$

Integrating this expression[3] gives the instantaneous current in the inductor as a function of time:

$$i_L = \frac{\Delta V_{\max}}{L} \int \sin \omega t \, dt = -\frac{\Delta V_{\max}}{\omega L} \cos \omega t \tag{33.7}$$

When we use the trigonometric identity $\cos \omega t = -\sin(\omega t - \pi/2)$, we can express Equation 33.7 as

$$i_L = \frac{\Delta V_{\max}}{\omega L} \sin\left(\omega t - \frac{\pi}{2}\right) \tag{33.8}$$

Comparing this result with Equation 33.6, we see that the instantaneous current i_L in the inductor and the instantaneous voltage Δv_L across the inductor are out of phase by $(\pi/2)$ rad $= 90°$.

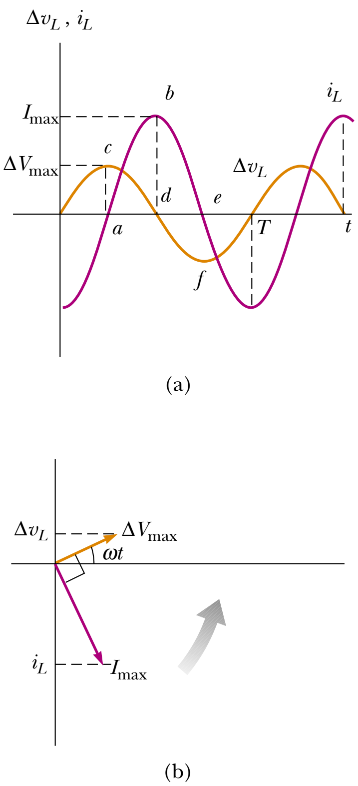


Figure 33.5 (a) Plots of the instantaneous current i_L and instantaneous voltage Δv_L across an inductor as functions of time. The current lags behind the voltage by 90°. (b) Phasor diagram for the inductive circuit, showing that the current lags behind the voltage by 90°.

In general, inductors in an ac circuit produce a current that is out of phase with the ac voltage. A plot of voltage and current versus time is provided in Figure 33.5a. At point *a*, the current begins to increase in the positive direction. At this instant the rate of change of current is at a maximum, and thus the voltage across the inductor is also at a maximum. As the current increases between points *a* and *b*, di/dt (the slope of the current curve) gradually decreases until it reaches zero at point *b*. As a result, the voltage across the inductor is decreasing during this same time interval, as the curve segment between *c* and *d* indicates. Immediately after point *b*, the current begins to decrease, although it still has the same direction it had during the previous quarter cycle (from *a* to *b*). As the current decreases to zero (from *b* to *e*), a voltage is again induced in the inductor (*d* to *f*), but the polarity of this voltage is opposite that of the voltage induced between *c* and *d* (because back emfs are always directed to oppose the change in the current). Note that the voltage reaches its maximum value one quarter of a period before the current reaches its maximum value. Thus, we see that

The current in an inductor lags the voltage by 90°

for a sinusoidal applied voltage, the current in an inductor always lags behind the voltage across the inductor by 90° (one-quarter cycle in time).

[3] We neglect the constant of integration here because it depends on the initial conditions, which are not important for this situation.

The phasor diagram for the inductive circuit of Figure 33.4 is shown in Figure 33.5b.

From Equation 33.7 we see that the current in an inductive circuit reaches its maximum value when $\cos \omega t = -1$:

$$I_{\max} = \frac{\Delta V_{\max}}{\omega L} = \frac{\Delta V_{\max}}{X_L} \tag{33.9}$$

Maximum current in an inductor

where the quantity X_L, called the **inductive reactance,** is

$$X_L = \omega L \tag{33.10}$$

Inductive reactance

Equation 33.9 indicates that, for a given applied voltage, the maximum current decreases as the inductive reactance increases. The expression for the rms current in an inductor is similar to Equation 33.9, with $I_{\max}$ replaced by I_{rms} and $\Delta V_{\max}$ replaced by ΔV_{rms}.

Inductive reactance, like resistance, has units of ohms. However, unlike resistance, reactance depends on frequency as well as on the characteristics of the inductor. Note that the reactance of an inductor in an ac circuit increases as the frequency of the current increases. This is because at higher frequencies, the instantaneous current must change more rapidly than it does at the lower frequencies; this causes an increase in the maximum induced emf associated with a given maximum current.

Using Equations 33.6 and 33.9, we find that the instantaneous voltage across the inductor is

$$\Delta v_L = -L\frac{di}{dt} = -\Delta V_{\max} \sin \omega t = -I_{\max} X_L \sin \omega t \tag{33.11}$$

Conceptual Example 33.2

Figure 33.6 shows a circuit consisting of a series combination of an alternating voltage source, a switch, an inductor, and a lightbulb. The switch is thrown closed, and the circuit is allowed to come to equilibrium so that the lightbulb glows steadily. An iron rod is then inserted into the interior of the inductor. What happens to the brightness of the lightbulb, and why?

Solution The bulb gets dimmer. As the rod is inserted, the inductance increases because the magnetic field inside the inductor increases. According to Equation 33.10, this increase in L means that the inductive reactance of the inductor also increases. The voltage across the inductor increases while the voltage across the lightbulb decreases. With less voltage across it, the lightbulb glows more dimly. In theatrical productions of the early 20th century, this method was used to dim the lights in the theater gradually.

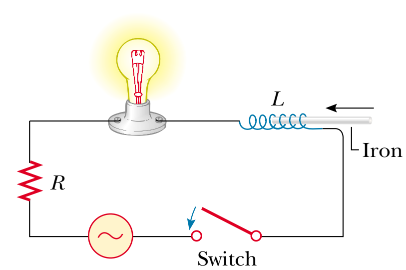


Figure 33.6

Example 33.3 A Purely Inductive ac Circuit

In a purely inductive ac circuit (see Fig. 33.4), $L = 25.0$ mH and the rms voltage is 150 V. Calculate the inductive reactance and rms current in the circuit if the frequency is 60.0 Hz.

Solution Equation 33.10 gives

$$X_L = \omega L = 2\pi f L = 2\pi(60.0\text{ Hz})(25.0 \times 10^{-3}\text{ H}) = \boxed{9.42\ \Omega}$$

From a modified version of Equation 33.9, the rms current is

$$I_{\text{rms}} = \frac{\Delta V_{L,\text{rms}}}{X_L} = \frac{150\text{ V}}{9.42\ \Omega} = \boxed{15.9\text{ A}}$$

Exercise Calculate the inductive reactance and rms current in the circuit if the frequency is 6.00 kHz.

Answer 942 Ω, 0.159 A.

Exercise Show that inductive reactance has SI units of ohms.

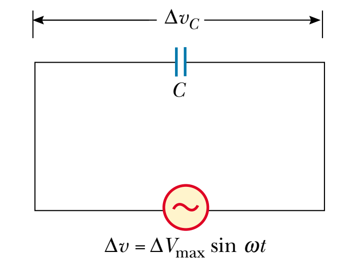


Figure 33.7 A circuit consisting of a capacitor of capacitance C connected to an ac generator.

33.4 CAPACITORS IN AN AC CIRCUIT

Figure 33.7 shows an ac circuit consisting of a capacitor connected across the terminals of an ac generator. Kirchhoff's loop rule applied to this circuit gives $\Delta v - \Delta v_C = 0$, or

$$\Delta v = \Delta v_C = \Delta V_{\max} \sin \omega t \qquad \textbf{(33.12)}$$

where Δv_C is the instantaneous voltage across the capacitor. We know from the definition of capacitance that $C = q/\Delta v_C$; hence, Equation 33.12 gives

$$q = C\,\Delta V_{\max} \sin \omega t \qquad \textbf{(33.13)}$$

where q is the instantaneous charge on the capacitor. Because $i = dq/dt$, differentiating Equation 33.13 gives the instantaneous current in the circuit:

$$i_C = \frac{dq}{dt} = \omega C\,\Delta V_{\max} \cos \omega t \qquad \textbf{(33.14)}$$

Using the trigonometric identity

$$\cos \omega t = \sin\left(\omega t + \frac{\pi}{2}\right)$$

we can express Equation 33.14 in the alternative form

$$i_C = \omega C\,\Delta V_{\max} \sin\left(\omega t + \frac{\pi}{2}\right) \qquad \textbf{(33.15)}$$

Comparing this expression with Equation 33.12, we see that the current is $\pi/2$ rad = 90° out of phase with the voltage across the capacitor. A plot of current and voltage versus time (Fig. 33.8a) shows that the current reaches its maximum value one quarter of a cycle sooner than the voltage reaches its maximum value.

Looking more closely, we see that the segment of the current curve from a to b indicates that the current starts out at a relatively high value. We can understand this by recognizing that there is no charge on the capacitor at $t = 0$; as a consequence, nothing in the circuit except the resistance of the wires can hinder the flow of charge at this instant. However, the current decreases as the voltage across the capacitor increases (from c to d on the voltage curve), and the capacitor is charging. When the voltage is at point d, the current reverses and begins to increase in the opposite direction (from b to e on the current curve). During this time, the voltage across the capacitor decreases from d to f because the plates are now losing the charge they accumulated earlier. During the second half of the cycle, the current is initially at its maximum value in the opposite direction (point e) and then decreases as the voltage across the capacitor builds up. The phasor diagram in Figure 33.8b also shows that

for a sinusoidally applied voltage, the current in a capacitor always leads the voltage across the capacitor by 90°.

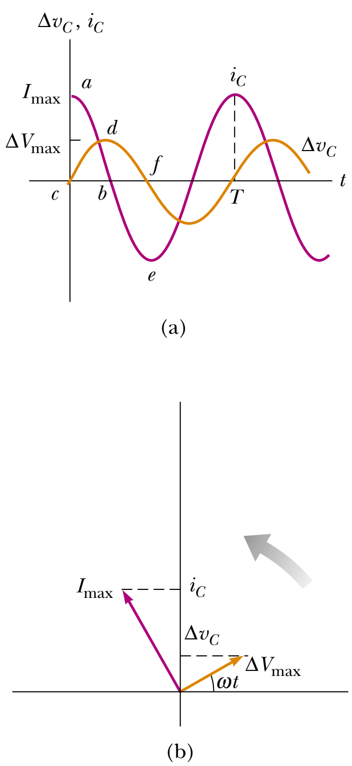


Figure 33.8 (a) Plots of the instantaneous current i_C and instantaneous voltage Δv_C across a capacitor as functions of time. The voltage lags behind the current by 90°. (b) Phasor diagram for the capacitive circuit, showing that the current leads the voltage by 90°.

From Equation 33.14, we see that the current in the circuit reaches its maximum value when $\cos \omega t = 1$:

$$I_{\max} = \omega C \Delta V_{\max} = \frac{\Delta V_{\max}}{X_C} \tag{33.16}$$

where X_C is called the **capacitive reactance:**

$$X_C = \frac{1}{\omega C} \tag{33.17}$$

Capacitive reactance

Note that capacitive reactance also has units of ohms.

The rms current is given by an expression similar to Equation 33.16, with $I_{\max}$ replaced by I_{rms} and $\Delta V_{\max}$ replaced by ΔV_{rms}.

Combining Equations 33.12 and 33.16, we can express the instantaneous voltage across the capacitor as

$$\Delta v_C = \Delta V_{\max} \sin \omega t = I_{\max} X_C \sin \omega t \tag{33.18}$$

Equations 33.16 and 33.17 indicate that as the frequency of the voltage source increases, the capacitive reactance decreases and therefore the maximum current increases. Again, note that the frequency of the current is determined by the frequency of the voltage source driving the circuit. As the frequency approaches zero, the capacitive reactance approaches infinity, and hence the current approaches zero. This makes sense because the circuit approaches direct-current conditions as ω approaches 0.

EXAMPLE 33.4 A Purely Capacitive ac Circuit

An 8.00-μF capacitor is connected to the terminals of a 60.0-Hz ac generator whose rms voltage is 150 V. Find the capacitive reactance and the rms current in the circuit.

Solution Using Equation 33.17 and the fact that $\omega = 2\pi f = 377\ \text{s}^{-1}$ gives

$$X_C = \frac{1}{\omega C} = \frac{1}{(377\ \text{s}^{-1})(8.00 \times 10^{-6}\ \text{F})} = \boxed{332\ \Omega}$$

Hence, from a modified Equation 33.16, the rms current is

$$I_{\text{rms}} = \frac{\Delta V_{\text{rms}}}{X_C} = \frac{150\ \text{V}}{332\ \Omega} = \boxed{0.452\ \text{A}}$$

Exercise If the frequency is doubled, what happens to the capacitive reactance and the current?

Answer X_C is halved, and $I_{\max}$ is doubled.

33.5 THE *RLC* SERIES CIRCUIT

13.7

Figure 33.9a shows a circuit that contains a resistor, an inductor, and a capacitor connected in series across an alternating-voltage source. As before, we assume that the applied voltage varies sinusoidally with time. It is convenient to assume that the instantaneous applied voltage is given by

$$\Delta v = \Delta V_{\max} \sin \omega t$$

while the current varies as

$$i = I_{\max} \sin(\omega t - \phi)$$

where ϕ is the **phase angle** between the current and the applied voltage. Our aim

Phase angle ϕ

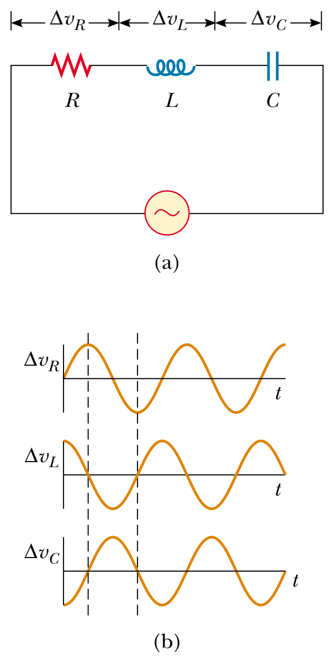


Figure 33.9 (a) A series circuit consisting of a resistor, an inductor, and a capacitor connected to an ac generator. (b) Phase relationships for instantaneous voltages in the series *RLC* circuit.

is to determine ϕ and I_{max}. Figure 33.9b shows the voltage versus time across each element in the circuit and their phase relationships.

To solve this problem, we must analyze the phasor diagram for this circuit. First, we note that because the elements are in series, the current everywhere in the circuit must be the same at any instant. That is, **the current at all points in a series ac circuit has the same amplitude and phase.** Therefore, as we found in the preceding sections, the voltage across each element has a different amplitude and phase, as summarized in Figure 33.10. In particular, the voltage across the resistor is in phase with the current, the voltage across the inductor leads the current by 90°, and the voltage across the capacitor lags behind the current by 90°. Using these phase relationships, we can express the instantaneous voltages across the three elements as

$$\Delta v_R = I_{max} R \sin \omega t = \Delta V_R \sin \omega t \qquad \textbf{(33.19)}$$

$$\Delta v_L = I_{max} X_L \sin\left(\omega t + \frac{\pi}{2}\right) = \Delta V_L \cos \omega t \qquad \textbf{(33.20)}$$

$$\Delta v_C = I_{max} X_C \sin\left(\omega t - \frac{\pi}{2}\right) = -\Delta V_C \cos \omega t \qquad \textbf{(33.21)}$$

where ΔV_R, ΔV_L, and ΔV_C are the maximum voltage values across the elements:

$$\Delta V_R = I_{max} R \qquad \Delta V_L = I_{max} X_L \qquad \Delta V_C = I_{max} X_C$$

At this point, we could proceed by noting that the instantaneous voltage Δv across the three elements equals the sum

$$\Delta v = \Delta v_R + \Delta v_L + \Delta v_C$$

Quick Quiz 33.2

For the circuit of Figure 33.9a, is the voltage of the ac source equal to (a) the sum of the maximum voltages across the elements, (b) the sum of the instantaneous voltages across the elements, or (c) the sum of the rms voltages across the elements?

Although this analytical approach is correct, it is simpler to obtain the sum by examining the phasor diagram. Because the current at any instant is the same in all

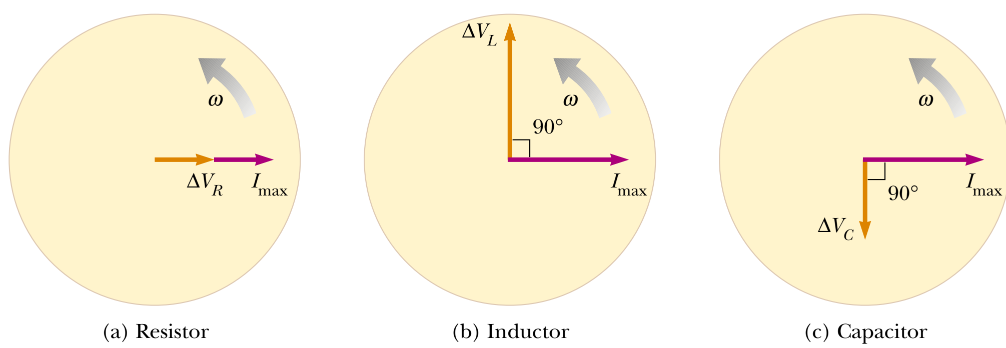


Figure 33.10 Phase relationships between the voltage and current phasors for (a) a resistor, (b) an inductor, and (c) a capacitor connected in series.

elements, we can obtain a phasor diagram for the circuit. We combine the three phasor pairs shown in Figure 33.10 to obtain Figure 33.11a, in which a single phasor I_{max} is used to represent the current in each element. To obtain the vector sum of the three voltage phasors in Figure 33.11a, we redraw the phasor diagram as in Figure 33.11b. From this diagram, we see that the vector sum of the voltage amplitudes ΔV_R, ΔV_L, and ΔV_C equals a phasor whose length is the maximum applied voltage ΔV_{max}, where the phasor ΔV_{max} makes an angle ϕ with the current phasor I_{max}. Note that the voltage phasors ΔV_L and ΔV_C are in opposite directions along the same line, and hence we can construct the difference phasor $\Delta V_L - \Delta V_C$, which is perpendicular to the phasor ΔV_R. From either one of the right triangles in Figure 33.11b, we see that

$$\Delta V_{max} = \sqrt{\Delta V_R^2 + (\Delta V_L - \Delta V_C)^2} = \sqrt{(I_{max}R)^2 + (I_{max}X_L - I_{max}X_C)^2}$$

$$\Delta V_{max} = I_{max}\sqrt{R^2 + (X_L - X_C)^2} \qquad \textbf{(33.22)}$$

Therefore, we can express the maximum current as

$$I_{max} = \frac{\Delta V_{max}}{\sqrt{R^2 + (X_L - X_C)^2}}$$

The **impedance** Z of the circuit is defined as

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2} \qquad \textbf{(33.23)}$$

where impedance also has units of ohms. Therefore, we can write Equation 33.22 in the form

$$\Delta V_{max} = I_{max}Z \qquad \textbf{(33.24)}$$

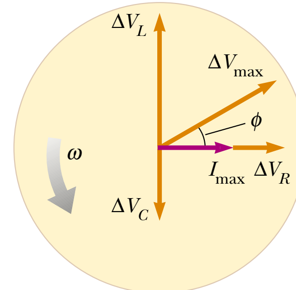
We can regard Equation 33.24 as the ac equivalent of Equation 27.8, which defined *resistance* in a dc circuit as the ratio of the voltage across a conductor to the current in that conductor. Note that the impedance and therefore the current in an ac circuit depend upon the resistance, the inductance, the capacitance, and the frequency (because the reactances are frequency-dependent).

By removing the common factor I_{max} from each phasor in Figure 33.11a, we can construct the *impedance triangle* shown in Figure 33.12. From this phasor diagram we find that the phase angle ϕ between the current and the voltage is

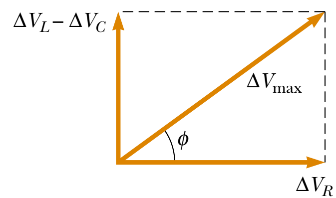
$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) \qquad \textbf{(33.25)}$$

Also, from Figure 33.12, we see that $\cos\phi = R/Z$. When $X_L > X_C$ (which occurs at high frequencies), the phase angle is positive, signifying that the current lags behind the applied voltage, as in Figure 33.11a. When $X_L < X_C$, the phase angle is negative, signifying that the current leads the applied voltage. When $X_L = X_C$, the phase angle is zero. In this case, the impedance equals the resistance and the current has its maximum value, given by $\Delta V_{max}/R$. The frequency at which this occurs is called the *resonance frequency;* it is described further in Section 33.7.

Table 33.1 gives impedance values and phase angles for various series circuits containing different combinations of elements.



(a)



(b)

Figure 33.11 (a) Phasor diagram for the series *RLC* circuit shown in Figure 33.9a. The phasor ΔV_R is in phase with the current phasor I_{max}, the phasor ΔV_L leads I_{max} by 90°, and the phasor ΔV_C lags I_{max} by 90°. The total voltage ΔV_{max} makes an angle ϕ with I_{max}. (b) Simplified version of the phasor diagram shown in (a).

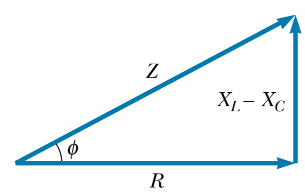
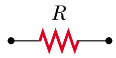
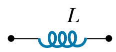
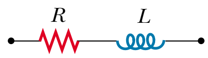
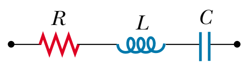


Figure 33.12 An impedance triangle for a series *RLC* circuit gives the relationship $Z = \sqrt{R^2 + (X_L - X_C)^2}$.

TABLE 33.1 Impedance Values and Phase Angles for Various Circuit-Element Combinations[a]

Circuit Elements	Impedance Z	Phase Angle ϕ
	R	$0°$
	X_C	$-90°$
	X_L	$+90°$
	$\sqrt{R^2 + X_C^2}$	Negative, between $-90°$ and $0°$
	$\sqrt{R^2 + X_L^2}$	Positive, between $0°$ and $90°$
	$\sqrt{R^2 + (X_L - X_C)^2}$	Negative if $X_C > X_L$ Positive if $X_C < X_L$

[a] In each case, an ac voltage (not shown) is applied across the elements.

Quick Quiz 33.3

Label each part of Figure 33.13 as being $X_L > X_C$, $X_L = X_C$, or $X_L < X_C$.

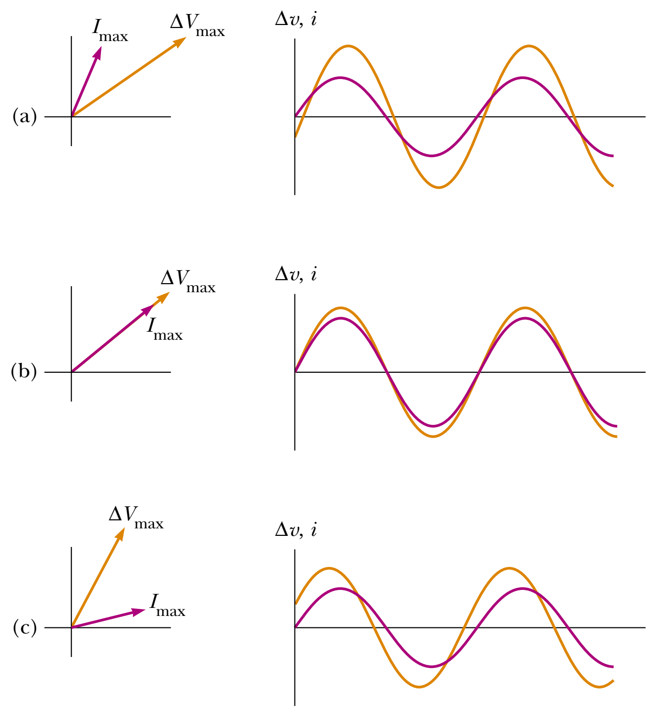


Figure 33.13

EXAMPLE 33.5 Finding *L* from a Phasor Diagram

In a series *RLC* circuit, the applied voltage has a maximum value of 120 V and oscillates at a frequency of 60.0 Hz. The circuit contains an inductor whose inductance can be varied, a 200-Ω resistor, and a 4.00-μF capacitor. What value of L

should an engineer analyzing the circuit choose such that the voltage across the capacitor lags the applied voltage by 30.0°?

Solution The phase relationships for the drops in voltage across the elements are shown in Figure 33.14. From the figure we see that the phase angle is $\phi = -60.0°$. This is because the phasors representing I_{max} and ΔV_R are in the same direction (they are in phase). From Equation 33.25, we find that

$$X_L = X_C + R \tan \phi$$

Substituting Equations 33.10 and 33.17 (with $\omega = 2\pi f$) into this expression gives

$$2\pi f L = \frac{1}{2\pi f C} + R \tan \phi$$

$$L = \frac{1}{2\pi f}\left[\frac{1}{2\pi f C} + R \tan \phi\right]$$

Substituting the given values into the equation gives $L =$ 0.84 H.

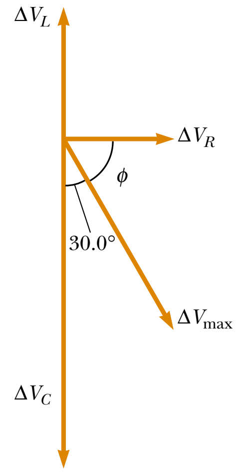


Figure 33.14

Example 33.6 Analyzing a Series *RLC* Circuit

A series *RLC* ac circuit has $R = 425\ \Omega$, $L = 1.25$ H, $C = 3.50\ \mu$F, $\omega = 377\ s^{-1}$, and $\Delta V_{max} = 150$ V. (a) Determine the inductive reactance, the capacitive reactance, and the impedance of the circuit.

Solution The reactances are $X_L = \omega L =$ 471 Ω and $X_C = 1/\omega C =$ 758 Ω. The impedance is

$$Z = \sqrt{R^2 + (X_L - X_C)^2}$$

$$= \sqrt{(425\ \Omega)^2 + (471\ \Omega - 758\ \Omega)^2} = 513\ \Omega$$

(b) Find the maximum current in the circuit.

Solution

$$I_{max} = \frac{V_{max}}{Z} = \frac{150\text{ V}}{513\ \Omega} = 0.292\text{ A}$$

(c) Find the phase angle between the current and voltage.

Solution

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) = \tan^{-1}\left(\frac{471\ \Omega - 758\ \Omega}{425\ \Omega}\right)$$

$$= -34.0°$$

Because the circuit is more capacitive than inductive, ϕ is negative and the current leads the applied voltage.

(d) Find both the maximum voltage and the instantaneous voltage across each element.

Solution The maximum voltages are

$$\Delta V_R = I_{max}R = (0.292\text{ A})(425\ \Omega) = 124\text{ V}$$

$$\Delta V_L = I_{max}X_L = (0.292\text{ A})(471\ \Omega) = 138\text{ V}$$

$$\Delta V_C = I_{max}X_C = (0.292\text{ A})(758\ \Omega) = 221\text{ V}$$

Using Equations 33.19, 33.20, and 33.21, we find that we can write the instantaneous voltages across the three elements as

$$\Delta v_R = (124\text{ V}) \sin 377t$$

$$\Delta v_L = (138\text{ V}) \cos 377t$$

$$\Delta v_C = (-221\text{ V}) \cos 377t$$

Comments The sum of the maximum voltages across the elements is $\Delta V_R + \Delta V_L + \Delta V_C = 483$ V. Note that this sum is much greater than the maximum voltage of the generator, 150 V. As we saw in Quick Quiz 33.2, the sum of the maximum voltages is a meaningless quantity because when sinusoidally varying quantities are added, *both their amplitudes and their phases* must be taken into account. We know that the

maximum voltages across the various elements occur at different times. That is, the voltages must be added in a way that takes account of the different phases. When this is done, Equation 33.22 is satisfied. You should verify this result.

Exercise Construct a phasor diagram to scale, showing the voltages across the elements and the applied voltage. From your diagram, verify that the phase angle is $-34.0°$.

33.6 POWER IN AN AC CIRCUIT

No power losses are associated with pure capacitors and pure inductors in an ac circuit. To see why this is true, let us first analyze the power in an ac circuit containing only a generator and a capacitor.

When the current begins to increase in one direction in an ac circuit, charge begins to accumulate on the capacitor, and a voltage drop appears across it. When this voltage drop reaches its maximum value, the energy stored in the capacitor is $\frac{1}{2}C(\Delta V_{\text{max}})^2$. However, this energy storage is only momentary. The capacitor is charged and discharged twice during each cycle: Charge is delivered to the capacitor during two quarters of the cycle and is returned to the voltage source during the remaining two quarters. Therefore, **the average power supplied by the source is zero.** In other words, **no power losses occur in a capacitor in an ac circuit.**

Similarly, the voltage source must do work against the back emf of the inductor. When the current reaches its maximum value, the energy stored in the inductor is a maximum and is given by $\frac{1}{2}LI_{\text{max}}^2$. When the current begins to decrease in the circuit, this stored energy is returned to the source as the inductor attempts to maintain the current in the circuit.

In Example 28.1 we found that the power delivered by a battery to a dc circuit is equal to the product of the current and the emf of the battery. Likewise, the instantaneous power delivered by an ac generator to a circuit is the product of the generator current and the applied voltage. For the *RLC* circuit shown in Figure 33.9a, we can express the instantaneous power $\mathcal{P}$ as

$$\begin{aligned}\mathcal{P} &= i\,\Delta v = I_{\text{max}}\sin(\omega t - \phi)\,\Delta V_{\text{max}}\sin\omega t \\ &= I_{\text{max}}\,\Delta V_{\text{max}}\sin\omega t\sin(\omega t - \phi)\end{aligned} \tag{33.26}$$

Clearly, this result is a complicated function of time and therefore is not very useful from a practical viewpoint. What is generally of interest is the average power over one or more cycles. Such an average can be computed by first using the trigonometric identity $\sin(\omega t - \phi) = \sin\omega t\cos\phi - \cos\omega t\sin\phi$. Substituting this into Equation 33.26 gives

$$\mathcal{P} = I_{\text{max}}\,\Delta V_{\text{max}}\sin^2\omega t\cos\phi - I_{\text{max}}\,\Delta V_{\text{max}}\sin\omega t\cos\omega t\sin\phi \tag{33.27}$$

We now take the time average of $\mathcal{P}$ over one or more cycles, noting that I_{max}, ΔV_{max}, ϕ, and ω are all constants. The time average of the first term on the right in Equation 33.27 involves the average value of $\sin^2\omega t$, which is $\frac{1}{2}$ (as shown in footnote 2). The time average of the second term on the right is identically zero because $\sin\omega t\cos\omega t = \frac{1}{2}\sin 2\omega t$, and the average value of $\sin 2\omega t$ is zero. Therefore, we can express the **average power** $\mathcal{P}_{\text{av}}$ as

$$\mathcal{P}_{\text{av}} = \tfrac{1}{2}I_{\text{max}}\,\Delta V_{\text{max}}\cos\phi \tag{33.28}$$

It is convenient to express the average power in terms of the rms current and rms voltage defined by Equations 33.4 and 33.5:

$$\boxed{\mathcal{P}_{\text{av}} = I_{\text{rms}}\,\Delta V_{\text{rms}}\cos\phi} \tag{33.29}$$

where the quantity $\cos\phi$ is called the **power factor.** By inspecting Figure 33.11b, we see that the maximum voltage drop across the resistor is given by $\Delta V_R = \Delta V_{max}\cos\phi = I_{max}R$. Using Equation 33.5 and the fact that $\cos\phi = I_{max}R/\Delta V_{max}$, we find that we can express $\mathcal{P}_{av}$ as

$$\mathcal{P}_{av} = I_{rms}\,\Delta V_{rms}\cos\phi = I_{rms}\left(\frac{\Delta V_{max}}{\sqrt{2}}\right)\frac{I_{max}R}{\Delta V_{max}} = I_{rms}\,\frac{I_{max}R}{\sqrt{2}}$$

After making the substitution $I_{max} = \sqrt{2}I_{rms}$ from Equation 33.4, we have

$$\boxed{\mathcal{P}_{av} = I_{rms}^2 R} \qquad \textbf{(33.30)}$$

Average power delivered to an *RLC* circuit

In words, the **average power delivered by the generator is converted to internal energy in the resistor,** just as in the case of a dc circuit. **No power loss occurs in an ideal inductor or capacitor.** When the load is purely resistive, then $\phi = 0$, $\cos\phi = 1$, and from Equation 33.29 we see that

$$\mathcal{P}_{av} = I_{rms}\,\Delta V_{rms}$$

Equation 33.29 shows that the power delivered by an ac source to any circuit depends on the phase, and this result has many interesting applications. For example, a factory that uses large motors in machines, generators, or transformers has a large inductive load (because of all the windings). To deliver greater power to such devices in the factory without using excessively high voltages, technicians introduce capacitance in the circuits to shift the phase.

EXAMPLE 33.7 Average Power in an *RLC* Series Circuit

Calculate the average power delivered to the series *RLC* circuit described in Example 33.6.

Solution First, let us calculate the rms voltage and rms current, using the values of ΔV_{max} and I_{max} from Example 33.6:

$$\Delta V_{rms} = \frac{\Delta V_{max}}{\sqrt{2}} = \frac{150\text{ V}}{\sqrt{2}} = 106\text{ V}$$

$$I_{rms} = \frac{I_{max}}{\sqrt{2}} = \frac{0.292\text{ A}}{\sqrt{2}} = 0.206\text{ A}$$

Because $\phi = -34.0°$, the power factor, $\cos\phi$, is 0.829; hence, the average power delivered is

$$\mathcal{P}_{av} = I_{rms}\,\Delta V_{rms}\cos\phi = (0.206\text{ A})(106\text{ V})(0.829)$$

$$= \boxed{18.1\text{ W}}$$

We can obtain the same result using Equation 33.30.

33.7 RESONANCE IN A SERIES *RLC* CIRCUIT

13.7 A series *RLC* circuit is said to be **in resonance** when the current has its maximum value. In general, the rms current can be written

$$I_{rms} = \frac{\Delta V_{rms}}{Z} \qquad \textbf{(33.31)}$$

where Z is the impedance. Substituting the expression for Z from Equation 33.23 into 33.31 gives

$$\boxed{I_{rms} = \frac{\Delta V_{rms}}{\sqrt{R^2 + (X_L - X_C)^2}}} \qquad \textbf{(33.32)}$$

Because the impedance depends on the frequency of the source, the current in the RLC circuit also depends on the frequency. The frequency ω_0 at which $X_L - X_C = 0$ is called the **resonance frequency** of the circuit. To find ω_0, we use the condition $X_L = X_C$, from which we obtain $\omega_0 L = 1/\omega_0 C$, or

Resonance frequency

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{33.33}$$

Note that this frequency also corresponds to the natural frequency of oscillation of an LC circuit (see Section 32.5). Therefore, the current in a series RLC circuit reaches its maximum value when the frequency of the applied voltage matches the natural oscillator frequency—which depends only on L and C. Furthermore, at this frequency the current is in phase with the applied voltage.

Quick Quiz 33.4

What is the impedance of a series RLC circuit at resonance? What is the current in the circuit at resonance?

A plot of rms current versus frequency for a series RLC circuit is shown in Figure 33.15a. The data assume a constant $\Delta V_{\text{rms}} = 5.0$ mV, that $L = 5.0\ \mu$H, and that $C = 2.0$ nF. The three curves correspond to three values of R. Note that in each case the current reaches its maximum value at the resonance frequency ω_0. Furthermore, the curves become narrower and taller as the resistance decreases.

By inspecting Equation 33.32, we must conclude that, when $R = 0$, the current becomes infinite at resonance. Although the equation predicts this, real circuits always have some resistance, which limits the value of the current.

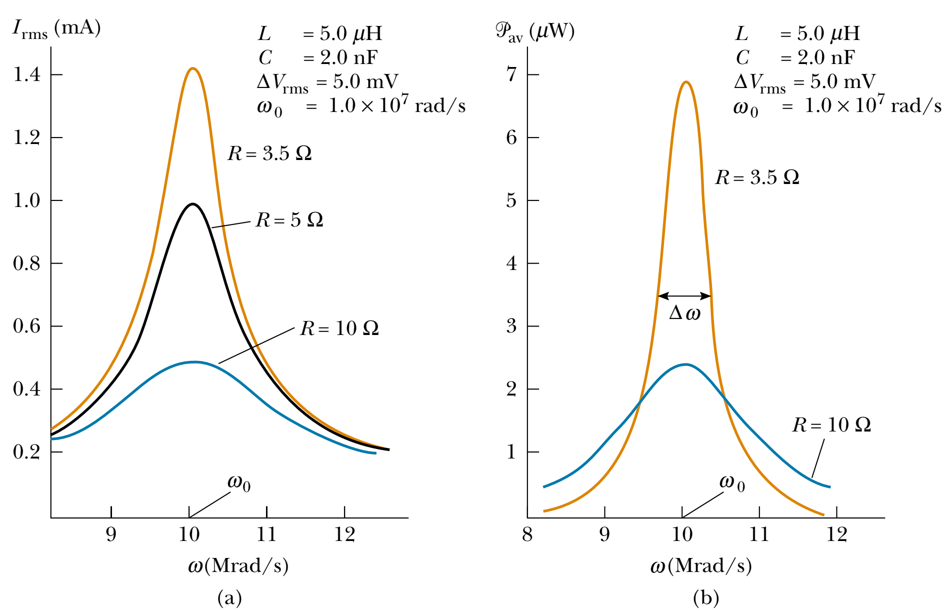


Figure 33.15 (a) The rms current versus frequency for a series RLC circuit, for three values of R. The current reaches its maximum value at the resonance frequency ω_0. (b) Average power versus frequency for the series RLC circuit, for two values of R.

It is also interesting to calculate the average power as a function of frequency for a series *RLC* circuit. Using Equations 33.30, 33.31, and 33.23, we find that

$$\mathcal{P}_{av} = I_{rms}^2 R = \frac{(\Delta V_{rms})^2}{Z^2} R = \frac{(\Delta V_{rms})^2 R}{R^2 + (X_L - X_C)^2} \quad \textbf{(33.34)}$$

Because $X_L = \omega L$, $X_C = 1/\omega C$, and $\omega_0{}^2 = 1/LC$, we can express the term $(X_L - X_C)^2$ as

$$(X_L - X_C)^2 = \left(\omega L - \frac{1}{\omega C}\right)^2 = \frac{L^2}{\omega^2}(\omega^2 - \omega_0{}^2)^2$$

Using this result in Equation 33.34 gives

$$\mathcal{P}_{av} = \frac{(\Delta V_{rms})^2 R\omega^2}{R^2\omega^2 + L^2(\omega^2 - \omega_0{}^2)^2} \quad \textbf{(33.35)}$$

Average power as a function of frequency in an *RLC* circuit

This expression shows that at resonance, when $\omega = \omega_0$, **the average power is a maximum** and has the value $(\Delta V_{rms})^2/R$. Figure 33.15b is a plot of average power versus frequency for two values of R in a series *RLC* circuit. As the resistance is made smaller, the curve becomes sharper in the vicinity of the resonance frequency. This curve sharpness is usually described by a dimensionless parameter known as the **quality factor,** denoted by Q:[4]

$$Q = \frac{\omega_0}{\Delta\omega}$$

Quality factor

where $\Delta\omega$ is the width of the curve measured between the two values of ω for which $\mathcal{P}_{av}$ has half its maximum value, called the *half-power points* (see Fig. 33.15b.) It is left as a problem (Problem 70) to show that the width at the half-power points has the value $\Delta\omega = R/L$, so

$$Q = \frac{\omega_0 L}{R} \quad \textbf{(33.36)}$$

The curves plotted in Figure 33.16 show that a high-Q circuit responds to only a very narrow range of frequencies, whereas a low-Q circuit can detect a much broader range of frequencies. Typical values of Q in electronic circuits range from 10 to 100.

The receiving circuit of a radio is an important application of a resonant circuit. One tunes the radio to a particular station (which transmits a specific electromagnetic wave or signal) by varying a capacitor, which changes the resonant frequency of the receiving circuit. When the resonance frequency of the circuit matches that of the incoming electromagnetic wave, the current in the receiving circuit increases. This signal caused by the incoming wave is then amplified and fed to a speaker. Because many signals are often present over a range of frequencies, it is important to design a high-Q circuit to eliminate unwanted signals. In this manner, stations whose frequencies are near but not equal to the resonance frequency give signals at the receiver that are negligibly small relative to the signal that matches the resonance frequency.

QuickLab

Tune a radio to your favorite station. Can you determine what the product of *LC* must be for the radio's tuning circuitry?

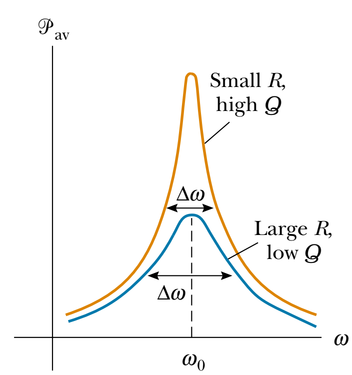


Figure 33.16 Average power versus frequency for a series *RLC* circuit. The width $\Delta\omega$ of each curve is measured between the two points where the power is half its maximum value. The power is a maximum at the resonance frequency ω_0.

[4] The quality factor is also defined as the ratio $2\pi E/\Delta E$, where E is the energy stored in the oscillating system and ΔE is the energy lost per cycle of oscillation. The quality factor for a mechanical system can also be defined, as noted in Section 13.7.

Quick Quiz 33.5

An airport metal detector (Fig. 33.17) is essentially a resonant circuit. The portal you step through is an inductor (a large loop of conducting wire) that is part of the circuit. The frequency of the circuit is tuned to the resonant frequency of the circuit when there is no metal in the inductor. Any metal on your body increases the effective inductance of the loop and changes the current in it. If you want the detector to be able to detect a small metallic object, should the circuit have a high quality factor or a low one?

Figure 33.17 When you pass through a metal detector, you become part of a resonant circuit. As you step through the detector, the inductance of the circuit changes, and thus the current in the circuit changes. *(Terry Qing/FPG International)*

Example 33.8 A Resonating Series *RLC* Circuit

Consider a series *RLC* circuit for which $R = 150\ \Omega$, $L = 20.0$ mH, $\Delta V_{\text{rms}} = 20.0$ V, and $\omega = 5\,000\ \text{s}^{-1}$. Determine the value of the capacitance for which the current is a maximum.

Solution The current has its maximum value at the resonance frequency ω_0, which should be made to match the "driving" frequency of $5\,000\ \text{s}^{-1}$:

$$\omega_0 = 5.00 \times 10^3\ \text{s}^{-1} = \frac{1}{\sqrt{LC}}$$

$$C = \frac{1}{\omega_0{}^2 L} = \frac{1}{(25.0 \times 10^6\ \text{s}^{-2})(20.0 \times 10^{-3}\ \text{H})}$$

$$= \boxed{2.00\ \mu\text{F}}$$

Exercise Calculate the maximum value of the rms current in the circuit as the frequency is varied.

Answer 0.133 A.

33.8 THE TRANSFORMER AND POWER TRANSMISSION

When electric power is transmitted over great distances, it is economical to use a high voltage and a low current to minimize the I^2R loss in the transmission lines.

Consequently, 350-kV lines are common, and in many areas even higher-voltage (765-kV) lines are under construction. At the receiving end of such lines, the consumer requires power at a low voltage (for safety and for efficiency in design). Therefore, a device is required that can change the alternating voltage and current without causing appreciable changes in the power delivered. The ac transformer is that device.

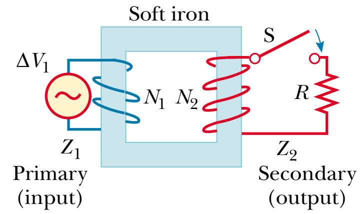


Figure 33.18 An ideal transformer consists of two coils wound on the same iron core. An alternating voltage ΔV_1 is applied to the primary coil, and the output voltage ΔV_2 is across the resistor of resistance R.

In its simplest form, the **ac transformer** consists of two coils of wire wound around a core of iron, as illustrated in Figure 33.18. The coil on the left, which is connected to the input alternating voltage source and has N_1 turns, is called the *primary winding* (or the *primary*). The coil on the right, consisting of N_2 turns and connected to a load resistor R, is called the *secondary winding* (or the *secondary*). The purpose of the iron core is to increase the magnetic flux through the coil and to provide a medium in which nearly all the flux through one coil passes through the other coil. Eddy current losses are reduced by using a laminated core. Iron is used as the core material because it is a soft ferromagnetic substance and hence reduces hysteresis losses. Transformation of energy to internal energy in the finite resistance of the coil wires is usually quite small. Typical transformers have power efficiencies from 90% to 99%. In the discussion that follows, we assume an *ideal transformer,* one in which the energy losses in the windings and core are zero.

First, let us consider what happens in the primary circuit when the switch in the secondary circuit is open. If we assume that the resistance of the primary is negligible relative to its inductive reactance, then the primary circuit is equivalent to a simple circuit consisting of an inductor connected to an ac generator. Because the current is 90° out of phase with the voltage, the power factor $\cos\phi$ is zero, and hence the average power delivered from the generator to the primary circuit is zero. Faraday's law states that the voltage ΔV_1 across the primary is

$$\Delta V_1 = -N_1 \frac{d\Phi_B}{dt} \tag{33.37}$$

where Φ_B is the magnetic flux through each turn. If we assume that all magnetic field lines remain within the iron core, the flux through each turn of the primary equals the flux through each turn of the secondary. Hence, the voltage across the secondary is

$$\Delta V_2 = -N_2 \frac{d\Phi_B}{dt} \tag{33.38}$$

Solving Equation 33.37 for $d\Phi_B/dt$ and substituting the result into Equation 33.38, we find that

$$\Delta V_2 = \frac{N_2}{N_1} \Delta V_1 \tag{33.39}$$

When $N_2 > N_1$, the output voltage ΔV_2 exceeds the input voltage ΔV_1. This setup is referred to as a *step-up transformer.* When $N_2 < N_1$, the output voltage is less than the input voltage, and we have a *step-down transformer.*

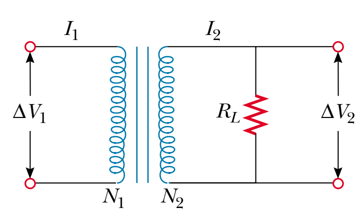


Figure 33.19 Circuit diagram for a transformer.

When the switch in the secondary circuit is thrown closed, a current I_2 is induced in the secondary. If the load in the secondary circuit is a pure resistance, the induced current is in phase with the induced voltage. The power supplied to the secondary circuit must be provided by the ac generator connected to the primary circuit, as shown in Figure 33.19. In an ideal transformer, where there are no losses, the power $I_1\,\Delta V_1$ supplied by the generator is equal to the power $I_2\,\Delta V_2$ in

This cylindrical step-down transformer drops the voltage from 4 000 V to 220 V for delivery to a group of residences. *(George Semple)*

the secondary circuit. That is,

$$I_1\,\Delta V_1 = I_2\,\Delta V_2 \qquad \textbf{(33.40)}$$

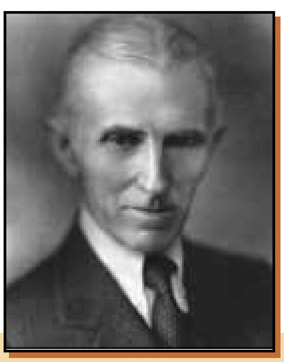
The value of the load resistance R_L determines the value of the secondary current because $I_2 = \Delta V_2/R_L$. Furthermore, the current in the primary is $I_1 = \Delta V_1/R_{\text{eq}}$, where

$$R_{\text{eq}} = \left(\frac{N_1}{N_2}\right)^2 R_L \qquad \textbf{(33.41)}$$

is the equivalent resistance of the load resistance when viewed from the primary side. From this analysis we see that a transformer may be used to match resistances between the primary circuit and the load. In this manner, maximum power transfer can be achieved between a given power source and the load resistance. For example, a transformer connected between the 1-kΩ output of an audio amplifier and an 8-Ω speaker ensures that as much of the audio signal as possible is transferred into the speaker. In stereo terminology, this is called *impedance matching*.

We can now also understand why transformers are useful for transmitting power over long distances. Because the generator voltage is stepped up, the current in the transmission line is reduced, and hence I^2R losses are reduced. In practice, the voltage is stepped up to around 230 000 V at the generating station, stepped down to around 20 000 V at a distributing station, then to 4 000 V for delivery to residential areas, and finally to 120–240 V at the customer's site. The power is supplied by a three-wire cable. In the United States, two of these wires are "hot," with voltages of 120 V with respect to a common ground wire. Home appliances operating on 120 V are connected in parallel between one of the hot wires and ground. Larger appliances, such as electric stoves and clothes dryers, require 240 V. This is obtained across the two hot wires, which are 180° out of phase so that the voltage difference between them is 240 V.

There is a practical upper limit to the voltages that can be used in transmission lines. Excessive voltages could ionize the air surrounding the transmission lines, which could result in a conducting path to ground or to other objects in the vicinity. This, of course, would present a serious hazard to any living creatures. For this reason, a long string of insulators is used to keep high-voltage wires away from their supporting metal towers. Other insulators are used to maintain separation between wires.



Nikola Tesla (1856–1943) Tesla was born in Croatia but spent most of his professional life as an inventor in the United States. He was a key figure in the development of alternating-current electricity, high-voltage transformers, and the transport of electric power via ac transmission lines. Tesla's viewpoint was at odds with the ideas of Thomas Edison, who committed himself to the use of direct current in power transmission. Tesla's ac approach won out. *(UPI/Bettmann)*

Figure 33.20 The primary winding in this transformer is directly attached to the prongs of the plug. The secondary winding is connected to the wire on the right, which runs to an electronic device. Many of these power-supply transformers also convert alternating current to direct current. *(George Semple)*

Many common household electronic devices require low voltages to operate properly. A small transformer that plugs directly into the wall, like the one illustrated in the photograph at the beginning of this chapter, can provide the proper voltage. Figure 33.20 shows the two windings wrapped around a common iron core that is found inside all these little "black boxes." This particular transformer converts the 120-V ac in the wall socket to 12.5-V ac. (Can you determine the ratio of the numbers of turns in the two coils?) Some black boxes also make use of diodes to convert the alternating current to direct current (see Section 33.9).

web

For information on how small transformers and hundreds of other everyday devices operate, visit **http://www.howstuffworks.com**

EXAMPLE 33.9 The Economics of ac Power

An electricity-generating station needs to deliver 20 MW of power to a city 1.0 km away. (a) If the resistance of the wires is 2.0 Ω and the electricity costs about 10¢/kWh, estimate what it costs the utility company to send the power to the city for one day. A common voltage for commercial power generators is 22 kV, but a step-up transformer is used to boost the voltage to 230 kV before transmission.

Solution The power losses in the transmission line are the result of the resistance of the line. We can determine the loss from Equation 27.23, $\mathcal{P} = I^2R$. Because this is an estimate, we can use dc equations and calculate I from Equation 27.22:

$$I = \frac{\mathcal{P}}{\Delta V} = \frac{20 \times 10^6 \text{ W}}{230 \times 10^3 \text{ V}} = 87 \text{ A}$$

Therefore,

$$\mathcal{P} = I^2R = (87 \text{ A})^2(2.0 \ \Omega) = 15 \text{ kW}$$

Over the course of a day, the energy loss due to the resistance of the wires is $(15 \text{ kW})(24 \text{ h}) = 360 \text{ kWh}$, at a cost of $\boxed{\$36.}$

(b) Repeat the calculation for the situation in which the power plant delivers the electricity at its original voltage of 22 kV.

Solution

$$I = \frac{\mathcal{P}}{\Delta V} = \frac{20 \times 10^6 \text{ W}}{22 \times 10^3 \text{ V}} = 910 \text{ A}$$

$$\mathcal{P} = I^2R = (910 \text{ A})^2(2.0 \ \Omega) = 1.7 \times 10^3 \text{ kW}$$

$$\text{Cost per day} = (1.7 \times 10^3 \text{ kW})(24 \text{ h})(\$0.10/\text{kWh})$$

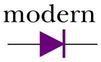
$$= \boxed{\$4\,100}$$

The tremendous savings that are possible through the use of transformers and high-voltage transmission lines, along with the efficiency of using alternating current to operate motors, led to the universal adoption of alternating current instead of direct current for commercial power grids.

Optional Section

33.9 RECTIFIERS AND FILTERS

Portable electronic devices such as radios and compact disc (CD) players are often powered by direct current supplied by batteries. Many devices come with ac–dc converters that provide a readily available direct-current source if the batteries are low. Such a converter contains a transformer that steps the voltage down from 120 V to typically 9 V and a circuit that converts alternating current to direct current. The process of converting alternating current to direct current is called **rectification,** and the converting device is called a **rectifier.**

The most important element in a rectifier circuit is a **diode,** a circuit element that conducts current in one direction but not the other. Most diodes used in modern electronics are semiconductor devices. The circuit symbol for a diode is , where the arrow indicates the direction of the current through the diode. A diode has low resistance to current in one direction (the direction of the arrow) and high resistance to current in the opposite direction. We can understand how a diode rectifies a current by considering Figure 33.21a, which shows a

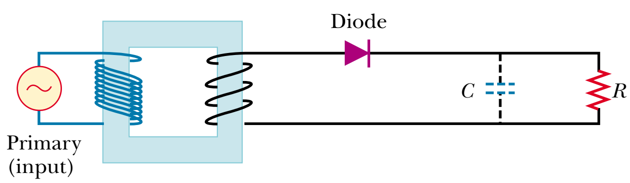
diode and a resistor connected to the secondary of a transformer. The transformer reduces the voltage from 120-V ac to the lower voltage that is needed for the device having a resistance R (the load resistance). Because current can pass through the diode in only one direction, the alternating current in the load resistor is reduced to the form shown by the solid curve in Figure 33.21b. The diode conducts current only when the side of the symbol containing the arrowhead has a positive potential relative to the other side. In this situation, the diode acts as a *half-wave rectifier* because current is present in the circuit during only half of each cycle.

When a capacitor is added to the circuit, as shown by the dashed lines and the capacitor symbol in Figure 33.21a, the circuit is a simple dc power supply. The time variation in the current in the load resistor (the dashed curve in Fig. 33.21b) is close to being zero, as determined by the RC time constant of the circuit.

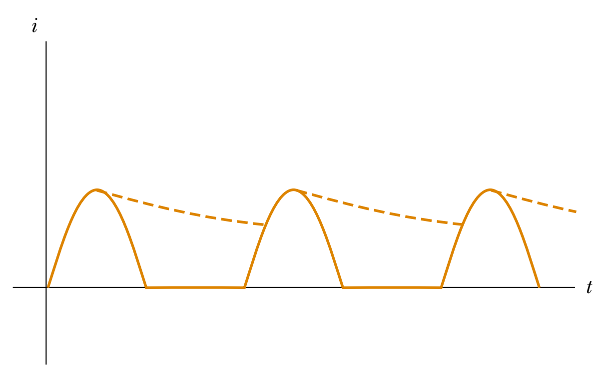
The RC circuit in Figure 33.21a is one example of a **filter circuit,** which is used to smooth out or eliminate a time-varying signal. For example, radios are usually powered by a 60-Hz alternating voltage. After rectification, the voltage still contains a small ac component at 60 Hz (sometimes called *ripple*), which must be filtered. By "filtered," we mean that the 60-Hz ripple must be reduced to a value much less than that of the audio signal to be amplified, because without filtering, the resulting audio signal includes an annoying hum at 60 Hz.

To understand how a filter works, let us consider the simple series RC circuit shown in Figure 33.22a. The input voltage is across the two elements and is represented by $\Delta V_{\text{max}} \sin \omega t$. Because we are interested only in maximum values, we can use Equation 33.24, taking $X_L = 0$ and substituting $X_C = 1/\omega C$. This shows that the maximum input voltage is related to the maximum current by

$$\Delta V_{\text{in}} = I_{\text{max}} Z = I_{\text{max}} \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$



(a)



(b)

Figure 33.21 (a) A half-wave rectifier with an optional filter capacitor. (b) Current versus time in the resistor. The solid curve represents the current with no filter capacitor, and the dashed curve is the current when the circuit includes the capacitor.

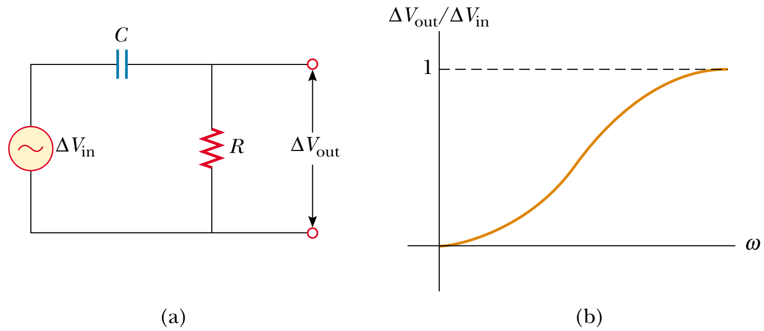


Figure 33.22 (a) A simple *RC* high-pass filter. (b) Ratio of output voltage to input voltage for an *RC* high-pass filter as a function of the angular frequency of the circuit.

If the voltage across the resistor is considered to be the output voltage, then the maximum output voltage is

$$\Delta V_{\text{out}} = I_{\max} R$$

Therefore, the ratio of the output voltage to the input voltage is

$$\frac{\Delta V_{\text{out}}}{\Delta V_{\text{in}}} = \frac{R}{\sqrt{R^2 + \left(\dfrac{1}{\omega C}\right)^2}} \tag{33.42}$$

High-pass filter

A plot of this ratio as a function of angular frequency (see Fig. 33.22b) shows that at low frequencies ΔV_{out} is much smaller than ΔV_{in}, whereas at high frequencies the two voltages are equal. Because the circuit preferentially passes signals of higher frequency while blocking low-frequency signals, the circuit is called an *RC* high-pass filter. Physically, a high-pass filter works because a capacitor "blocks out" direct current and ac current at low frequencies.

Now let us consider the circuit shown in Figure 33.23a, where the output voltage is taken across the capacitor. In this case, the maximum voltage equals the voltage across the capacitor. Because the impedance across the capacitor is $X_C = 1/\omega C$, we have

$$\Delta V_{\text{out}} = I_{\max} X_C = \frac{I_{\max}}{\omega C}$$

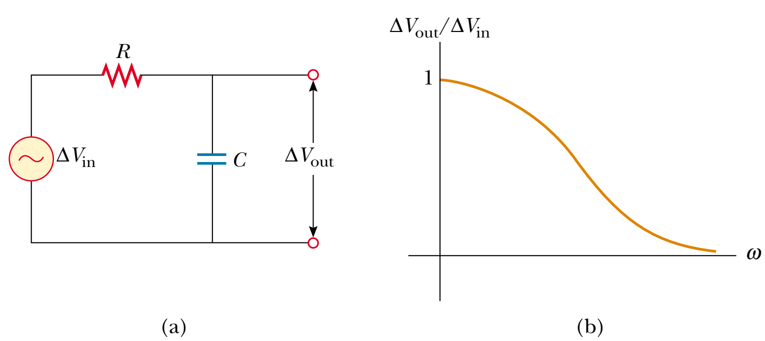


Figure 33.23 (a) A simple *RC* low-pass filter. (b) Ratio of output voltage to input voltage for an *RC* low-pass filter as a function of the angular frequency of the circuit.

Therefore, the ratio of the output voltage to the input voltage is

Low-pass filter

$$\frac{\Delta V_{\text{out}}}{\Delta V_{\text{in}}} = \frac{1/\omega C}{\sqrt{R^2 + \left(\dfrac{1}{\omega C}\right)^2}} \tag{33.43}$$

This ratio, plotted as a function of ω in Figure 33.23b, shows that in this case the circuit preferentially passes signals of low frequency. Hence, the circuit is called an *RC* low-pass filter.

You may be familiar with crossover networks, which are an important part of the speaker systems for high-fidelity audio systems. These networks utilize low-pass filters to direct low frequencies to a special type of speaker, the "woofer," which is designed to reproduce the low notes accurately. The high frequencies are sent to the "tweeter" speaker.

Quick Quiz 33.6

Suppose you are designing a high-fidelity system containing both large loudspeakers (woofers) and small loudspeakers (tweeters). (a) What circuit element would you place in series with a woofer, which passes low-frequency signals? (b) What circuit element would you place in series with a tweeter, which passes high-frequency signals?

SUMMARY

If an ac circuit consists of a generator and a resistor, the current is in phase with the voltage. That is, the current and voltage reach their maximum values at the same time.

The **rms current** and **rms voltage** in an ac circuit in which the voltages and current vary sinusoidally are given by the expressions

$$I_{\text{rms}} = \frac{I_{\text{max}}}{\sqrt{2}} = 0.707 I_{\text{max}} \tag{33.4}$$

$$\Delta V_{\text{rms}} = \frac{\Delta V_{\text{max}}}{\sqrt{2}} = 0.707 \Delta V_{\text{max}} \tag{33.5}$$

where I_{max} and ΔV_{max} are the maximum values.

If an ac circuit consists of a generator and an inductor, the current lags behind the voltage by 90°. That is, the voltage reaches its maximum value one quarter of a period before the current reaches its maximum value.

If an ac circuit consists of a generator and a capacitor, the current leads the voltage by 90°. That is, the current reaches its maximum value one quarter of a period before the voltage reaches its maximum value.

In ac circuits that contain inductors and capacitors, it is useful to define the **inductive reactance** X_L and the **capacitive reactance** X_C as

$$X_L = \omega L \tag{33.10}$$

$$X_C = \frac{1}{\omega C} \tag{33.17}$$

where ω is the angular frequency of the ac generator. The SI unit of reactance is the ohm.

The **impedance** Z of an RLC series ac circuit, which also has the ohm as its unit, is

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2} \tag{33.23}$$

This expression illustrates that we cannot simply add the resistance and reactances in a circuit. We must account for the fact that the applied voltage and current are out of phase, with the **phase angle** ϕ between the current and voltage being

$$\phi = \tan^{-1}\left(\frac{X_L - X_C}{R}\right) \tag{33.25}$$

The sign of ϕ can be positive or negative, depending on whether X_L is greater or less than X_C. The phase angle is zero when $X_L = X_C$.

The **average power** delivered by the generator in an RLC ac circuit is

$$\mathcal{P}_{\text{av}} = I_{\text{rms}}\,\Delta V_{\text{rms}} \cos\phi \tag{33.29}$$

An equivalent expression for the average power is

$$\mathcal{P}_{\text{av}} = I_{\text{rms}}^2 R \tag{33.30}$$

The average power delivered by the generator results in increasing internal energy in the resistor. No power loss occurs in an ideal inductor or capacitor.

The rms current in a series RLC circuit is

$$I_{\text{rms}} = \frac{\Delta V_{\text{rms}}}{\sqrt{R^2 + (X_L - X_C)^2}} \tag{33.32}$$

A series RLC circuit is in resonance when the inductive reactance equals the capacitive reactance. When this condition is met, the current given by Equation 33.32 reaches its maximum value. When $X_L = X_C$ in a circuit, the **resonance frequency** ω_0 of the circuit is

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{33.33}$$

The current in a series RLC circuit reaches its maximum value when the frequency of the generator equals ω_0—that is, when the "driving" frequency matches the resonance frequency.

Transformers allow for easy changes in alternating voltage. Because energy (and therefore power) are conserved, we can write

$$I_1\,\Delta V_1 = I_2\,\Delta V_2 \tag{33.40}$$

to relate the currents and voltages in the primary and secondary windings of a transformer.

QUESTIONS

1. Fluorescent lights flicker on and off 120 times every second. Explain what causes this. Why can't you see it happening?

2. Why does a capacitor act as a short circuit at high frequencies? Why does it act as an open circuit at low frequencies?

3. Explain how the acronyms in the mnemonic "ELI the ICE man" can be used to recall whether current leads voltage or voltage leads current in RLC circuits. (Note that "E" represents voltage.)

4. Why is the sum of the maximum voltages across the elements in a series RLC circuit usually greater than the maximum applied voltage? Doesn't this violate Kirchhoff's second rule?

5. Does the phase angle depend on frequency? What is the phase angle when the inductive reactance equals the capacitive reactance?

6. Energy is delivered to a series RLC circuit by a generator. This energy appears as internal energy in the resistor. What is the source of this energy?

7. Explain why the average power delivered to an *RLC* circuit by the generator depends on the phase between the current and the applied voltage.
8. A particular experiment requires a beam of light of very stable intensity. Why would an ac voltage be unsuitable for powering the light source?
9. Consider a series *RLC* circuit in which *R* is an incandescent lamp, *C* is some fixed capacitor, and *L* is a variable inductance. The source is 120-V ac. Explain why the lamp glows brightly for some values of *L* and does not glow at all for other values.
10. What determines the maximum voltage that can be used on a transmission line?
11. Will a transformer operate if a battery is used for the input voltage across the primary? Explain.
12. How can the average value of a current be zero and yet the square root of the average squared current not be zero?
13. What is the time average of the "square-wave" voltage shown in Figure Q33.13? What is its rms voltage?

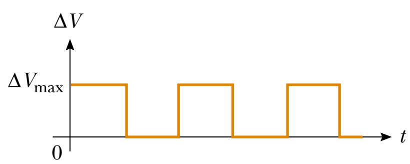


Figure Q33.13

14. Explain how the quality factor is related to the response characteristics of a radio receiver. Which variable most strongly determines the quality factor?
15. Why are the primary and secondary windings of a transformer wrapped on an iron core that passes through both coils?
16. With reference to Figure Q33.16, explain why the capacitor prevents a dc signal from passing between circuits A and B, yet allows an ac signal to pass from circuit A to circuit B. (The circuits are said to be capacitively coupled.)

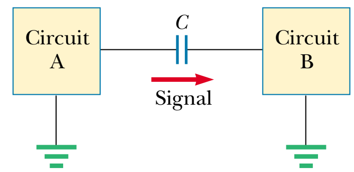


Figure Q33.16

17. With reference to Figure Q33.17, if *C* is made sufficiently large, an ac signal passes from circuit A to ground rather than from circuit A to circuit B. Hence, the capacitor acts as a filter. Explain.

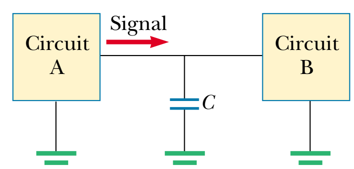


Figure Q33.17

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging ☐ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
☐ = paired numerical/symbolic problems

Note: Assume that all ac voltages and currents are sinusoidal unless stated otherwise.

Section 33.1 ac Sources and Phasors

Section 33.2 Resistors in an ac Circuit

1. The rms output voltage of an ac generator is 200 V, and the operating frequency is 100 Hz. Write the equation giving the output voltage as a function of time.
2. (a) What is the resistance of a lightbulb that uses an average power of 75.0 W when connected to a 60.0-Hz power source having a maximum voltage of 170 V? (b) What is the resistance of a 100-W bulb?
3. An ac power supply produces a maximum voltage $\Delta V_{max} = 100$ V. This power supply is connected to a 24.0-Ω resistor, and the current and resistor voltage are measured with an ideal ac ammeter and voltmeter, as shown in Figure P33.3. What does each meter read?

Note that an ideal ammeter has zero resistance and that an ideal voltmeter has infinite resistance.

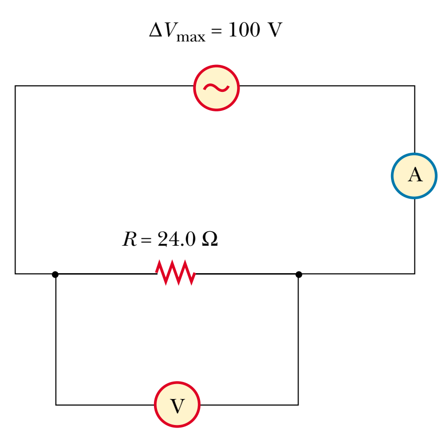


Figure P33.3

4. In the simple ac circuit shown in Figure 33.1, $R = 70.0\ \Omega$ and $\Delta v = \Delta V_{max} \sin \omega t$. (a) If $\Delta v_R = 0.250\ \Delta V_{max}$ for the first time at $t = 0.010\ 0$ s, what is the angular frequency of the generator? (b) What is the next value of t for which $\Delta v_R = 0.250 \Delta V_{max}$?

5. The current in the circuit shown in Figure 33.1 equals 60.0% of the peak current at $t = 7.00$ ms. What is the smallest frequency of the generator that gives this current?

6. Figure P33.6 shows three lamps connected to a 120-V ac (rms) household supply voltage. Lamps 1 and 2 have 150-W bulbs; lamp 3 has a 100-W bulb. Find the rms current and the resistance of each bulb.

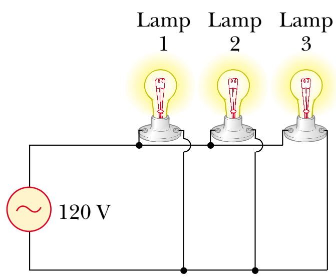


Figure P33.6

7. An audio amplifier, represented by the ac source and resistor in Figure P33.7, delivers to the speaker alternating voltage at audio frequencies. If the source voltage has an amplitude of 15.0 V, $R = 8.20\ \Omega$, and the speaker is equivalent to a resistance of $10.4\ \Omega$, what time-averaged power is transferred to it?

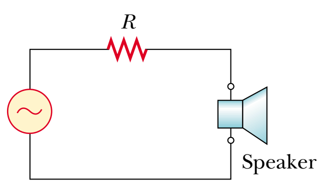


Figure P33.7

Section 33.3 Inductors in an ac Circuit

8. An inductor is connected to a 20.0-Hz power supply that produces a 50.0-V rms voltage. What inductance is needed to keep the instantaneous current in the circuit below 80.0 mA?

9. In a purely inductive ac circuit, such as that shown in Figure 33.4, $\Delta V_{max} = 100$ V. (a) If the maximum current is 7.50 A at 50.0 Hz, what is the inductance L? (b) At what angular frequency ω is the maximum current 2.50 A?

10. An inductor has a 54.0-Ω reactance at 60.0 Hz. What is the maximum current when this inductor is connected to a 50.0-Hz source that produces a 100-V rms voltage?

WEB 11. For the circuit shown in Figure 33.4, $\Delta V_{max} = 80.0$ V, $\omega = 65.0\pi$ rad/s, and $L = 70.0$ mH. Calculate the current in the inductor at $t = 15.5$ ms.

12. A 20.0-mH inductor is connected to a standard outlet ($\Delta V_{rms} = 120$ V, $f = 60.0$ Hz). Determine the energy stored in the inductor at $t = (1/180)$ s, assuming that this energy is zero at $t = 0$.

13. **Review Problem.** Determine the maximum magnetic flux through an inductor connected to a standard outlet ($\Delta V_{rms} = 120$ V, $f = 60.0$ Hz).

Section 33.4 Capacitors in an ac Circuit

14. (a) For what frequencies does a 22.0-μF capacitor have a reactance below 175 Ω? (b) Over this same frequency range, what is the reactance of a 44.0-μF capacitor?

15. What maximum current is delivered by a 2.20-μF capacitor when it is connected across (a) a North American outlet having $\Delta V_{rms} = 120$ V and $f = 60.0$ Hz? (b) a European outlet having $\Delta V_{rms} = 240$ V and $f = 50.0$ Hz?

16. A capacitor C is connected to a power supply that operates at a frequency f and produces an rms voltage ΔV. What is the maximum charge that appears on either of the capacitor plates?

17. What maximum current is delivered by an ac generator with $\Delta V_{max} = 48.0$ V and $f = 90.0$ Hz when it is connected across a 3.70-μF capacitor?

18. A 1.00-mF capacitor is connected to a standard outlet ($\Delta V_{rms} = 120$ V, $f = 60.0$ Hz). Determine the current in the capacitor at $t = (1/180)$ s, assuming that at $t = 0$ the energy stored in the capacitor is zero.

Section 33.5 The *RLC* Series Circuit

19. An inductor ($L = 400$ mH), a capacitor ($C = 4.43\ \mu$F), and a resistor ($R = 500\ \Omega$) are connected in series. A 50.0-Hz ac generator produces a peak current of 250 mA in the circuit. (a) Calculate the required peak voltage ΔV_{max}. (b) Determine the phase angle by which the current leads or lags the applied voltage.

20. At what frequency does the inductive reactance of a 57.0-μH inductor equal the capacitive reactance of a 57.0-μF capacitor?

21. A series ac circuit contains the following components: $R = 150\ \Omega$, $L = 250$ mH, $C = 2.00\ \mu$F, and a generator with $\Delta V_{max} = 210$ V operating at 50.0 Hz. Calculate the (a) inductive reactance, (b) capacitive reactance, (c) impedance, (d) maximum current, and (e) phase angle between current and generator voltage.

22. A sinusoidal voltage $\Delta v(t) = (40.0\text{ V}) \sin(100t)$ is applied to a series *RLC* circuit with $L = 160$ mH, $C = 99.0\ \mu$F, and $R = 68.0\ \Omega$. (a) What is the impedance of the circuit? (b) What is the maximum current? (c) Determine the numerical values for I_{max}, ω, and ϕ in the equation $i(t) = I_{max} \sin(\omega t - \phi)$.

WEB 23. An *RLC* circuit consists of a 150-Ω resistor, a 21.0-μF capacitor, and a 460-mH inductor, connected in series with a 120-V, 60.0-Hz power supply. (a) What is the

phase angle between the current and the applied voltage? (b) Which reaches its maximum earlier, the current or the voltage?

24. A person is working near the secondary of a transformer, as shown in Figure P33.24. The primary voltage is 120 V at 60.0 Hz. The capacitance C_s, which is the stray capacitance between the person's hand and the secondary winding, is 20.0 pF. Assuming that the person has a body resistance to ground $R_b = 50.0\ \text{k}\Omega$, determine the rms voltage across the body. (*Hint:* Redraw the circuit with the secondary of the transformer as a simple ac source.)

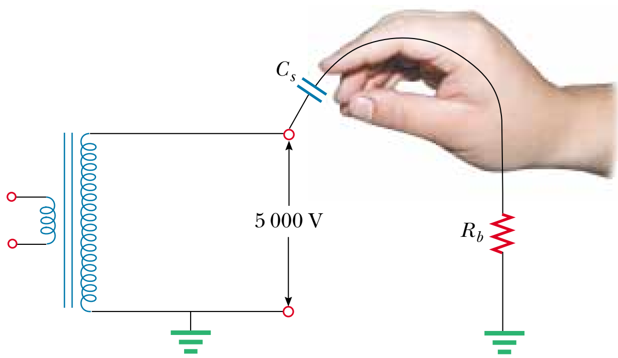


Figure P33.24

25. An ac source with $\Delta V_{\text{max}} = 150$ V and $f = 50.0$ Hz is connected between points a and d in Figure P33.25. Calculate the maximum voltages between points (a) a and b, (b) b and c, (c) c and d, and (d) b and d.

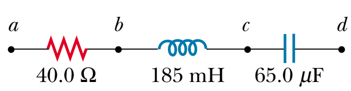


Figure P33.25 Problems 25 and 64.

26. Draw to scale a phasor diagram showing Z, X_L, X_C, and ϕ for an ac series circuit for which $R = 300\ \Omega$, $C = 11.0\ \mu\text{F}$, $L = 0.200$ H, and $f = (500/\pi)$ Hz.

27. A coil of resistance 35.0 Ω and inductance 20.5 H is in series with a capacitor and a 200-V (rms), 100-Hz source. The rms current in the circuit is 4.00 A. (a) Calculate the capacitance in the circuit. (b) What is ΔV_{rms} across the coil?

Section 33.6 Power in an ac Circuit

28. The voltage source in Figure P33.28 has an output $\Delta V_{\text{rms}} = 100$ V at $\omega = 1\ 000$ rad/s. Determine (a) the current in the circuit and (b) the power supplied by the source. (c) Show that the power delivered to the resistor is equal to the power supplied by the source.

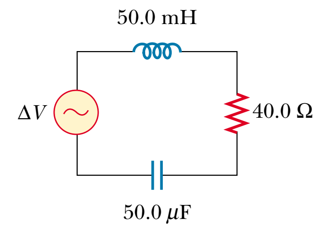


Figure P33.28

WEB $\boxed{29.}$ An ac voltage of the form $\Delta v = (100\ \text{V}) \sin(1\ 000t)$ is applied to a series RLC circuit. If $R = 400\ \Omega$, $C = 5.00\ \mu\text{F}$, and $L = 0.500$ H, what is the average power delivered to the circuit?

30. A series RLC circuit has a resistance of 45.0 Ω and an impedance of 75.0 Ω. What average power is delivered to this circuit when $\Delta V_{\text{rms}} = 210$ V?

$\boxed{31.}$ In a certain series RLC circuit, $I_{\text{rms}} = 9.00$ A, $\Delta V_{\text{rms}} = 180$ V, and the current leads the voltage by 37.0°. (a) What is the total resistance of the circuit? (b) What is the reactance of the circuit $(X_L - X_C)$?

32. Suppose you manage a factory that uses many electric motors. The motors create a large inductive load to the electric power line, as well as a resistive load. The electric company builds an extra-heavy distribution line to supply you with a component of current that is 90° out of phase with the voltage, as well as with current in phase with the voltage. The electric company charges you an extra fee for "reactive volt-amps" in addition to the amount you pay for the energy you use. You can avoid the extra fee by installing a capacitor between the power line and your factory. The following problem models this solution.

In an LR circuit, a 120-V (rms), 60.0-Hz source is in series with a 25.0-mH inductor and a 20.0-Ω resistor. What are (a) the rms current and (b) the power factor? (c) What capacitor must be added in series to make the power factor 1? (d) To what value can the supply voltage be reduced if the power supplied is to be the same as that provided before installation of the capacitor?

33. **Review Problem.** Over a distance of 100 km, power of 100 MW is to be transmitted at 50.0 kV with only 1.00% loss. Copper wire of what diameter should be used for each of the two conductors of the transmission line? Assume that the current density in the conductors is uniform.

34. **Review Problem.** Suppose power $\mathcal{P}$ is to be transmitted over a distance d at a voltage ΔV, with only 1.00% loss. Copper wire of what diameter should be used for each of the two conductors of the transmission line? Assume that the current density in the conductors is uniform.

35. A diode is a device that allows current to pass in only one direction (the direction indicated by the arrowhead in its circuit-diagram symbol). Find, in terms of ΔV and

R, the average power delivered to the diode circuit shown in Figure P33.35.

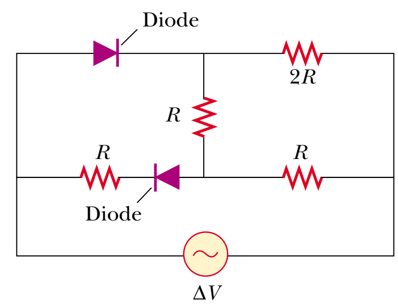


Figure P33.35

Section 33.7 Resonance in a Series *RLC* Circuit

36. The tuning circuit of an AM radio contains an LC combination. The inductance is 0.200 mH, and the capacitor is variable, so the circuit can resonate at any frequency between 550 kHz and 1 650 kHz. Find the range of values required for C.

37. An RLC circuit is used in a radio to tune in to an FM station broadcasting at 99.7 MHz. The resistance in the circuit is 12.0 Ω, and the inductance is 1.40 μH. What capacitance should be used?

38. A series RLC circuit has the following values: $L =$ 20.0 mH, $C =$ 100 nF, $R =$ 20.0 Ω, and $\Delta V_{max} =$ 100 V, with $\Delta v = \Delta V_{max} \sin \omega t$. Find (a) the resonant frequency, (b) the amplitude of the current at the resonant frequency, (c) the Q of the circuit, and (d) the amplitude of the voltage across the inductor at resonance.

39. A 10.0-Ω resistor, a 10.0-mH inductor, and a 100-μF capacitor are connected in series to a 50.0-V (rms) source having variable frequency. What is the energy delivered to the circuit during one period if the operating frequency is twice the resonance frequency?

40. A resistor R, an inductor L, and a capacitor C are connected in series to an ac source of rms voltage ΔV and variable frequency. What is the energy delivered to the circuit during one period if the operating frequency is twice the resonance frequency?

41. Compute the quality factor for the circuits described in Problems 22 and 23. Which circuit has the sharper resonance?

Section 33.8 The Transformer and Power Transmission

42. A step-down transformer is used for recharging the batteries of portable devices such as tape players. The turns ratio inside the transformer is 13:1, and it is used with 120-V (rms) household service. If a particular ideal transformer draws 0.350 A from the house outlet, what (a) voltage and (b) current are supplied to a tape player from the transformer? (c) How much power is delivered?

43. A transformer has $N_1 = 350$ turns and $N_2 = 2\,000$ turns. If the input voltage is $\Delta v(t) = (170 \text{ V}) \cos \omega t$, what rms voltage is developed across the secondary coil?

44. A step-up transformer is designed to have an output voltage of 2 200 V (rms) when the primary is connected across a 110-V (rms) source. (a) If there are 80 turns on the primary winding, how many turns are required on the secondary? (b) If a load resistor across the secondary draws a current of 1.50 A, what is the current in the primary under ideal conditions? (c) If the transformer actually has an efficiency of 95.0%, what is the current in the primary when the secondary current is 1.20 A?

45. In the transformer shown in Figure P33.45, the load resistor is 50.0 Ω. The turns ratio $N_1 : N_2$ is 5:2, and the source voltage is 80.0 V (rms). If a voltmeter across the load measures 25.0 V (rms), what is the source resistance R_s?

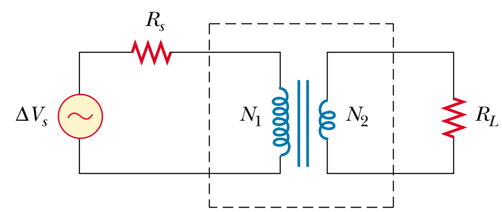


Figure P33.45

46. The secondary voltage of an ignition transformer in a furnace is 10.0 kV. When the primary operates at an rms voltage of 120 V, the primary impedance is 24.0 Ω and the transformer is 90.0% efficient. (a) What turns ratio is required? What are (b) the current in the secondary and (c) the impedance in the secondary?

47. A transmission line that has a resistance per unit length of 4.50×10^{-4} Ω/m is to be used to transmit 5.00 MW over 400 mi (6.44×10^5 m). The output voltage of the generator is 4.50 kV. (a) What is the power loss if a transformer is used to step up the voltage to 500 kV? (b) What fraction of the input power is lost to the line under these circumstances? (c) What difficulties would be encountered on attempting to transmit the 5.00 MW at the generator voltage of 4.50 kV?

(Optional)
Section 33.9 Rectifiers and Filters

48. The RC low-pass filter shown in Figure 33.23 has a resistance $R =$ 90.0 Ω and a capacitance $C =$ 8.00 nF. Calculate the gain $(\Delta V_{out}/\Delta V_{in})$ for input frequencies of (a) 600 Hz and (b) 600 kHz.

WEB **49.** The RC high-pass filter shown in Figure 33.22 has a resistance $R =$ 0.500 Ω. (a) What capacitance gives an output signal that has one-half the amplitude of a 300-Hz input signal? (b) What is the gain $(\Delta V_{out}/\Delta V_{in})$ for a 600-Hz signal?

50. The circuit in Figure P33.50 represents a high-pass filter in which the inductor has internal resistance. What is the source frequency if the output voltage ΔV_2 is one-half the input voltage?

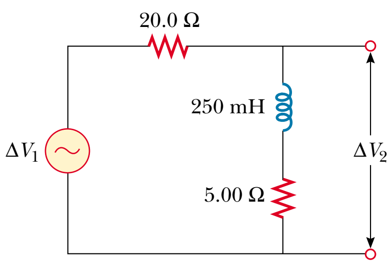


Figure P33.50

51. The resistor in Figure P33.51 represents the midrange speaker in a three-speaker system. Assume that its resistance is constant at 8.00 Ω. The source represents an audio amplifier producing signals of uniform amplitude $\Delta V_{in} = 10.0$ V at all audio frequencies. The inductor and capacitor are to function as a bandpass filter with $\Delta V_{out}/\Delta V_{in} = \frac{1}{2}$ at 200 Hz and at 4 000 Hz. (a) Determine the required values of L and C. (b) Find the maximum value of the gain ratio $\Delta V_{out}/\Delta V_{in}$. (c) Find the frequency f_0 at which the gain ratio has its maximum value. (d) Find the phase shift between ΔV_{in} and ΔV_{out} at 200 Hz, at f_0, and at 4 000 Hz. (e) Find the average power transferred to the speaker at 200 Hz, at f_0, and at 4 000 Hz. (f) Treating the filter as a resonant circuit, find its quality factor.

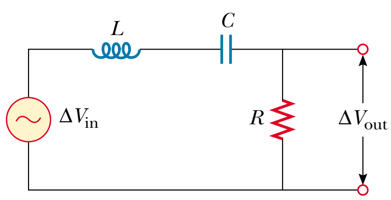


Figure P33.51

52. Show that two successive high-pass filters having the same values of R and C give a combined gain

$$\frac{\Delta V_{out}}{\Delta V_{in}} = \frac{1}{1 + (1/\omega RC)^2}$$

53. Consider a low-pass filter followed by a high-pass filter, as shown in Figure P33.53. If $R = 1\,000\ \Omega$ and $C = 0.050\,0\ \mu$F, determine $\Delta V_{out}/\Delta V_{in}$ for a 2.00-kHz input frequency.

ADDITIONAL PROBLEMS

54. Show that the rms value for the sawtooth voltage shown in Figure P33.54 is $\Delta V_{max}/\sqrt{3}$.

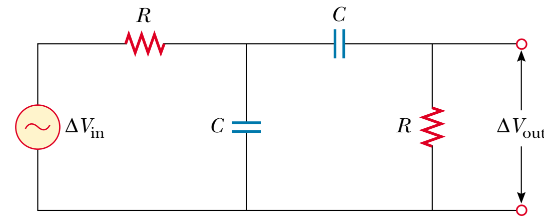


Figure P33.53

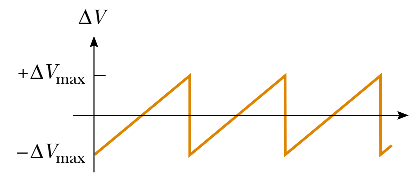


Figure P33.54

WEB **55.** A series RLC circuit consists of an 8.00-Ω resistor, a 5.00-μF capacitor, and a 50.0-mH inductor. A variable frequency source applies an emf of 400 V (rms) across the combination. Determine the power delivered to the circuit when the frequency is equal to one-half the resonance frequency.

56. To determine the inductance of a coil used in a research project, a student first connects the coil to a 12.0-V battery and measures a current of 0.630 A. The student then connects the coil to a 24.0-V (rms), 60.0-Hz generator and measures an rms current of 0.570 A. What is the inductance?

57. In Figure P33.57, find the current delivered by the 45.0-V (rms) power supply (a) when the frequency is very large and (b) when the frequency is very small.

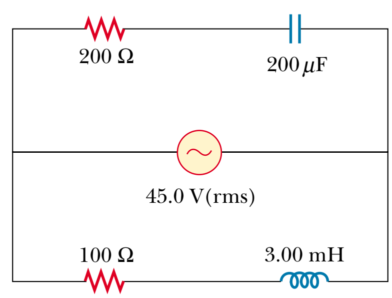


Figure P33.57

58. In the circuit shown in Figure P33.58, assume that all parameters except C are given. (a) Find the current as a function of time. (b) Find the power delivered to the circuit. (c) Find the current as a function of time after *only* switch 1 is opened. (d) After switch 2 is *also* opened, the current and voltage are in phase. Find the capacitance C. (e) Find the impedance of the circuit when both switches are open. (f) Find the maximum

energy stored in the capacitor during oscillations. (g) Find the maximum energy stored in the inductor during oscillations. (h) Now the frequency of the voltage source is doubled. Find the phase difference between the current and the voltage. (i) Find the frequency that makes the inductive reactance one-half the capacitive reactance.

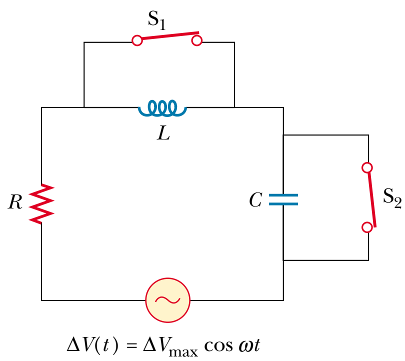


Figure P33.58

59. As an alternative to the *RC* filters described in Section 33.9, *LC* filters are used as both high- and low-pass filters. However, all real inductors have resistance, as indicated in Figure P33.59, and this resistance must be taken into account. (a) Determine which circuit in Figure P33.59 is the high-pass filter and which is the low-pass filter. (b) Derive the output/input ratio for each circuit, following the procedure used for the *RC* filters in Section 33.9.

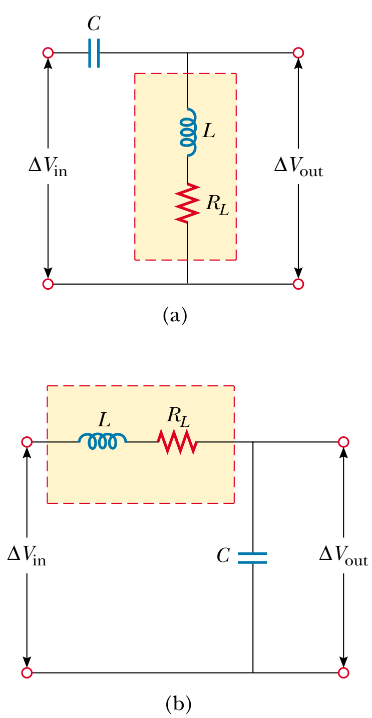


Figure P33.59

60. An 80.0-Ω resistor and a 200-mH inductor are connected in *parallel* across a 100-V (rms), 60.0-Hz source. (a) What is the rms current in the resistor? (b) By what angle does the total current lead or lag behind the voltage?

61. Make an order-of-magnitude estimate of the electric current that the electric company delivers to a town from a remote generating station. State the data that you measure or estimate. If you wish, you may consider a suburban bedroom community of 20 000 people.

62. A voltage $\Delta v = (100\text{ V}) \sin \omega t$ (in SI units) is applied across a series combination of a 2.00-H inductor, a 10.0-μF capacitor, and a 10.0-Ω resistor. (a) Determine the angular frequency ω_0 at which the power delivered to the resistor is a maximum. (b) Calculate the power at that frequency. (c) Determine the two angular frequencies ω_1 and ω_2 at which the power delivered is one-half the maximum value. [The Q of the circuit is approximately $\omega_0/(\omega_2 - \omega_1)$.]

63. Consider a series *RLC* circuit having the following circuit parameters: $R = 200\ \Omega$, $L = 663$ mH, and $C = 26.5\ \mu$F. The applied voltage has an amplitude of 50.0 V and a frequency of 60.0 Hz. Find the following: (a) the current I_{max}, including its phase constant ϕ relative to the applied voltage Δv; (b) the voltage ΔV_R across the resistor and its phase relative to the current; (c) the voltage ΔV_C across the capacitor and its phase relative to the current; and (d) the voltage ΔV_L across the inductor and its phase relative to the current.

64. A power supply with $\Delta V_{rms} = 120$ V is connected between points *a* and *d* in Figure P33.25. At what frequency will it deliver a power of 250 W?

65. Example 28.2 showed that maximum power is transferred when the internal resistance of a dc source is equal to the resistance of the load. A transformer may be used to provide maximum power transfer between two ac circuits that have different impedances. (a) Show that the ratio of turns N_1/N_2 needed to meet this condition is

$$\frac{N_1}{N_2} = \sqrt{\frac{Z_1}{Z_2}}$$

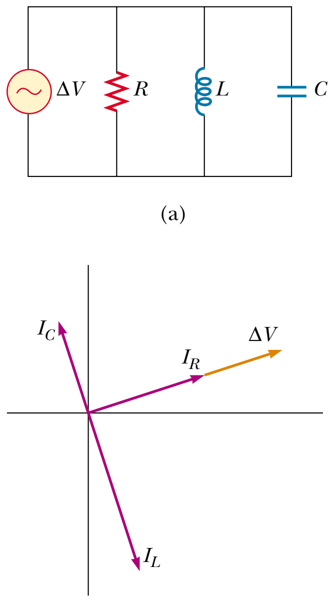
(b) Suppose you want to use a transformer as an impedance-matching device between an audio amplifier that has an output impedance of 8.00 kΩ and a speaker that has an input impedance of 8.00 Ω. What should your N_1/N_2 ratio be?

66. Figure P33.66a shows a parallel *RLC* circuit, and the corresponding phasor diagram is provided in Figure P33.66b. The instantaneous voltages and rms voltages across the three circuit elements are the same, and each is in phase with the current through the resistor. The currents in *C* and *L* lead or lag behind the current in the resistor, as shown in Figure P33.66b. (a) Show that the rms current delivered by the source is

$$I_{rms} = \Delta V_{rms}\left[\frac{1}{R^2} + \left(\omega C - \frac{1}{\omega L}\right)^2\right]^{1/2}$$

(b) Show that the phase angle ϕ between ΔV_{rms} and I_{rms} is

$$\tan\phi = R\left(\frac{1}{X_C} - \frac{1}{X_L}\right)$$



(a)

(b)

Figure P33.66

67. An 80.0-Ω resistor, a 200-mH inductor, and a 0.150-μF capacitor are connected *in parallel* across a 120-V (rms) source operating at 374 rad/s. (a) What is the resonant frequency of the circuit? (b) Calculate the rms current in the resistor, the inductor, and the capacitor. (c) What rms current is delivered by the source? (d) Is the current leading or lagging behind the voltage? By what angle?

68. Consider the phase-shifter circuit shown in Figure P33.68. The input voltage is described by the expression $\Delta v = (10.0 \text{ V}) \sin 200t$ (in SI units). Assuming that $L = 500$ mH, find (a) the value of R such that the output voltage lags behind the input voltage by 30.0° and (b) the amplitude of the output voltage.

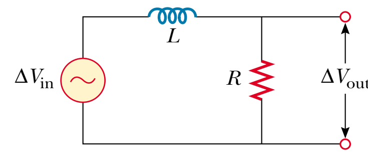


Figure P33.68

69. A series RLC circuit is operating at 2 000 Hz. At this frequency, $X_L = X_C = 1\,884\ \Omega$. The resistance of the circuit is 40.0 Ω. (a) Prepare a table showing the values of X_L, X_C, and Z for f = 300, 600, 800, 1 000, 1 500, 2 000, 3 000, 4 000, 6 000, and 10 000 Hz. (b) Plot on the same set of axes X_L, X_C, and Z as functions of ln f.

70. A series RLC circuit in which $R = 1.00\ \Omega$, $L = 1.00$ mH, and $C = 1.00$ nF is connected to an ac generator delivering 1.00 V (rms). Make a precise graph of the power delivered to the circuit as a function of the frequency, and verify that the full width of the resonance peak at half-maximum is $R/2\pi L$.

71. Suppose the high-pass filter shown in Figure 33.22 has $R = 1\,000\ \Omega$ and $C = 0.050\,0\ \mu$F. (a) At what frequency does $\Delta V_{\text{out}}/\Delta V_{\text{in}} = \frac{1}{2}$? (b) Plot $\log_{10}(\Delta V_{\text{out}}/\Delta V_{\text{in}})$ versus $\log_{10}(f)$ over the frequency range from 1 Hz to 1 MHz. (This log–log plot of gain versus frequency is known as a *Bode plot.*)

Answers to Quick Quizzes

33.1 (c) $\mathcal{P}_{\text{av}} > 0$ and $i_{\text{av}} = 0$. The average power is proportional to the rms current—which, as Figure 33.3 shows, is nonzero even though the average current is zero. Condition (a) is valid only for an open circuit, and conditions (b) and (d) can never be true because $i_{\text{av}} = 0$ for ac circuits even though $i_{\text{rms}} > 0$.

33.2 (b) Sum of instantaneous voltages across elements. Choices (a) and (c) are incorrect because the unaligned sine curves in Figure 33.9b mean that the voltages are out of phase, so we cannot simply add the maximum (or rms) voltages across the elements. (In other words, $\Delta V \neq \Delta V_R + \Delta V_L + \Delta V_C$ even though it is true that $\Delta v = \Delta v_R + \Delta v_L + \Delta v_C$.)

33.3 (a) $X_L < X_C$. (b) $X_L = X_C$. (c) $X_L > X_C$.

33.4 Equation 33.23 indicates that at resonance (when $X_L = X_C$) the impedance is due strictly to the resistor, $Z = R$. At resonance, the current is given by the expression $I_{\text{rms}} = \Delta V_{\text{rms}}/R$.

33.5 High. The higher the quality factor, the more sensitive the detector. As you can see from Figure 33.15a, when $Q = \omega_0/\Delta\omega$ is high, as it is in the $R = 3.5\ \Omega$ case, a slight change in the resonance frequency (as might happen when a small piece of metal passes through the portal) causes a large change in current that can be detected easily.

33.6 (a) An inductor. The current in an inductive circuit decreases with increasing frequency (see Eq. 33.9). Thus, an inductor connected in series with a woofer blocks high-frequency signals and passes low-frequency signals. (b) A capacitor. The current in a capacitive circuit decreases with decreasing frequency (see Eq. 33.16). When a capacitor is connected in series with a tweeter, the capacitor blocks low-frequency signals and passes high-frequency signals.

PUZZLER

This person is exposed to very bright sunlight at the beach. If he is wearing the wrong kind of sunglasses, he may be causing more permanent harm to his vision than he would be if he took the glasses off and squinted. What determines whether certain types of sunglasses are good for your eyes?
(Ron Chapple/FPG International)

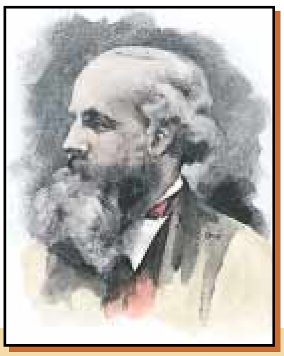
chapter 34

Electromagnetic Waves

Chapter Outline

The waves described in Chapters 16, 17, and 18 are mechanical waves. By definition, the propagation of mechanical disturbances—such as sound waves, water waves, and waves on a string—requires the presence of a medium. This chapter is concerned with the properties of electromagnetic waves, which (unlike mechanical waves) can propagate through empty space.

In Section 31.7 we gave a brief description of Maxwell's equations, which form the theoretical basis of all electromagnetic phenomena. The consequences of Maxwell's equations are far-reaching and dramatic. The Ampère–Maxwell law predicts that a time-varying electric field produces a magnetic field, just as Faraday's law tells us that a time-varying magnetic field produces an electric field. Maxwell's introduction of the concept of displacement current as a new source of a magnetic field provided the final important link between electric and magnetic fields in classical physics.



James Clerk Maxwell **Scottish theoretical physicist (1831–1879)** Maxwell developed the electromagnetic theory of light and the kinetic theory of gases, and he explained the nature of color vision and of Saturn's rings. His successful interpretation of the electromagnetic field produced the field equations that bear his name. Formidable mathematical ability combined with great insight enabled Maxwell to lead the way in the study of electromagnetism and kinetic theory. He died of cancer before he was 50. *(North Wind Picture Archives)*

Astonishingly, Maxwell's equations also predict the existence of electromagnetic waves that propagate through space at the speed of light c. This chapter begins with a discussion of how Heinrich Hertz confirmed Maxwell's prediction when he generated and detected electromagnetic waves in 1887. That discovery has led to many practical communication systems, including radio, television, and radar. On a conceptual level, Maxwell unified the subjects of light and electromagnetism by developing the idea that light is a form of electromagnetic radiation.

Next, we learn how electromagnetic waves are generated by oscillating electric charges. The waves consist of oscillating electric and magnetic fields that are at right angles to each other and to the direction of wave propagation. Thus, electromagnetic waves are transverse waves. Maxwell's prediction of electromagnetic radiation shows that the amplitudes of the electric and magnetic fields in an electromagnetic wave are related by the expression $E = cB$. The waves radiated from the oscillating charges can be detected at great distances. Furthermore, electromagnetic waves carry energy and momentum and hence can exert pressure on a surface.

The chapter concludes with a look at the wide range of frequencies covered by electromagnetic waves. For example, radio waves (frequencies of about 10^7 Hz) are electromagnetic waves produced by oscillating currents in a radio tower's transmitting antenna. Light waves are a high-frequency form of electromagnetic radiation (about 10^{14} Hz) produced by oscillating electrons in atoms.

34.1 MAXWELL'S EQUATIONS AND HERTZ'S DISCOVERIES

In his unified theory of electromagnetism, Maxwell showed that electromagnetic waves are a natural consequence of the fundamental laws expressed in the following four equations (see Section 31.7):

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q}{\epsilon_0} \tag{34.1}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{A} = 0 \tag{34.2}$$

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt} \tag{34.3}$$

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 I + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \tag{34.4}$$

As we shall see in the next section, Equations 34.3 and 34.4 can be combined to obtain a wave equation for both the electric field and the magnetic field. In empty space ($Q = 0$, $I = 0$), the solution to these two equations shows that the speed at which electromagnetic waves travel equals the measured speed of light. This result led Maxwell to predict that light waves are a form of electromagnetic radiation.

The experimental apparatus that Hertz used to generate and detect electromagnetic waves is shown schematically in Figure 34.1. An induction coil is connected to a transmitter made up of two spherical electrodes separated by a narrow gap. The coil provides short voltage surges to the electrodes, making one positive and the other negative. A spark is generated between the spheres when the electric field near either electrode surpasses the dielectric strength for air (3×10^6 V/m; see Table 26.1). In a strong electric field, the acceleration of free electrons provides them with enough energy to ionize any molecules they strike. This ionization provides more electrons, which can accelerate and cause further ionizations. As the air in the gap is ionized, it becomes a much better conductor, and the discharge between the electrodes exhibits an oscillatory behavior at a very high frequency. From an electric-circuit viewpoint, this is equivalent to an LC circuit in which the inductance is that of the coil and the capacitance is due to the spherical electrodes.

Because L and C are quite small in Hertz's apparatus, the frequency of oscillation is very high, ≈ 100 MHz. (Recall from Eq. 32.22 that $\omega = 1/\sqrt{LC}$ for an LC circuit.) Electromagnetic waves are radiated at this frequency as a result of the oscillation (and hence acceleration) of free charges in the transmitter circuit. Hertz was able to detect these waves by using a single loop of wire with its own spark gap (the receiver). Such a receiver loop, placed several meters from the transmitter, has its own effective inductance, capacitance, and natural frequency of oscillation. In Hertz's experiment, sparks were induced across the gap of the receiving electrodes when the frequency of the receiver was adjusted to match that of the transmitter. Thus, Hertz demonstrated that the oscillating current induced in the receiver was produced by electromagnetic waves radiated by the transmitter. His experiment is analogous to the mechanical phenomenon in which a tuning fork responds to acoustic vibrations from an identical tuning fork that is oscillating.

Heinrich Rudolf Hertz German physicist (1857–1894) Hertz made his most important discovery—radio waves—in 1887. After finding that the speed of a radio wave was the same as that of light, he showed that radio waves, like light waves, could be reflected, refracted, and diffracted. Hertz died of blood poisoning at age 36. He made many contributions to science during his short life. The hertz, equal to one complete vibration or cycle per second, is named after him. *(The Bettmann Archive)*

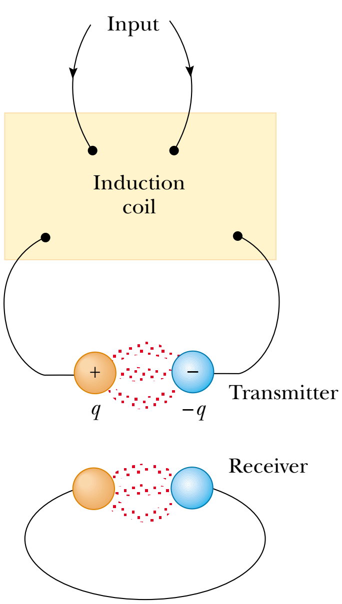
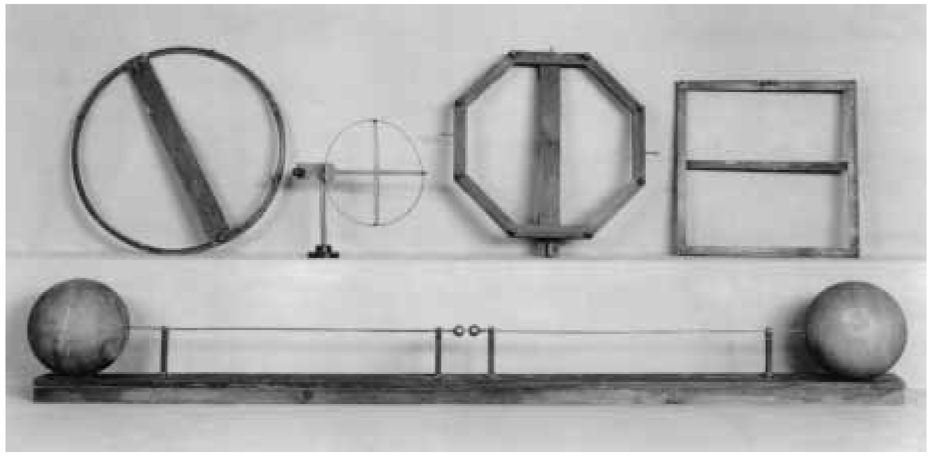


Figure 34.1 Schematic diagram of Hertz's apparatus for generating and detecting electromagnetic waves. The transmitter consists of two spherical electrodes connected to an induction coil, which provides short voltage surges to the spheres, setting up oscillations in the discharge between the electrodes (suggested by the red dots). The receiver is a nearby loop of wire containing a second spark gap.



A large oscillator (bottom) and circular, octagonal, and square receivers used by Heinrich Hertz.

QuickLab

Some electric motors use commutators that make and break electrical contact, creating sparks reminiscent of Hertz's method for generating electromagnetic waves. Try running an electric shaver or kitchen mixer near an AM radio. What happens to the reception?

Additionally, Hertz showed in a series of experiments that the radiation generated by his spark-gap device exhibited the wave properties of interference, diffraction, reflection, refraction, and polarization, all of which are properties exhibited by light. Thus, it became evident that the radio-frequency waves Hertz was generating had properties similar to those of light waves and differed only in frequency and wavelength. Perhaps his most convincing experiment was the measurement of the speed of this radiation. Radio-frequency waves of known frequency were reflected from a metal sheet and created a standing-wave interference pattern whose nodal points could be detected. The measured distance between the nodal points enabled determination of the wavelength λ. Using the relationship $v = \lambda f$ (Eq. 16.14), Hertz found that v was close to 3×10^8 m/s, the known speed c of visible light.

34.2 PLANE ELECTROMAGNETIC WAVES

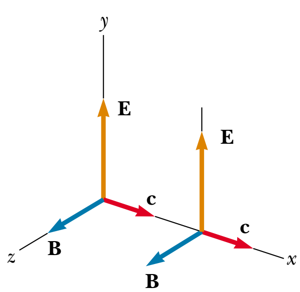


Figure 34.2 An electromagnetic wave traveling at velocity **c** in the positive x direction. The electric field is along the y direction, and the magnetic field is along the z direction. These fields depend only on x and t.

The properties of electromagnetic waves can be deduced from Maxwell's equations. One approach to deriving these properties is to solve the second-order differential equation obtained from Maxwell's third and fourth equations. A rigorous mathematical treatment of that sort is beyond the scope of this text. To circumvent this problem, we assume that the vectors for the electric field and magnetic field in an electromagnetic wave have a specific space–time behavior that is simple but consistent with Maxwell's equations.

To understand the prediction of electromagnetic waves more fully, let us focus our attention on an electromagnetic wave that travels in the x direction (the *direction of propagation*). In this wave, the electric field **E** is in the y direction, and the magnetic field **B** is in the z direction, as shown in Figure 34.2. Waves such as this one, in which the electric and magnetic fields are restricted to being parallel to a pair of perpendicular axes, are said to be **linearly polarized waves.**[1] Furthermore, we assume that at any point P, the magnitudes E and B of the fields depend

[1] Waves having other particular patterns of vibration of the electric and magnetic fields include circularly polarized waves. The most general polarization pattern is elliptical.

upon x and t only, and not upon the y or z coordinate. A collection of such waves from individual sources is called a **plane wave.** A surface connecting points of equal phase on all waves, which we call a **wave front,** would be a geometric plane. In comparison, a point source of radiation sends waves out in all directions. A surface connecting points of equal phase is a sphere for this situation, so we call this a **spherical wave.**

We can relate E and B to each other with Equations 34.3 and 34.4. In empty space, where $Q = 0$ and $I = 0$, Equation 34.3 remains unchanged and Equation 34.4 becomes

$$\oint \mathbf{B} \cdot d\mathbf{s} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} \tag{34.5}$$

Using Equations 34.3 and 34.5 and the plane-wave assumption, we obtain the following differential equations relating E and B. (We shall derive these equations formally later in this section.) For simplicity, we drop the subscripts on the components E_y and B_z:

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t} \tag{34.6}$$

$$\frac{\partial B}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E}{\partial t} \tag{34.7}$$

Note that the derivatives here are partial derivatives. For example, when we evaluate $\partial E/\partial x$, we assume that t is constant. Likewise, when we evaluate $\partial B/\partial t$, x is held constant. Taking the derivative of Equation 34.6 with respect to x and combining the result with Equation 34.7, we obtain

$$\frac{\partial^2 E}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{\partial B}{\partial t}\right) = -\frac{\partial}{\partial t}\left(\frac{\partial B}{\partial x}\right) = -\frac{\partial}{\partial t}\left(-\mu_0 \epsilon_0 \frac{\partial E}{\partial t}\right)$$

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \tag{34.8}$$

In the same manner, taking the derivative of Equation 34.7 with respect to x and combining it with Equation 34.6, we obtain

$$\frac{\partial^2 B}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2} \tag{34.9}$$

Equations 34.8 and 34.9 both have the form of the general wave equation[2] with the wave speed v replaced by c, where

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{34.10}$$

Speed of electromagnetic waves

Taking $\mu_0 = 4\pi \times 10^{-7}$ T·m/A and $\epsilon_0 = 8.854\,19 \times 10^{-12}$ C^2/N·m^2 in Equation 34.10, we find that $c = 2.997\,92 \times 10^8$ m/s. Because this speed is precisely the same as the speed of light in empty space, we are led to believe (correctly) that light is an electromagnetic wave.

[2] The general wave equation is of the form $(\partial^2 y/\partial x^2) = (1/v^2)(\partial^2 y/\partial t^2)$, where v is the speed of the wave and y is the wave function. The general wave equation was introduced as Equation 16.26, and it would be useful for you to review Section 16.9.

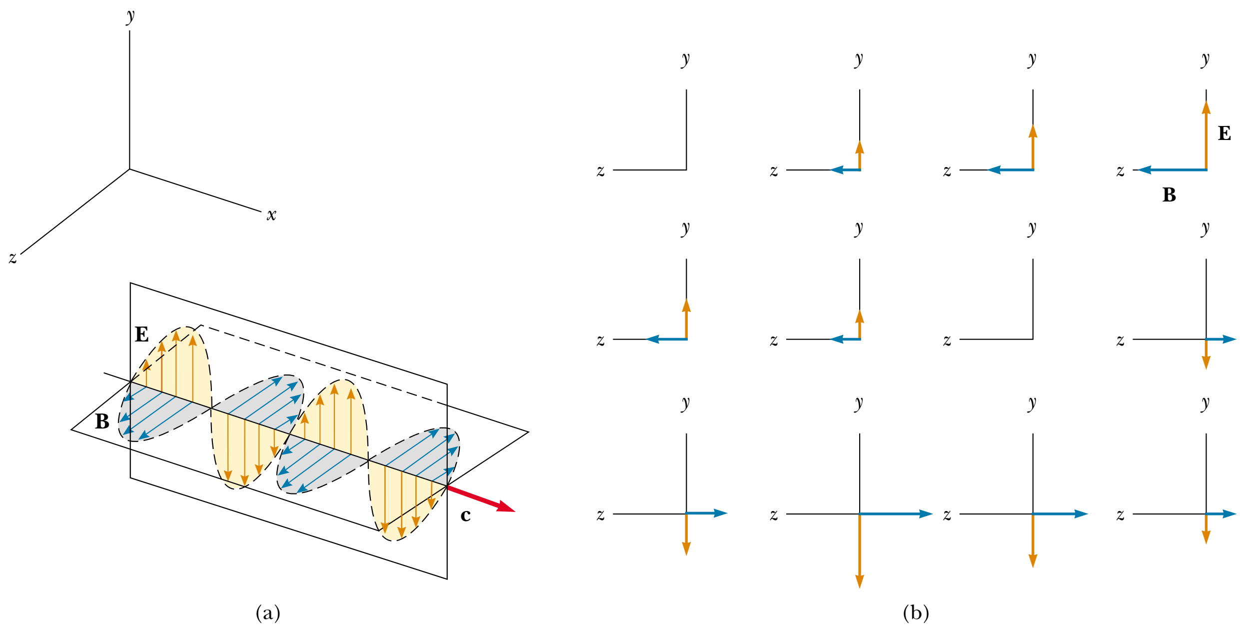


Figure 34.3 Representation of a sinusoidal, linearly polarized plane electromagnetic wave moving in the positive x direction with velocity $\mathbf{c}$. (a) The wave at some instant. Note the sinusoidal variations of E and B with x. (b) A time sequence illustrating the electric and magnetic field vectors present in the yz plane, as seen by an observer looking in the negative x direction. Note the sinusoidal variations of E and B with t.

The simplest solution to Equations 34.8 and 34.9 is a sinusoidal wave, for which the field magnitudes E and B vary with x and t according to the expressions

Sinusoidal electric and magnetic fields

$$E = E_{\max} \cos(kx - \omega t) \tag{34.11}$$

$$B = B_{\max} \cos(kx - \omega t) \tag{34.12}$$

where $E_{\max}$ and $B_{\max}$ are the maximum values of the fields. The angular wave number is the constant $k = 2\pi/\lambda$, where λ is the wavelength. The angular frequency is $\omega = 2\pi f$, where f is the wave frequency. The ratio ω/k equals the speed c:

$$\frac{\omega}{k} = \frac{2\pi f}{2\pi/\lambda} = \lambda f = c$$

We have used Equation 16.14, $v = c = \lambda f$, which relates the speed, frequency, and wavelength of any continuous wave. Figure 34.3a is a pictorial representation, at one instant, of a sinusoidal, linearly polarized plane wave moving in the positive x direction. Figure 34.3b shows how the electric and magnetic field vectors at a fixed location vary with time.

Quick Quiz 34.1

What is the phase difference between B and E in Figure 34.3?

Taking partial derivatives of Equations 34.11 (with respect to x) and 34.12

(with respect to t), we find that

$$\frac{\partial E}{\partial x} = -kE_{\max}\sin(kx - \omega t)$$

$$\frac{\partial B}{\partial t} = \omega B_{\max}\sin(kx - \omega t)$$

Substituting these results into Equation 34.6, we find that at any instant

$$kE_{\max} = \omega B_{\max}$$

$$\frac{E_{\max}}{B_{\max}} = \frac{\omega}{k} = c$$

Using these results together with Equations 34.11 and 34.12, we see that

$$\frac{E_{\max}}{B_{\max}} = \frac{E}{B} = c \qquad (34.13)$$

That is, **at every instant the ratio of the magnitude of the electric field to the magnitude of the magnetic field in an electromagnetic wave equals the speed of light.**

Finally, note that electromagnetic waves obey the superposition principle (which we discussed in Section 16.4 with respect to mechanical waves) because the differential equations involving E and B are linear equations. For example, we can add two waves with the same frequency simply by adding the magnitudes of the two electric fields algebraically.

Properties of electromagnetic waves

- The solutions of Maxwell's third and fourth equations are wave-like, with both E and B satisfying a wave equation.
- Electromagnetic waves travel through empty space at the speed of light $c = 1/\sqrt{\mu_0\epsilon_0}$.
- The components of the electric and magnetic fields of plane electromagnetic waves are perpendicular to each other and perpendicular to the direction of wave propagation. We can summarize the latter property by saying that electromagnetic waves are transverse waves.
- The magnitudes of **E** and **B** in empty space are related by the expression $E/B = c$.
- Electromagnetic waves obey the principle of superposition.

EXAMPLE 34.1 An Electromagnetic Wave

A sinusoidal electromagnetic wave of frequency 40.0 MHz travels in free space in the x direction, as shown in Figure 34.4. (a) Determine the wavelength and period of the wave.

Solution Using Equation 16.14 for light waves, $c = \lambda f$, and given that $f = 40.0\text{ MHz} = 4.00 \times 10^7\text{ s}^{-1}$, we have

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^8\text{ m/s}}{4.00 \times 10^7\text{ s}^{-1}} = \boxed{7.50\text{ m}}$$

The period T of the wave is the inverse of the frequency:

$$T = \frac{1}{f} = \frac{1}{4.00 \times 10^7\text{ s}^{-1}} = \boxed{2.50 \times 10^{-8}\text{ s}}$$

(b) At some point and at some instant, the electric field has its maximum value of 750 N/C and is along the y axis. Calculate the magnitude and direction of the magnetic field at this position and time.

Solution From Equation 34.13 we see that

$$B_{\max} = \frac{E_{\max}}{c} = \frac{750\text{ N/C}}{3.00 \times 10^8\text{ m/s}} = \boxed{2.50 \times 10^{-6}\text{ T}}$$

Because **E** and **B** must be perpendicular to each other and perpendicular to the direction of wave propagation (x in this case), we conclude that **B** is in the z direction.

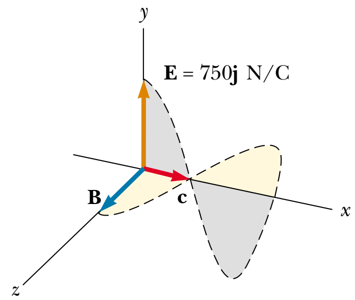


Figure 34.4 At some instant, a plane electromagnetic wave moving in the x direction has a maximum electric field of 750 N/C in the positive y direction. The corresponding magnetic field at that point has a magnitude E/c and is in the z direction.

(c) Write expressions for the space-time variation of the components of the electric and magnetic fields for this wave.

Solution We can apply Equations 34.11 and 34.12 directly:

$$E = E_{\text{max}} \cos(kx - \omega t) = (750 \text{ N/C}) \cos(kx - \omega t)$$

$$B = B_{\text{max}} \cos(kx - \omega t) = (2.50 \times 10^{-6} \text{ T}) \cos(kx - \omega t)$$

where

$$\omega = 2\pi f = 2\pi(4.00 \times 10^{7} \text{ s}^{-1}) = 2.51 \times 10^{8} \text{ rad/s}$$

$$k = \frac{2\pi}{\lambda} = \frac{2\pi}{7.50 \text{ m}} = 0.838 \text{ rad/m}$$

Let us summarize the properties of electromagnetic waves as we have described them:

Optional Section

Derivation of Equations 34.6 and 34.7

To derive Equation 34.6, we start with Faraday's law, Equation 34.3:

$$\oint \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_B}{dt}$$

Let us again assume that the electromagnetic wave is traveling in the x direction, with the electric field $\mathbf{E}$ in the positive y direction and the magnetic field $\mathbf{B}$ in the positive z direction.

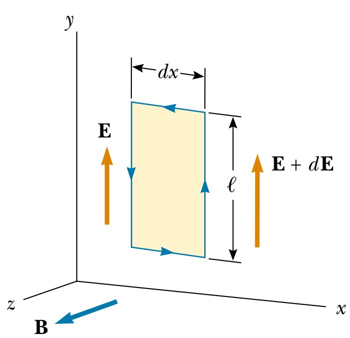


Figure 34.5 As a plane wave passes through a rectangular path of width dx lying in the xy plane, the electric field in the y direction varies from $\mathbf{E}$ to $\mathbf{E} + d\mathbf{E}$. This spatial variation in $\mathbf{E}$ gives rise to a time-varying magnetic field along the z direction, according to Equation 34.6.

Consider a rectangle of width dx and height ℓ lying in the xy plane, as shown in Figure 34.5. To apply Equation 34.3, we must first evaluate the line integral of $\mathbf{E} \cdot d\mathbf{s}$ around this rectangle. The contributions from the top and bottom of the rectangle are zero because $\mathbf{E}$ is perpendicular to $d\mathbf{s}$ for these paths. We can express the electric field on the right side of the rectangle as

$$E(x + dx, t) \approx E(x, t) + \left.\frac{dE}{dx}\right]_{t \text{ constant}} dx = E(x, t) + \frac{\partial E}{\partial x} dx$$

while the field on the left side is simply $E(x, t)$.[3] Therefore, the line integral over this rectangle is approximately

$$\oint \mathbf{E} \cdot d\mathbf{s} = E(x + dx, t) \cdot \ell - E(x, t) \cdot \ell \approx (\partial E/\partial x)\, dx \cdot \ell \qquad \textbf{(34.14)}$$

Because the magnetic field is in the z direction, the magnetic flux through the rectangle of area $\ell\ dx$ is approximately $\Phi_B = B\ell\, dx$. (This assumes that dx is very small compared with the wavelength of the wave.) Taking the time derivative of

[3] Because dE/dx in this equation is expressed as the change in E with x at a given instant t, dE/dx is equivalent to the partial derivative $\partial E/\partial x$. Likewise, dB/dt means the change in B with time at a particular position x, so in Equation 34.15 we can replace dB/dt with $\partial B/\partial t$.

the magnetic flux gives

$$\frac{d\Phi_B}{dt} = \ell\ dx \frac{dB}{dt}\bigg]_{x\ \text{constant}} = \ell\ dx \frac{\partial B}{\partial t} \tag{34.15}$$

Substituting Equations 34.14 and 34.15 into Equation 34.3, we obtain

$$\left(\frac{\partial E}{\partial x}\right) dx \cdot \ell = -\ell\ dx \frac{\partial B}{\partial t}$$

$$\frac{\partial E}{\partial x} = -\frac{\partial B}{\partial t}$$

This expression is Equation 34.6.

In a similar manner, we can verify Equation 34.7 by starting with Maxwell's fourth equation in empty space (Eq. 34.5). In this case, we evaluate the line integral of $\mathbf{B} \cdot d\mathbf{s}$ around a rectangle lying in the xz plane and having width dx and length ℓ, as shown in Figure 34.6. Noting that the magnitude of the magnetic field changes from $B(x, t)$ to $B(x + dx, t)$ over the width dx, we find the line integral over this rectangle to be approximately

$$\oint \mathbf{B} \cdot d\mathbf{s} = B(x, t) \cdot \ell - B(x + dx, t) \cdot \ell \approx -(\partial B/\partial x)\ dx \cdot \ell \tag{34.16}$$

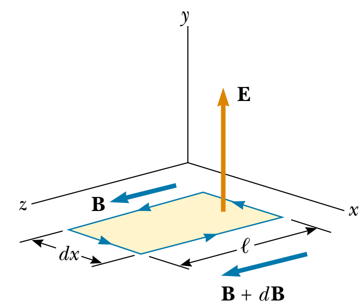


Figure 34.6 As a plane wave passes through a rectangular path of width dx lying in the xz plane, the magnetic field in the z direction varies from $\mathbf{B}$ to $\mathbf{B} + d\mathbf{B}$. This spatial variation in $\mathbf{B}$ gives rise to a time-varying electric field along the y direction, according to Equation 34.7.

The electric flux through the rectangle is $\Phi_E = E\ell\ dx$, which, when differentiated with respect to time, gives

$$\frac{\partial \Phi_E}{\partial t} = \ell\ dx \frac{\partial E}{\partial t} \tag{34.17}$$

Substituting Equations 34.16 and 34.17 into Equation 34.5 gives

$$-(\partial B/\partial x)\ dx \cdot \ell = \mu_0 \epsilon_0 \ell\ dx(\partial E/\partial t)$$

$$\frac{\partial B}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

which is Equation 34.7.

34.3 ENERGY CARRIED BY ELECTROMAGNETIC WAVES

Electromagnetic waves carry energy, and as they propagate through space they can transfer energy to objects placed in their path. The rate of flow of energy in an electromagnetic wave is described by a vector $\mathbf{S}$, called the **Poynting vector,** which is defined by the expression

Poynting vector

$$\boxed{\mathbf{S} \equiv \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}} \tag{34.18}$$

The magnitude of the Poynting vector represents the rate at which energy flows through a unit surface area perpendicular to the direction of wave propagation. Thus, the magnitude of the Poynting vector represents *power per unit area.* The direction of the vector is along the direction of wave propagation (Fig. 34.7). The SI units of the Poynting vector are $\text{J/s}\cdot\text{m}^2 = \text{W/m}^2$.

Magnitude of the Poynting vector for a plane wave

As an example, let us evaluate the magnitude of **S** for a plane electromagnetic wave where $|\mathbf{E} \times \mathbf{B}| = EB$. In this case,

$$S = \frac{EB}{\mu_0} \qquad \textbf{(34.19)}$$

Because $B = E/c$, we can also express this as

$$S = \frac{E^2}{\mu_0 c} = \frac{c}{\mu_0} B^2$$

These equations for S apply at any instant of time and represent the *instantaneous* rate at which energy is passing through a unit area.

What is of greater interest for a sinusoidal plane electromagnetic wave is the time average of S over one or more cycles, which is called the *wave intensity I*. (We discussed the intensity of sound waves in Chapter 17.) When this average is taken, we obtain an expression involving the time average of $\cos^2(kx - \omega t)$, which equals $\frac{1}{2}$. Hence, the average value of S (in other words, the intensity of the wave) is

Wave intensity

$$I = S_{\text{av}} = \frac{E_{\max} B_{\max}}{2\mu_0} = \frac{E_{\max}^2}{2\mu_0 c} = \frac{c}{2\mu_0} B_{\max}^2 \qquad \textbf{(34.20)}$$

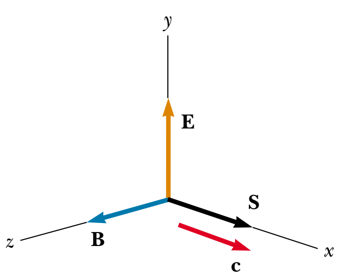


Figure 34.7 The Poynting vector **S** for a plane electromagnetic wave is along the direction of wave propagation.

Recall that the energy per unit volume, which is the instantaneous energy density u_E associated with an electric field, is given by Equation 26.13,

$$u_E = \tfrac{1}{2}\epsilon_0 E^2$$

and that the instantaneous energy density u_B associated with a magnetic field is given by Equation 32.14:

$$u_B = \frac{B^2}{2\mu_0}$$

Because E and B vary with time for an electromagnetic wave, the energy densities also vary with time. When we use the relationships $B = E/c$ and $c = 1/\sqrt{\mu_0\epsilon_0}$, Equation 32.14 becomes

$$u_B = \frac{(E/c)^2}{2\mu_0} = \frac{\mu_0\epsilon_0}{2\mu_0} E^2 = \tfrac{1}{2}\epsilon_0 E^2$$

Comparing this result with the expression for u_E, we see that

$$u_B = u_E = \tfrac{1}{2}\epsilon_0 E^2 = \frac{B^2}{2\mu_0}$$

That is, **for an electromagnetic wave, the instantaneous energy density associated with the magnetic field equals the instantaneous energy density associated with the electric field.** Hence, in a given volume the energy is equally shared by the two fields.

Total instantaneous energy density

The **total instantaneous energy density** u is equal to the sum of the energy densities associated with the electric and magnetic fields:

$$u = u_E + u_B = \epsilon_0 E^2 = \frac{B^2}{\mu_0}$$

Average energy density of an electromagnetic wave

When this total instantaneous energy density is averaged over one or more cycles of an electromagnetic wave, we again obtain a factor of $\frac{1}{2}$. Hence, for any electromagnetic wave, the total average energy per unit volume is

$$u_{\text{av}} = \epsilon_0 (E^2)_{\text{av}} = \tfrac{1}{2}\,\epsilon_0 E_{\text{max}}^2 = \frac{B_{\text{max}}^2}{2\mu_0} \tag{34.21}$$

EXAMPLE 34.2 Fields on the Page

Estimate the maximum magnitudes of the electric and magnetic fields of the light that is incident on this page because of the visible light coming from your desk lamp. Treat the bulb as a point source of electromagnetic radiation that is about 5% efficient at converting electrical energy to visible light.

Solution Recall from Equation 17.8 that the wave intensity I a distance r from a point source is $I = \mathcal{P}_{\text{av}}/4\pi r^2$, where $\mathcal{P}_{\text{av}}$ is the average power output of the source and $4\pi r^2$ is the area of a sphere of radius r centered on the source. Because the intensity of an electromagnetic wave is also given by Equation 34.20, we have

$$I = \frac{\mathcal{P}_{\text{av}}}{4\pi r^2} = \frac{E_{\text{max}}^2}{2\mu_0 c}$$

We must now make some assumptions about numbers to enter in this equation. If we have a 60-W lightbulb, its output at 5% efficiency is approximately 3.0 W in the form of visible light. (The remaining energy transfers out of the bulb by conduction and invisible radiation.) A reasonable distance from the bulb to the page might be 0.30 m. Thus, we have

$$E_{\text{max}} = \sqrt{\frac{\mu_0 c \mathcal{P}_{\text{av}}}{2\pi r^2}}$$

$$= \sqrt{\frac{(4\pi \times 10^{-7}\ \text{T}\cdot\text{m/A})(3.00 \times 10^8\ \text{m/s})(3.0\ \text{W})}{2\pi (0.30\ \text{m})^2}}$$

$$= \boxed{45\ \text{V/m}}$$

From Equation 34.13,

$$B_{\text{max}} = \frac{E_{\text{max}}}{c} = \frac{45\ \text{V/m}}{3.00 \times 10^8\ \text{m/s}} = \boxed{1.5 \times 10^{-7}\ \text{T}}$$

This value is two orders of magnitude smaller than the Earth's magnetic field, which, unlike the magnetic field in the light wave from your desk lamp, is not oscillating.

Exercise Estimate the energy density of the light wave just before it strikes this page.

Answer $9.0 \times 10^{-9}\ \text{J/m}^3$.

Comparing this result with Equation 34.20 for the average value of S, we see that

$$I = S_{\text{av}} = c u_{\text{av}} \tag{34.22}$$

In other words, **the intensity of an electromagnetic wave equals the average energy density multiplied by the speed of light.**

34.4 MOMENTUM AND RADIATION PRESSURE

Momentum transported to a perfectly absorbing surface

Electromagnetic waves transport linear momentum as well as energy. It follows that, as this momentum is absorbed by some surface, pressure is exerted on the surface. We shall assume in this discussion that the electromagnetic wave strikes the surface at normal incidence and transports a total energy U to the surface in a time t. Maxwell showed that, if the surface absorbs all the incident energy U in this time (as does a black body, introduced in Chapter 20), the total momentum $\mathbf{p}$ transported to the surface has a magnitude

$$p = \frac{U}{c} \quad \text{(complete absorption)} \tag{34.23}$$

The pressure exerted on the surface is defined as force per unit area F/A. Let us combine this with Newton's second law:

$$P = \frac{F}{A} = \frac{1}{A}\frac{dp}{dt}$$

If we now replace p, the momentum transported to the surface by light, from Equation 34.23, we have

Radiation pressure exerted on a perfectly absorbing surface

$$P = \frac{1}{A}\frac{dp}{dt} = \frac{1}{A}\frac{d}{dt}\left(\frac{U}{c}\right) = \frac{1}{c}\frac{(dU/dt)}{A}$$

We recognize $(dU/dt)/A$ as the rate at which energy is arriving at the surface per unit area, which is the magnitude of the Poynting vector. Thus, the radiation pressure P exerted on the perfectly absorbing surface is

$$P = \frac{S}{c} \qquad \textbf{(34.24)}$$

Note that Equation 34.24 is an expression for uppercase P, the pressure, while Equation 34.23 is an expression for lowercase p, linear momentum.

If the surface is a perfect reflector (such as a mirror) and incidence is normal, then the momentum transported to the surface in a time t is twice that given by Equation 34.23. That is, the momentum transferred to the surface by the incoming light is $p = U/c$, and that transferred by the reflected light also is $p = U/c$. Therefore,

$$p = \frac{2U}{c} \qquad \text{(complete reflection)} \qquad \textbf{(34.25)}$$

Radiation pressure exerted on a perfectly reflecting surface

The momentum delivered to a surface having a reflectivity somewhere between these two extremes has a value between U/c and $2U/c$, depending on the properties of the surface. Finally, the radiation pressure exerted on a perfectly reflecting surface for normal incidence of the wave is[4]

$$P = \frac{2S}{c} \qquad \textbf{(34.26)}$$

Although radiation pressures are very small (about 5×10^{-6} N/m^2 for direct sunlight), they have been measured with torsion balances such as the one shown in Figure 34.8. A mirror (a perfect reflector) and a black disk (a perfect absorber) are connected by a horizontal rod suspended from a fine fiber. Normal-incidence light striking the black disk is completely absorbed, so all of the momentum of the

QuickLab

Using Example 34.2 as a starting point, estimate the total force exerted on this page by the light from your desk lamp. Does it make a difference if the page contains large, dark photographs instead of mostly white space?

web

Visit **http://pds.jpl.nasa.gov** for more information about missions to the planets. You may also want to read Arthur C. Clarke's 1963 science fiction story *The Wind from the Sun* about a solar yacht race.

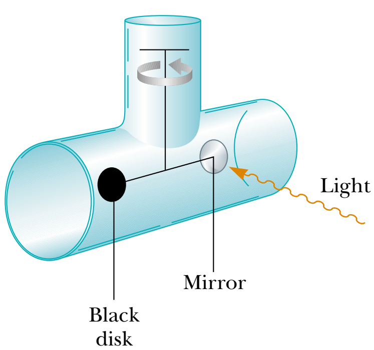


Figure 34.8 An apparatus for measuring the pressure exerted by light. In practice, the system is contained in a high vacuum.

Figure 34.9 Mariner 10 used its solar panels to "sail on sunlight."

[4] For oblique incidence on a perfectly reflecting surface, the momentum transferred is $(2U\cos\theta)/c$ and the pressure is $P = (2S\cos^2\theta)/c$, where θ is the angle between the normal to the surface and the direction of wave propagation.

light is transferred to the disk. Normal-incidence light striking the mirror is totally reflected, and hence the momentum transferred to the mirror is twice as great as that transferred to the disk. The radiation pressure is determined by measuring the angle through which the horizontal connecting rod rotates. The apparatus

CONCEPTUAL EXAMPLE 34.3 Sweeping the Solar System

A great amount of dust exists in interplanetary space. Although in theory these dust particles can vary in size from molecular size to much larger, very little of the dust in our solar system is smaller than about 0.2 μm. Why?

Solution The dust particles are subject to two significant forces—the gravitational force that draws them toward the Sun and the radiation-pressure force that pushes them away from the Sun. The gravitational force is proportional to the cube of the radius of a spherical dust particle because it is proportional to the mass and therefore to the volume $4\pi r^3/3$ of the particle. The radiation pressure is proportional to the square of the radius because it depends on the planar cross-section of the particle. For large particles, the gravitational force is greater than the force from radiation pressure. For particles having radii less than about 0.2 μm, the radiation-pressure force is greater than the gravitational force, and as a result these particles are swept out of the Solar System.

EXAMPLE 34.4 Pressure from a Laser Pointer

Many people giving presentations use a laser pointer to direct the attention of the audience. If a 3.0-mW pointer creates a spot that is 2.0 mm in diameter, determine the radiation pressure on a screen that reflects 70% of the light that strikes it. The power 3.0 mW is a time-averaged value.

Solution We certainly do not expect the pressure to be very large. Before we can calculate it, we must determine the Poynting vector of the beam by dividing the time-averaged power delivered via the electromagnetic wave by the cross-sectional area of the beam:

$$S = \frac{\mathcal{P}}{A} = \frac{\mathcal{P}}{\pi r^2} = \frac{3.0 \times 10^{-3}\,\text{W}}{\pi\left(\dfrac{2.0 \times 10^{-3}\,\text{m}}{2}\right)^2} = 955\ \text{W/m}^2$$

This is about the same as the intensity of sunlight at the Earth's surface. (Thus, it is not safe to shine the beam of a laser pointer into a person's eyes; that may be more dangerous than looking directly at the Sun.)

Now we can determine the radiation pressure from the laser beam. Equation 34.26 indicates that a completely reflected beam would apply a pressure of $P = 2S/c$. We can model the actual reflection as follows: Imagine that the surface absorbs the beam, resulting in pressure $P = S/c$. Then the surface emits the beam, resulting in additional pressure $P = S/c$. If the surface emits only a fraction f of the beam (so that f is the amount of the incident beam reflected), then the pressure due to the emitted beam is $P = fS/c$. Thus, the total pressure on the surface due to absorption and re-emission (reflection) is

$$P = \frac{S}{c} + f\frac{S}{c} = (1 + f)\frac{S}{c}$$

Notice that if $f = 1$, which represents complete reflection, this equation reduces to Equation 34.26. For a beam that is 70% reflected, the pressure is

$$P = (1 + 0.70)\frac{955\ \text{W/m}^2}{3.0 \times 10^8\ \text{m/s}} = \boxed{5.4 \times 10^{-6}\ \text{N/m}^2}$$

This is an extremely small value, as expected. (Recall from Section 15.2 that atmospheric pressure is approximately $10^5\ \text{N/m}^2$.)

EXAMPLE 34.5 Solar Energy

As noted in the preceding example, the Sun delivers about 1 000 W/m^2 of energy to the Earth's surface via electromagnetic radiation. (a) Calculate the total power that is incident on a roof of dimensions 8.00 m $\times$ 20.0 m.

Solution The magnitude of the Poynting vector for solar radiation at the surface of the Earth is $S = 1\,000\ \text{W/m}^2$; this represents the power per unit area, or the light intensity. Assuming that the radiation is incident normal to the roof, we obtain

$$\mathcal{P} = SA = (1\,000\ \text{W/m}^2)(8.00 \times 20.0\ \text{m}^2)$$
$$= \boxed{1.60 \times 10^5\ \text{W}}$$

If all of this power could be converted to electrical energy, it would provide more than enough power for the average home. However, solar energy is not easily harnessed, and the prospects for large-scale conversion are not as bright as may appear from this calculation. For example, the efficiency of conversion from solar to electrical energy is typically 10% for photovoltaic cells. Roof systems for converting solar energy to internal energy are approximately 50% efficient; however, solar energy is associated with other practical problems, such as overcast days, geographic location, and methods of energy storage.

(b) Determine the radiation pressure and the radiation force exerted on the roof, assuming that the roof covering is a perfect absorber.

Solution Using Equation 34.24 with $S = 1\,000\ \text{W/m}^2$, we find that the radiation pressure is

$$P = \frac{S}{c} = \frac{1\,000\ \text{W/m}^2}{3.00 \times 10^8\ \text{m/s}} = \boxed{3.33 \times 10^{-6}\ \text{N/m}^2}$$

Because pressure equals force per unit area, this corresponds to a radiation force of

$$F = PA = (3.33 \times 10^{-6}\ \text{N/m}^2)(160\ \text{m}^2) = \boxed{5.33 \times 10^{-4}\ \text{N}}$$

Exercise How much solar energy is incident on the roof in 1 h?

Answer 5.76×10^8 J.

must be placed in a high vacuum to eliminate the effects of air currents.

NASA is exploring the possibility of *solar sailing* as a low-cost means of sending spacecraft to the planets. Large reflective sheets would be used in much the way canvas sheets are used on earthbound sailboats. In 1973 NASA engineers took advantage of the momentum of the sunlight striking the solar panels of Mariner 10 (Fig. 34.9) to make small course corrections when the spacecraft's fuel supply was running low. (This procedure was carried out when the spacecraft was in the vicinity of the planet Mercury. Would it have worked as well near Pluto?)

Optional Section

34.5 RADIATION FROM AN INFINITE CURRENT SHEET

In this section, we describe the electric and magnetic fields radiated by a flat conductor carrying a time-varying current. In the symmetric plane geometry employed here, the mathematics is less complex than that required in lower-symmetry situations.

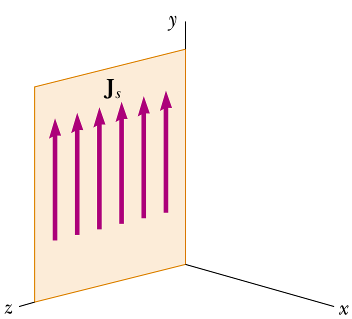


Figure 34.10 A portion of an infinite current sheet lying in the yz plane. The current density is sinusoidal and is given by the expression $J_s = J_{\max} \cos \omega t$. The magnetic field is everywhere parallel to the sheet and lies along z.

Consider an infinite conducting sheet lying in the yz plane and carrying a surface current in the y direction, as shown in Figure 34.10. The current is distributed across the z direction such that the current per unit length is $\mathbf{J}_s$. Let us assume that J_s varies sinusoidally with time as

$$J_s = J_{\max} \cos \omega t$$

where $J_{\max}$ is the amplitude of the current variation and ω is the angular frequency of the variation. A similar problem concerning the case of a steady current was treated in Example 30.6, in which we found that the magnetic field outside the sheet is everywhere parallel to the sheet and lies along the z axis. The magnetic field was found to have a magnitude

Radiated magnetic field

$$B_z = \mu_0 \frac{J_s}{2}$$

[5] Note that the solution could also be written in the form $\cos(\omega t - kx)$, which is equivalent to $\cos(kx - \omega t)$. That is, $\cos\theta$ is an even function, which means that $\cos(-\theta) = \cos\theta$.

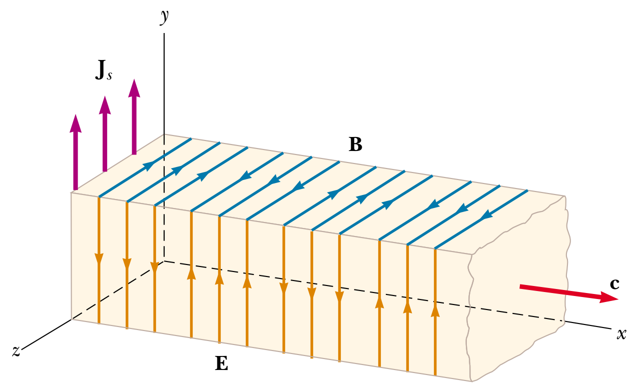


Figure 34.11 Representation of the plane electromagnetic wave radiated by an infinite current sheet lying in the yz plane. The vector $\mathbf{B}$ is in the z direction, the vector $\mathbf{E}$ is in the y direction, and the direction of wave motion is along x. Both vector $\mathbf{B}$ and vector $\mathbf{E}$ behave according to the expression $\cos(kx - \omega t)$. Compare this drawing with Figure 34.3a.

In the present situation, where J_s varies with time, this equation for B_z is valid only for distances close to the sheet. Substituting the expression for J_s, we have

$$B_z = \frac{\mu_0}{2} J_{\max} \cos \omega t \qquad \text{(for small values of } x\text{)}$$

To obtain the expression valid for B_z for arbitrary values of x, we can investigate the solution:[5]

$$B_z = \frac{\mu_0 J_{\max}}{2} \cos(kx - \omega t) \tag{34.27}$$

Radiated electric field

You should note two things about this solution, which is unique to the geometry under consideration. First, when x is very small, it agrees with our original solution. Second, it satisfies the wave equation as expressed in Equation 34.9. We conclude that the magnetic field lies along the z axis, varies with time, and is characterized by a transverse traveling wave having an angular frequency ω and an angular wave number $k = 2\pi/\lambda$.

We can obtain the electric field radiating from our infinite current sheet by using Equation 34.13:

$$E_y = cB_z = \frac{\mu_0 J_{\max} c}{2} \cos(kx - \omega t) \tag{34.28}$$

That is, the electric field is in the y direction, perpendicular to $\mathbf{B}$, and has the same space and time dependencies. These expressions for B_z and E_y show that the radiation field of an infinite current sheet carrying a sinusoidal current is a plane electromagnetic wave propagating with a speed c along the x axis, as shown in Figure 34.11.

We can calculate the Poynting vector for this wave from Equations 34.19,

EXAMPLE 34.6 An Infinite Sheet Carrying a Sinusoidal Current

An infinite current sheet lying in the yz plane carries a sinusoidal current that has a maximum density of 5.00 A/m. (a) Find the maximum values of the radiated magnetic and electric fields.

Solution From Equations 34.27 and 34.28, we see that the maximum values of B_z and E_y are

$$B_{\max} = \frac{\mu_0 J_{\max}}{2} \qquad \text{and} \qquad E_{\max} = \frac{\mu_0 J_{\max} c}{2}$$

Using the values $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$, $J_{\max} = 5.00\ \mathrm{A/m}$, and $c = 3.00 \times 10^8\ \mathrm{m/s}$, we get

$$B_{\max} = \frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})(5.00\ \mathrm{A/m})}{2}$$

$$= \boxed{3.14 \times 10^{-6}\ \mathrm{T}}$$

$$E_{\max} = \frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})(5.00\ \mathrm{A/m})(3.00 \times 10^8\ \mathrm{m/s})}{2}$$

$$= \boxed{942\ \mathrm{V/m}}$$

(b) What is the average power incident on a flat surface that is parallel to the sheet and has an area of $3.00\ \mathrm{m^2}$? (The length and width of this surface are both much greater than the wavelength of the radiation.)

Solution The intensity, or power per unit area, radiated in each direction by the current sheet is given by Equation 34.30:

$$I = \frac{\mu_0 J_{\max}^2 c}{8}$$

$$= \frac{(4\pi \times 10^{-7}\ \mathrm{T \cdot m/A})(5.00\ \mathrm{A/m})^2(3.00 \times 10^8\ \mathrm{m/s})}{8}$$

$$= 1.18 \times 10^3\ \mathrm{W/m^2}$$

Multiplying this by the area of the surface, we obtain the incident power:

$$\mathcal{P} = IA = (1.18 \times 10^3\ \mathrm{W/m^2})(3.00\ \mathrm{m^2})$$

$$= \boxed{3.54 \times 10^3\ \mathrm{W}}$$

The result is independent of the distance from the current sheet because we are dealing with a plane wave.

34.27, and 34.28:

$$S = \frac{EB}{\mu_0} = \frac{\mu_0 J_{\max}^2 c}{4} \cos^2(kx - \omega t) \tag{34.29}$$

The intensity of the wave, which equals the average value of S, is

$$I = S_{\mathrm{av}} = \frac{\mu_0 J_{\max}^2 c}{8} \tag{34.30}$$

Accelerating charges produce electromagnetic radiation

This intensity represents the power per unit area of the outgoing wave on each side of the sheet. The total rate of energy emitted per unit area of the conductor is $2S_{\mathrm{av}} = \mu_0 J_{\max}^2 c/4$.

Optional Section

34.6 PRODUCTION OF ELECTROMAGNETIC WAVES BY AN ANTENNA

Neither stationary charges nor steady currents can produce electromagnetic waves. Whenever the current through a wire changes with time, however, the wire emits

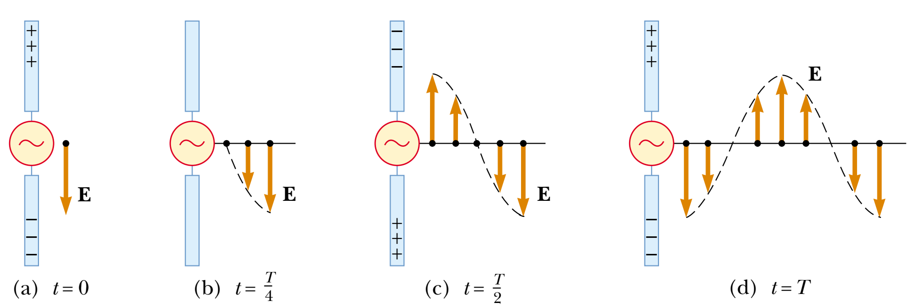


Figure 34.12 The electric field set up by charges oscillating in an antenna. The field moves away from the antenna with the speed of light.

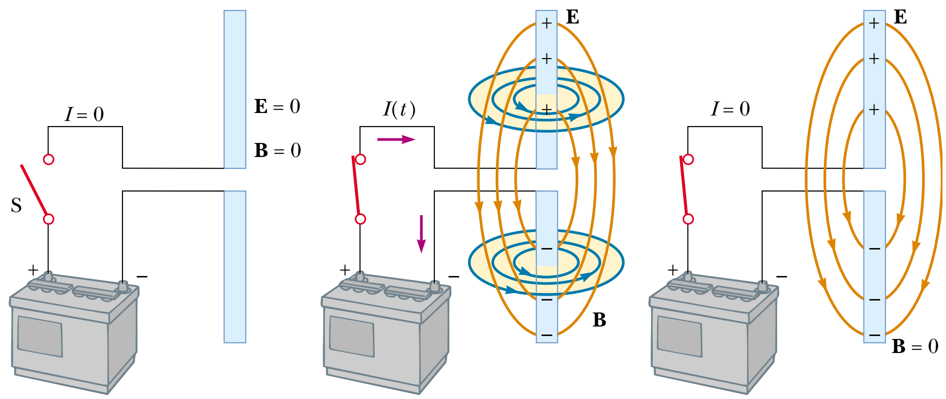


Figure 34.13 A pair of metal rods connected to a battery. (a) When the switch is open and no current exists, the electric and magnetic fields are both zero. (b) Immediately after the switch is closed, the rods are being charged (so a current exists). Because the current is changing, the rods generate changing electric and magnetic fields. (c) When the rods are fully charged, the current is zero, the electric field is a maximum, and the magnetic field is zero.

electromagnetic radiation. **The fundamental mechanism responsible for this radiation is the acceleration of a charged particle. Whenever a charged particle accelerates, it must radiate energy.**

An alternating voltage applied to the wires of an antenna forces an electric charge in the antenna to oscillate. This is a common technique for accelerating charges and is the source of the radio waves emitted by the transmitting antenna of a radio station. Figure 34.12 shows how this is done. Two metal rods are connected to a generator that provides a sinusoidally oscillating voltage. This causes charges to oscillate in the two rods. At $t = 0$, the upper rod is given a maximum positive charge and the bottom rod an equal negative charge, as shown in Figure 34.12a. The electric field near the antenna at this instant is also shown in Figure 34.12a. As the positive and negative charges decrease from their maximum values, the rods become less charged, the field near the rods decreases in strength, and the downward-directed maximum electric field produced at $t = 0$ moves away from the rod. (A magnetic field oscillating in a direction perpendicular to the plane of the diagram in Fig. 34.12 accompanies the oscillating electric field, but it is not shown for the sake of clarity.) When the charges on the rods are momentarily zero (Fig. 34.12b), the electric field at the rod has dropped to zero. This occurs at a time equal to one quarter of the period of oscillation.

As the generator charges the rods in the opposite sense from that at the beginning, the upper rod soon obtains a maximum negative charge and the lower rod a maximum positive charge (Fig. 34.12c); this results in an electric field near the rod that is directed upward after a time equal to one-half the period of oscillation. The oscillations continue as indicated in Figure 34.12d. The electric field near the antenna oscillates in phase with the charge distribution. That is, the field points down when the upper rod is positive and up when the upper rod is negative. Furthermore, the magnitude of the field at any instant depends on the amount of charge on the rods at that instant.

As the charges continue to oscillate (and accelerate) between the rods, the

[6] We have neglected the fields caused by the wires leading to the rods. This is a good approximation if the circuit dimensions are much less than the length of the rods.

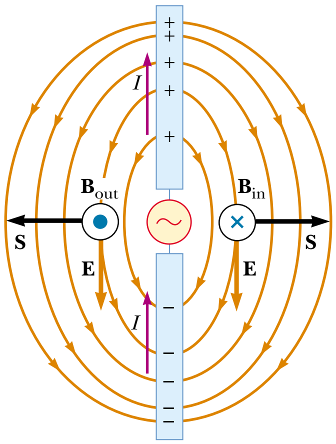


Figure 34.14 A half-wave antenna consists of two metal rods connected to an alternating voltage source. This diagram shows **E** and **B** at an instant when the current is upward. Note that the electric field lines resemble those of a dipole (shown in Fig. 23.21).

electric field they set up moves away from the antenna at the speed of light. As you can see from Figure 34.12, one cycle of charge oscillation produces one wavelength in the electric-field pattern.

Next, consider what happens when two conducting rods are connected to the terminals of a battery (Fig. 34.13). Before the switch is closed, the current is zero, so no fields are present (Fig. 34.13a). Just after the switch is closed, positive charge begins to build up on one rod and negative charge on the other (Fig. 34.13b), a situation that corresponds to a time-varying current. The changing charge distribution causes the electric field to change; this in turn produces a magnetic field around the rods.[6] Finally, when the rods are fully charged, the current is zero; hence, no magnetic field exists at that instant (Fig. 34.13c).

Now let us consider the production of electromagnetic waves by a *half-wave antenna.* In this arrangement, two conducting rods are connected to a source of alternating voltage (such as an LC oscillator), as shown in Figure 34.14. The length of each rod is equal to one quarter of the wavelength of the radiation that will be emitted when the oscillator operates at frequency f. The oscillator forces charges to accelerate back and forth between the two rods. Figure 34.14 shows the configuration of the electric and magnetic fields at some instant when the current is upward. The electric field lines resemble those of an electric dipole. (As a result, this type of antenna is sometimes called a *dipole antenna.*) Because these charges are continuously oscillating between the two rods, the antenna can be approximated by an oscillating electric dipole. The magnetic field lines form concentric circles around the antenna and are perpendicular to the electric field lines at all points. The magnetic field is zero at all points along the axis of the antenna. Furthermore, **E** and **B** are 90° out of phase in time because the current is zero when the charges at the outer ends of the rods are at a maximum.

At the two points where the magnetic field is shown in Figure 34.14, the Poynting vector **S** is directed radially outward. This indicates that energy is flowing away from the antenna at this instant. At later times, the fields and the Poynting vector change direction as the current alternates. Because **E** and **B** are 90° out of phase at points near the dipole, the net energy flow is zero. From this, we might conclude (incorrectly) that no energy is radiated by the dipole.

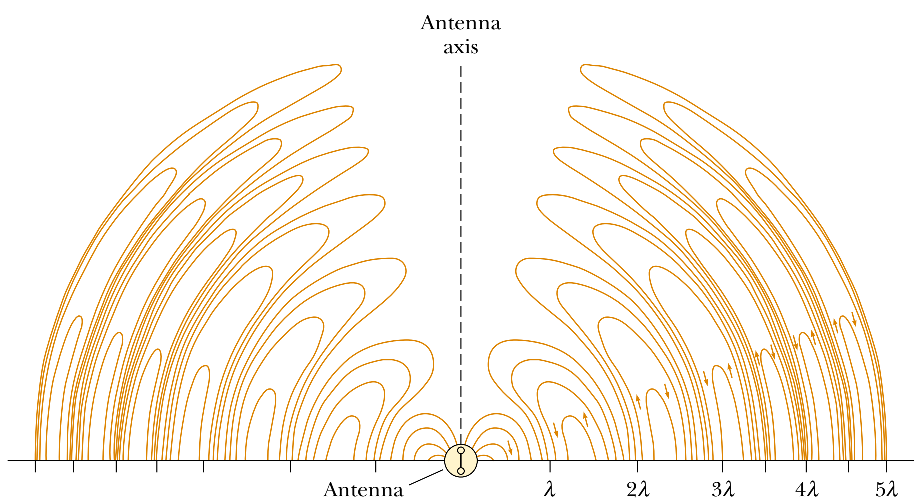


Figure 34.15 Electric field lines surrounding a dipole antenna at a given instant. The radiation fields propagate outward from the antenna with a speed c.

However, we find that energy is indeed radiated. Because the dipole fields fall off as $1/r^3$ (as shown in Example 23.6 for the electric field of a static dipole), they are not important at great distances from the antenna. However, at these great distances, something else causes a type of radiation different from that close to the antenna. The source of this radiation is the continuous induction of an electric field by the time-varying magnetic field and the induction of a magnetic field by the time-varying electric field, predicted by Equations 34.3 and 34.4. The electric and magnetic fields produced in this manner are in phase with each other and vary as $1/r$. The result is an outward flow of energy at all times.

The electric field lines produced by a dipole antenna at some instant are shown in Figure 34.15 as they propagate away from the antenna. Note that the intensity and the power radiated are a maximum in a plane that is perpendicular to the antenna and passing through its midpoint. Furthermore, the power radiated is zero along the antenna's axis. A mathematical solution to Maxwell's equations for the dipole antenna shows that the intensity of the radiation varies as $(\sin^2\theta)/r^2$, where θ is measured from the axis of the antenna. The angular dependence of the radiation intensity is sketched in Figure 34.16.

Electromagnetic waves can also induce currents in a receiving antenna. The response of a dipole receiving antenna at a given position is a maximum when the antenna axis is parallel to the electric field at that point and zero when the axis is perpendicular to the electric field.

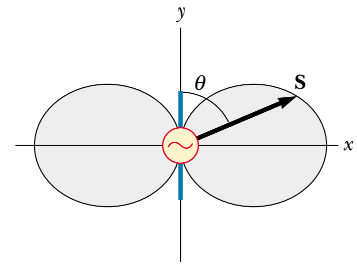


Figure 34.16 Angular dependence of the intensity of radiation produced by an oscillating electric dipole.

QuickLab

Rotate a portable radio (with a telescoping antenna) about a horizontal axis while it is tuned to a weak station. Can you use what you learn from this movement to verify the answer to Quick Quiz 34.2?

Quick Quiz 34.2

If the plane electromagnetic wave in Figure 34.11 represents the signal from a distant radio station, what would be the best orientation for your portable radio antenna—(a) along the x axis, (b) along the y axis, or (c) along the z axis?

34.7 THE SPECTRUM OF ELECTROMAGNETIC WAVES

The various types of electromagnetic waves are listed in Figure 34.17, which shows the **electromagnetic spectrum.** Note the wide ranges of frequencies and wavelengths. No sharp dividing point exists between one type of wave and the next. Remember that **all forms of the various types of radiation are produced by the same phenomenon—accelerating charges.** The names given to the types of waves are simply for convenience in describing the region of the spectrum in which they lie.

Radio waves are the result of charges accelerating through conducting wires. Ranging from more than 10^4 m to about 0.1 m in wavelength, they are generated by such electronic devices as *LC* oscillators and are used in radio and television communication systems.

Microwaves have wavelengths ranging from approximately 0.3 m to 10^{-4} m and are also generated by electronic devices. Because of their short wavelengths, they are well suited for radar systems and for studying the atomic and molecular properties of matter. Microwave ovens (in which the wavelength of the radiation is $\lambda = 0.122$ m) are an interesting domestic application of these waves. It has been suggested that solar energy could be harnessed by beaming microwaves to the Earth from a solar collector in space.[7]

Radio waves

Microwaves

Infrared waves

Visible light waves

[7] P. Glaser, "Solar Power from Satellites," *Phys. Today*, February 1977, p. 30.

Satellite-dish television antennas receive television-station signals from satellites in orbit around the Earth.

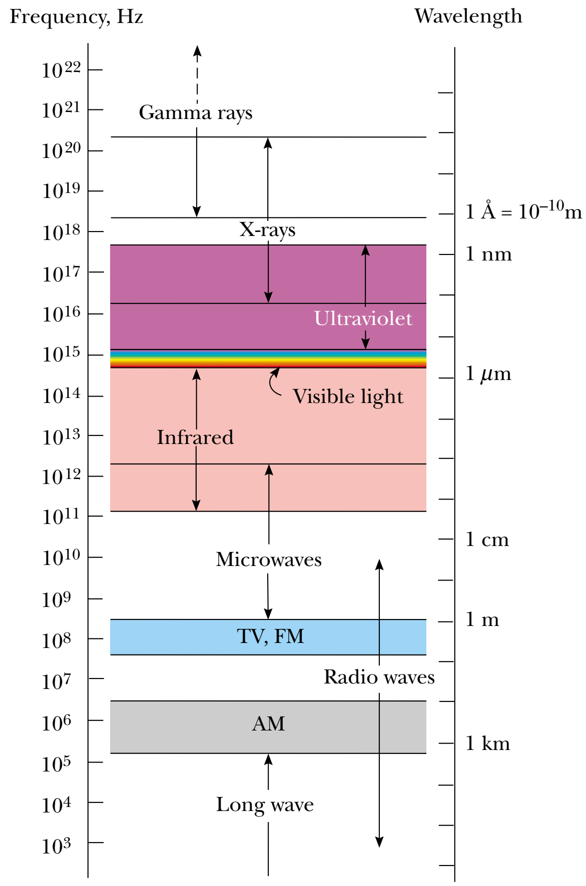


Figure 34.17 The electromagnetic spectrum. Note the overlap between adjacent wave types.

Infrared waves have wavelengths ranging from 10^{-3} m to the longest wavelength of visible light, 7×10^{-7} m. These waves, produced by molecules and room-temperature objects, are readily absorbed by most materials. The infrared (IR) energy absorbed by a substance appears as internal energy because the energy agitates the atoms of the object, increasing their vibrational or translational motion, which results in a temperature increase. Infrared radiation has practical and scientific applications in many areas, including physical therapy, IR photography, and vibrational spectroscopy.

Visible light, the most familiar form of electromagnetic waves, is the part of the electromagnetic spectrum that the human eye can detect. Light is produced by the rearrangement of electrons in atoms and molecules. The various wavelengths of visible light, which correspond to different colors, range from red ($\lambda \approx 7 \times 10^{-7}$ m) to violet ($\lambda \approx 4 \times 10^{-7}$ m). The sensitivity of the human eye is a function of wavelength, being a maximum at a wavelength of about 5.5×10^{-7} m. With this in mind, why do you suppose tennis balls often have a yellow-green color?

Ultraviolet waves

Ultraviolet waves cover wavelengths ranging from approximately 4×10^{-7} m to 6×10^{-10} m. The Sun is an important source of ultraviolet (UV) light, which is the main cause of sunburn. Sunscreen lotions are transparent to visible light but absorb most UV light. The higher a sunscreen's solar protection factor (SPF), the greater the percentage of UV light absorbed. Ultraviolet rays have also been impli-

cated in the formation of cataracts, a clouding of the lens inside the eye. Wearing sunglasses that do not block UV light is worse for your eyes than wearing no sunglasses. The lenses of any sunglasses absorb some visible light, thus causing the wearer's pupils to dilate. If the glasses do not also block UV light, then more damage may be done to the lens of the eye because of the dilated pupils. If you wear no sunglasses at all, your pupils are contracted, you squint, and a lot less UV light enters your eyes. High-quality sunglasses block nearly all the eye-damaging UV light.

Most of the UV light from the Sun is absorbed by ozone (O_3) molecules in the Earth's upper atmosphere, in a layer called the stratosphere. This ozone shield converts lethal high-energy UV radiation to infrared radiation, which in turn warms the stratosphere. Recently, a great deal of controversy has arisen concerning the possible depletion of the protective ozone layer as a result of the chemicals emitted from aerosol spray cans and used as refrigerants.

X-rays

X-rays have wavelengths in the range from approximately 10^{-8} m to 10^{-12} m. The most common source of x-rays is the deceleration of high-energy electrons bombarding a metal target. X-rays are used as a diagnostic tool in medicine and as a treatment for certain forms of cancer. Because x-rays damage or destroy living tissues and organisms, care must be taken to avoid unnecessary exposure or overexposure. X-rays are also used in the study of crystal structure because x-ray wavelengths are comparable to the atomic separation distances in solids (about 0.1 nm).

EXAMPLE 34.7 A Half-Wave Antenna

A half-wave antenna works on the principle that the optimum length of the antenna is one-half the wavelength of the radiation being received. What is the optimum length of a car antenna when it receives a signal of frequency 94.0 MHz?

Solution Equation 16.14 tells us that the wavelength of the signal is

$$\lambda = \frac{c}{f} = \frac{3.00 \times 10^8 \text{ m/s}}{9.40 \times 10^7 \text{ Hz}} = 3.19 \text{ m}$$

Thus, to operate most efficiently, the antenna should have a length of $(3.19 \text{ m})/2 = 1.60$ m. For practical reasons, car antennas are usually one-quarter wavelength in size.

Gamma rays

Gamma rays are electromagnetic waves emitted by radioactive nuclei (such as ^{60}Co and ^{137}Cs) and during certain nuclear reactions. High-energy gamma rays are a component of cosmic rays that enter the Earth's atmosphere from space. They have wavelengths ranging from approximately 10^{-10} m to less than 10^{-14} m. They are highly penetrating and produce serious damage when absorbed by living tissues. Consequently, those working near such dangerous radiation must be protected with heavily absorbing materials, such as thick layers of lead.

Quick Quiz 34.3

The *AM* in *AM radio* stands for *amplitude modulation,* and *FM* stands for *frequency modulation.* (The word *modulate* means "to change.") If our eyes could see the electromagnetic waves from a radio antenna, how could you tell an AM wave from an FM wave?

SUMMARY

Electromagnetic waves, which are predicted by Maxwell's equations, have the

following properties:

- The electric field and the magnetic field each satisfy a wave equation. These two wave equations, which can be obtained from Maxwell's third and fourth equations, are

$$\frac{\partial^2 E}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} \qquad \textbf{(34.8)}$$

$$\frac{\partial^2 B}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 B}{\partial t^2} \qquad \textbf{(34.9)}$$

- The waves travel through a vacuum with the speed of light c, where

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3.00 \times 10^8 \text{ m/s} \qquad \textbf{(34.10)}$$

- The electric and magnetic fields are perpendicular to each other and perpendicular to the direction of wave propagation. (Hence, electromagnetic waves are transverse waves.)
- The instantaneous magnitudes of **E** and **B** in an electromagnetic wave are related by the expression

$$\frac{E}{B} = c \qquad \textbf{(34.13)}$$

- The waves carry energy. The rate of flow of energy crossing a unit area is described by the Poynting vector **S**, where

$$\mathbf{S} \equiv \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \qquad \textbf{(34.18)}$$

- They carry momentum and hence exert pressure on surfaces. If an electromagnetic wave whose Poynting vector is **S** is completely absorbed by a surface upon which it is normally incident, the radiation pressure on that surface is

$$P = \frac{S}{c} \qquad \text{(complete absorption)} \qquad \textbf{(34.24)}$$

If the surface totally reflects a normally incident wave, the pressure is doubled.

The electric and magnetic fields of a sinusoidal plane electromagnetic wave propagating in the positive x direction can be written

$$E = E_{\max} \cos(kx - \omega t) \qquad \textbf{(34.11)}$$

$$B = B_{\max} \cos(kx - \omega t) \qquad \textbf{(34.12)}$$

where ω is the angular frequency of the wave and k is the angular wave number. These equations represent special solutions to the wave equations for E and B. Be-

QUESTIONS

1. For a given incident energy of an electromagnetic wave, why is the radiation pressure on a perfectly reflecting surface twice as great as that on a perfectly absorbing surface?
2. Describe the physical significance of the Poynting vector.
3. Do all current-carrying conductors emit electromagnetic waves? Explain.
4. What is the fundamental cause of electromagnetic radiation?
5. Electrical engineers often speak of the radiation resistance of an antenna. What do you suppose they mean by this phrase?
6. If a high-frequency current is passed through a solenoid containing a metallic core, the core warms up by induc-

tion. This process also cooks foods in microwave ovens. Explain why the materials warm up in these situations.

7. Before the advent of cable television and satellite dishes, homeowners either mounted a television antenna on the roof or used "rabbit ears" atop their sets (Fig. Q34.7). Certain orientations of the receiving antenna on a television set gave better reception than others. Furthermore, the best orientation varied from station to station. Explain.

Figure Q34.7 Questions 7, 12, 13, and 14. The V-shaped antenna is the VHF antenna. *(George Semple)*

8. Does a wire connected to the terminals of a battery emit an electromagnetic wave? Explain.

9. If you charge a comb by running it through your hair and then hold the comb next to a bar magnet, do the electric and magnetic fields that are produced constitute an electromagnetic wave?

10. An empty plastic or glass dish is cool to the touch right after it is removed from a microwave oven. How can this be possible? (Assume that your electric bill has been paid.)

11. Often when you touch the indoor antenna on a radio or television receiver, the reception instantly improves. Why?

12. Explain how the (dipole) VHF antenna of a television set works. (See Fig. Q34.7.)

13. Explain how the UHF (loop) antenna of a television set works. (See Fig. Q34.7.)

14. Explain why the voltage induced in a UHF (loop) antenna depends on the frequency of the signal, whereas the voltage in a VHF (dipole) antenna does not. (See Fig. Q34.7.)

15. List as many similarities and differences between sound waves and light waves as you can.

16. What does a radio wave do to the charges in the receiving antenna to provide a signal for your car radio?

17. What determines the height of an AM radio station's broadcast antenna?

18. Some radio transmitters use a "phased array" of antennas. What is their purpose?

19. What happens to the radio reception in an airplane as it flies over the (vertical) dipole antenna of the control tower?

20. When light (or other electromagnetic radiation) travels across a given region, what oscillates?

21. Why should an infrared photograph of a person look different from a photograph of that person taken with visible light?

22. Suppose a creature from another planet had eyes that were sensitive to infrared radiation. Describe what the creature would see if it looked around the room you are now in. That is, what would be bright and what would be dim?

PROBLEMS

1, 2, 3 = straightforward, intermediate, challenging □ = full solution available in the *Student Solutions Manual and Study Guide*
WEB = solution posted at **http://www.saunderscollege.com/physics/** = Computer useful in solving problem = Interactive Physics
 = paired numerical/symbolic problems

Section 34.1 Maxwell's Equations and Hertz's Discoveries

Section 34.2 Plane Electromagnetic Waves

Note: Assume that the medium is vacuum unless specified otherwise.

1. If the North Star, Polaris, were to burn out today, in what year would it disappear from our vision? Take the distance from the Earth to Polaris as 6.44×10^{18} m.

2. The speed of an electromagnetic wave traveling in a transparent nonmagnetic substance is $v = 1/\sqrt{\kappa\mu_0\epsilon_0}$, where κ is the dielectric constant of the substance. Determine the speed of light in water, which has a dielectric constant at optical frequencies of 1.78.

3. An electromagnetic wave in vacuum has an electric field amplitude of 220 V/m. Calculate the amplitude of the corresponding magnetic field.

4. Calculate the maximum value of the magnetic field of an electromagnetic wave in a medium where the speed of light is two thirds of the speed of light in vacuum and where the electric field amplitude is 7.60 mV/m.

WEB **5.** Figure 34.3a shows a plane electromagnetic sinusoidal wave propagating in what we choose as the x direction. Suppose that the wavelength is 50.0 m, and the electric field vibrates in the xy plane with an amplitude of 22.0 V/m. Calculate (a) the frequency of the wave and (b) the magnitude and direction of $\mathbf{B}$ when the electric field has its maximum value in the negative y direction. (c) Write an expression for B in the form

$$B = B_{\max} \cos(kx - \omega t)$$

with numerical values for $B_{\max}$, k, and ω.

6. Write down expressions for the electric and magnetic fields of a sinusoidal plane electromagnetic wave having a frequency of 3.00 GHz and traveling in the positive x direction. The amplitude of the electric field is 300 V/m.

7. In SI units, the electric field in an electromagnetic wave is described by

$$E_y = 100 \sin(1.00 \times 10^7 x - \omega t)$$

Find (a) the amplitude of the corresponding magnetic field, (b) the wavelength λ, and (c) the frequency f.

8. Verify by substitution that the following equations are solutions to Equations 34.8 and 34.9, respectively:

$$E = E_{\max} \cos(kx - \omega t)$$

$$B = B_{\max} \cos(kx - \omega t)$$

9. **Review Problem.** A standing-wave interference pattern is set up by radio waves between two metal sheets 2.00 m apart. This is the shortest distance between the plates that will produce a standing-wave pattern. What is the fundamental frequency?

10. A microwave oven is powered by an electron tube called a magnetron, which generates electromagnetic waves of frequency 2.45 GHz. The microwaves enter the oven and are reflected by the walls. The standing-wave pattern produced in the oven can cook food unevenly, with hot spots in the food at antinodes and cool spots at nodes, so a turntable is often used to rotate the food and distribute the energy. If a microwave oven intended for use with a turntable is instead used with a cooking dish in a fixed position, the antinodes can appear as burn marks on foods such as carrot strips or cheese. The separation distance between the burns is measured to be 6 cm $\pm$ 5%. From these data, calculate the speed of the microwaves.

Section 34.3 Energy Carried by Electromagnetic Waves

11. How much electromagnetic energy per cubic meter is contained in sunlight, if the intensity of sunlight at the Earth's surface under a fairly clear sky is 1 000 W/m^2?

12. An AM radio station broadcasts isotropically (equally in all directions) with an average power of 4.00 kW. A dipole receiving antenna 65.0 cm long is at a location 4.00 miles from the transmitter. Compute the emf that is induced by this signal between the ends of the receiving antenna.

13. What is the average magnitude of the Poynting vector 5.00 miles from a radio transmitter broadcasting isotropically with an average power of 250 kW?

14. A monochromatic light source emits 100 W of electromagnetic power uniformly in all directions. (a) Calculate the average electric-field energy density 1.00 m from the source. (b) Calculate the average magnetic-field energy density at the same distance from the source. (c) Find the wave intensity at this location.

WEB **15.** A community plans to build a facility to convert solar radiation to electric power. They require 1.00 MW of power, and the system to be installed has an efficiency of 30.0% (that is, 30.0% of the solar energy incident on the surface is converted to electrical energy). What must be the effective area of a perfectly absorbing surface used in such an installation, assuming a constant intensity of 1 000 W/m^2?

16. Assuming that the antenna of a 10.0-kW radio station radiates spherical electromagnetic waves, compute the maximum value of the magnetic field 5.00 km from the antenna, and compare this value with the surface magnetic field of the Earth.

WEB **17.** The filament of an incandescent lamp has a 150-Ω resistance and carries a direct current of 1.00 A. The filament is 8.00 cm long and 0.900 mm in radius. (a) Calculate the Poynting vector at the surface of the filament. (b) Find the magnitude of the electric and magnetic fields at the surface of the filament.

18. In a region of free space the electric field at an instant of time is $\mathbf{E} = (80.0\mathbf{i} + 32.0\mathbf{j} - 64.0\mathbf{k})$ N/C and the magnetic field is $\mathbf{B} = (0.200\mathbf{i} + 0.080\,0\mathbf{j} + 0.290\mathbf{k})$ μT. (a) Show that the two fields are perpendicular to each other. (b) Determine the Poynting vector for these fields.

19. A lightbulb filament has a resistance of 110 Ω. The bulb is plugged into a standard 120-V (rms) outlet and emits 1.00% of the electric power delivered to it as electromagnetic radiation of frequency f. Assuming that the bulb is covered with a filter that absorbs all other frequencies, find the amplitude of the magnetic field 1.00 m from the bulb.

20. A certain microwave oven contains a magnetron that has an output of 700 W of microwave power for an electrical input power of 1.40 kW. The microwaves are entirely transferred from the magnetron into the oven chamber through a waveguide, which is a metal tube of rectangular cross-section with a width of 6.83 cm and a height of 3.81 cm. (a) What is the efficiency of the magnetron? (b) Assuming that the food is absorbing all the microwaves produced by the magnetron and that no energy is reflected back into the waveguide, find the direction and magnitude of the Poynting vector, averaged over time, in the waveguide near the entrance to the oven chamber. (c) What is the maximum electric field magnitude at this point?

21. High-power lasers in factories are used to cut through cloth and metal (Fig. P34.21). One such laser has a beam diameter of 1.00 mm and generates an electric field with an amplitude of 0.700 MV/m at the target. Find (a) the amplitude of the magnetic field produced, (b) the intensity of the laser, and (c) the power delivered by the laser.

Figure P34.21 A laser cutting device mounted on a robot arm is being used to cut through a metallic plate. *(Philippe Plailly/SPL/Photo Researchers)*

22. At what distance from a 100-W electromagnetic-wave point source does $E_{max} = 15.0$ V/m?

23. A 10.0-mW laser has a beam diameter of 1.60 mm. (a) What is the intensity of the light, assuming it is uniform across the circular beam? (b) What is the average energy density of the beam?

24. At one location on the Earth, the rms value of the magnetic field caused by solar radiation is 1.80 μT. From this value, calculate (a) the average electric field due to solar radiation, (b) the average energy density of the solar component of electromagnetic radiation at this location, and (c) the magnitude of the Poynting vector for the Sun's radiation. (d) Compare the value found in part (c) with the value of the solar intensity given in Example 34.5.

Section 34.4 Momentum and Radiation Pressure

25. A radio wave transmits 25.0 W/m^2 of power per unit area. A flat surface of area A is perpendicular to the direction of propagation of the wave. Calculate the radiation pressure on it if the surface is a perfect absorber.

26. A plane electromagnetic wave of intensity 6.00 W/m^2 strikes a small pocket mirror, of area 40.0 cm^2, held perpendicular to the approaching wave. (a) What momentum does the wave transfer to the mirror each second? (b) Find the force that the wave exerts on the mirror.

27. A possible means of space flight is to place a perfectly reflecting aluminized sheet into orbit around the Earth and then use the light from the Sun to push this "solar sail." Suppose a sail of area 6.00×10^5 m^2 and mass 6 000 kg is placed in orbit facing the Sun. (a) What force is exerted on the sail? (b) What is the sail's acceleration? (c) How long does it take the sail to reach the Moon, 3.84×10^8 m away? Ignore all gravitational effects, assume that the acceleration calculated in part (b) remains constant, and assume a solar intensity of 1 340 W/m^2.

28. A 100-mW laser beam is reflected back upon itself by a mirror. Calculate the force on the mirror.

WEB **29.** A 15.0-mW helium–neon laser ($\lambda = 632.8$ nm) emits a beam of circular cross-section with a diameter of 2.00 mm. (a) Find the maximum electric field in the beam. (b) What total energy is contained in a 1.00-m length of the beam? (c) Find the momentum carried by a 1.00-m length of the beam.

30. Given that the intensity of solar radiation incident on the upper atmosphere of the Earth is 1 340 W/m^2, determine (a) the solar radiation incident on Mars, (b) the total power incident on Mars, and (c) the total force acting on the planet. (d) Compare this force to the gravitational attraction between Mars and the Sun (see Table 14.2).

31. A plane electromagnetic wave has an intensity of 750 W/m^2. A flat rectangular surface of dimensions 50.0 cm $\times$ 100 cm is placed perpendicular to the direction of the wave. If the surface absorbs half of the energy and reflects half, calculate (a) the total energy absorbed by the surface in 1.00 min and (b) the momentum absorbed in this time.

(Optional)
Section 34.5 Radiation from an Infinite Current Sheet

32. A large current-carrying sheet emits radiation in each direction (normal to the plane of the sheet) with an intensity of 570 W/m^2. What maximum value of sinusoidal current density is required?

33. A rectangular surface of dimensions 120 cm $\times$ 40.0 cm is parallel to and 4.40 m away from a much larger conducting sheet in which a sinusoidally varying surface current exists that has a maximum value of 10.0 A/m. (a) Calculate the average power that is incident on the smaller sheet. (b) What power per unit area is radiated by the larger sheet?

(Optional)
Section 34.6 Production of Electromagnetic Waves by an Antenna

34. Two hand-held radio transceivers with dipole antennas are separated by a great fixed distance. Assuming that the transmitting antenna is vertical, what fraction of the

maximum received power will occur in the receiving antenna when it is inclined from the vertical by (a) 15.0°? (b) 45.0°? (c) 90.0°?

35. Two radio-transmitting antennas are separated by half the broadcast wavelength and are driven in phase with each other. In which directions are (a) the strongest and (b) the weakest signals radiated?

36. Figure 34.14 shows a Hertz antenna (also known as a half-wave antenna, since its length is $\lambda/2$). The antenna is far enough from the ground that reflections do not significantly affect its radiation pattern. Most AM radio stations, however, use a Marconi antenna, which consists of the top half of a Hertz antenna. The lower end of this (quarter-wave) antenna is connected to earth ground, and the ground itself serves as the missing lower half. What are the heights of the Marconi antennas for radio stations broadcasting at (a) 560 kHz and (b) 1 600 kHz?

37. Review Problem. Accelerating charges radiate electromagnetic waves. Calculate the wavelength of radiation produced by a proton in a cyclotron with a radius of 0.500 m and a magnetic field with a magnitude of 0.350 T.

38. Review Problem. Accelerating charges radiate electromagnetic waves. Calculate the wavelength of radiation produced by a proton in a cyclotron of radius R and magnetic field B.

Section 34.7 The Spectrum of Electromagnetic Waves

39. (a) Classify waves with frequencies of 2 Hz, 2 kHz, 2 MHz, 2 GHz, 2 THz, 2 PHz, 2 EHz, 2 ZHz, and 2 YHz on the electromagnetic spectrum. (b) Classify waves with wavelengths of 2 km, 2 m, 2 mm, 2 μm, 2 nm, 2 pm, 2 fm, and 2 am.

40. Compute an order-of-magnitude estimate for the frequency of an electromagnetic wave with a wavelength equal to (a) your height; (b) the thickness of this sheet of paper. How is each wave classified on the electromagnetic spectrum?

41. The human eye is most sensitive to light having a wavelength of 5.50×10^{-7} m, which is in the green–yellow region of the visible electromagnetic spectrum. What is the frequency of this light?

42. Suppose you are located 180 m from a radio transmitter. (a) How many wavelengths are you from the transmitter if the station calls itself 1150 AM? (The AM band frequencies are in kilohertz.) (b) What if this station were 98.1 FM? (The FM band frequencies are in megahertz.)

43. What are the wavelengths of electromagnetic waves in free space that have frequencies of (a) 5.00×10^{19} Hz and (b) 4.00×10^{9} Hz?

44. A radar pulse returns to the receiver after a total travel time of 4.00×10^{-4} s. How far away is the object that reflected the wave?

45. *This just in!* An important news announcement is transmitted by radio waves to people sitting next to their radios, 100 km from the station, and by sound waves to people sitting across the newsroom, 3.00 m from the newscaster. Who receives the news first? Explain. Take the speed of sound in air to be 343 m/s.

46. The U.S. Navy has long proposed the construction of extremely low-frequency (ELF) communication systems. Such waves could penetrate the oceans to reach distant submarines. Calculate the length of a quarter-wavelength antenna for a transmitter generating ELF waves with a frequency of 75.0 Hz. How practical is this?

47. What are the wavelength ranges in (a) the AM radio band (540–1 600 kHz), and (b) the FM radio band (88.0–108 MHz)?

48. There are 12 VHF television channels (Channels 2–13) that lie in the range of frequencies between 54.0 MHz and 216 MHz. Each channel is assigned a width of 6.0 MHz, with the two ranges 72.0–76.0 MHz and 88.0–174 MHz reserved for non-TV purposes. (Channel 2, for example, lies between 54.0 and 60.0 MHz.) Calculate the wavelength ranges for (a) Channel 4, (b) Channel 6, and (c) Channel 8.

ADDITIONAL PROBLEMS

49. Assume that the intensity of solar radiation incident on the cloud tops of Earth is 1 340 W/m^2. (a) Calculate the total power radiated by the Sun, taking the average Earth–Sun separation to be 1.496×10^{11} m. (b) Determine the maximum values of the electric and magnetic fields at the Earth's location due to solar radiation.

50. The intensity of solar radiation at the top of the Earth's atmosphere is 1 340 W/m^2. Assuming that 60% of the incoming solar energy reaches the Earth's surface and assuming that you absorb 50% of the incident energy, make an order-of-magnitude estimate of the amount of solar energy you absorb in a 60-min sunbath.

WEB **51. Review Problem.** In the absence of cable input or a satellite dish, a television set can use a dipole-receiving antenna for VHF channels and a loop antenna for UHF channels (see Fig. Q34.7). The UHF antenna produces an emf from the changing magnetic flux through the loop. The TV station broadcasts a signal with a frequency f, and the signal has an electric-field amplitude E_{max} and a magnetic-field amplitude B_{max} at the location of the receiving antenna. (a) Using Faraday's law, derive an expression for the amplitude of the emf that appears in a single-turn circular loop antenna with a radius r, which is small compared to the wavelength of the wave. (b) If the electric field in the signal points vertically, what should be the orientation of the loop for best reception?

52. Consider a small, spherical particle of radius r located in space a distance R from the Sun. (a) Show that the ratio F_{rad}/F_{grav} is proportional to $1/r$, where F_{rad} is the

force exerted by solar radiation and F_{grav} is the force of gravitational attraction. (b) The result of part (a) means that, for a sufficiently small value of r, the force exerted on the particle by solar radiation exceeds the force of gravitational attraction. Calculate the value of r for which the particle is in equilibrium under the two forces. (Assume that the particle has a perfectly absorbing surface and a mass density of 1.50 g/cm^3. Let the particle be located 3.75×10^{11} m from the Sun, and use 214 W/m^2 as the value of the solar intensity at that point.)

53. A dish antenna with a diameter of 20.0 m receives (at normal incidence) a radio signal from a distant source, as shown in Figure P34.53. The radio signal is a continuous sinusoidal wave with amplitude $E_{max} = 0.200\ \mu$V/m. Assume that the antenna absorbs all the radiation that falls on the dish. (a) What is the amplitude of the magnetic field in this wave? (b) What is the intensity of the radiation received by this antenna? (c) What power is received by the antenna? (d) What force is exerted on the antenna by the radio waves?

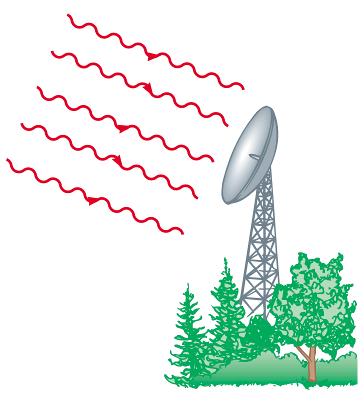


Figure P34.53

54. A parallel-plate capacitor has circular plates of radius r separated by distance ℓ. It has been charged to voltage ΔV and is being discharged as current i is drawn from it. Assume that the plate separation ℓ is very small compared to r, so the electric field is essentially constant in the volume between the plates and is zero outside this volume. Note that the displacement current between the capacitor plates creates a magnetic field. (a) Determine the magnitude and direction of the Poynting vector at the cylindrical surface surrounding the electric field volume. (b) Use the value of the Poynting vector and the lateral surface area of the cylinder to find the total power transfer for the capacitor. (c) What are the changes to these results if the direction of the current is reversed, so the capacitor is charging?

55. A section of a very long air-core solenoid, far from either end, forms an inductor with radius r, length ℓ, and n turns of wire per unit length. At a particular instant, the solenoid current is i and is increasing at the rate di/dt. Ignore the resistance of the wire. (a) Find the magnitude and direction of the Poynting vector over the interior surface of this section of solenoid. (b) Find the rate at which the energy stored in the magnetic field of the inductor is increasing. (c) Express the power in terms of the voltage ΔV across the inductor.

56. A goal of the Russian space program is to illuminate dark northern cities with sunlight reflected to Earth from a 200-m-diameter mirrored surface in orbit. Several smaller prototypes have already been constructed and put into orbit. (a) Assume that sunlight with an intensity of 1 340 W/m^2 falls on the mirror nearly perpendicularly, and that the atmosphere of the Earth allows 74.6% of the energy of sunlight to pass through it in clear weather. What power is received by a city when the space mirror is reflecting light to it? (b) The plan is for the reflected sunlight to cover a circle with a diameter of 8.00 km. What is the intensity of the light (the average magnitude of the Poynting vector) received by the city? (c) This intensity is what percentage of the vertical component of sunlight at Saint Petersburg in January, when the sun reaches an angle of 7.00° above the horizon at noon?

57. In 1965 Arno Penzias and Robert Wilson discovered the cosmic microwave radiation that was left over from the Big Bang expansion of the Universe. Suppose the energy density of this background radiation is equal to 4.00×10^{-14} J/m^3. Determine the corresponding electric-field amplitude.

58. A hand-held cellular telephone operates in the 860- to 900-MHz band and has a power output of 0.600 W from an antenna 10.0 cm long (Fig. P34.58). (a) Find the average magnitude of the Poynting vector 4.00 cm from the antenna, at the location of a typical person's head. Assume that the antenna emits energy with cylindrical wave fronts. (The actual radiation from antennas follows a more complicated pattern, as suggested by Fig. 34.15.) (b) The ANSI/IEEE C95.1-1991 maximum exposure standard is 0.57 mW/cm^2 for persons living near

Figure P34.58. *(©1998 Adam Smith/FPG International)*

cellular telephone base stations, who would be continuously exposed to the radiation. Compare the answer to part (a) with this standard.

59. A linearly polarized microwave with a wavelength of 1.50 cm is directed along the positive x axis. The electric field vector has a maximum value of 175 V/m and vibrates in the xy plane. (a) Assume that the magnetic-field component of the wave can be written in the form $B = B_{max} \sin(kx - \omega t)$, and give values for B_{max}, k, and ω. Also, determine in which plane the magnetic-field vector vibrates. (b) Calculate the magnitude of the Poynting vector for this wave. (c) What maximum radiation pressure would this wave exert if it were directed at normal incidence onto a perfectly reflecting sheet? (d) What maximum acceleration would be imparted to a 500-g sheet (perfectly reflecting and at normal incidence) with dimensions of $1.00 \text{ m} \times 0.750 \text{ m}$?

60. *Review Section 20.7 on thermal radiation.* (a) An elderly couple have installed a solar water heater on the roof of their house (Fig. P34.60). The solar-energy collector consists of a flat closed box with extraordinarily good thermal insulation. Its interior is painted black, and its front face is made of insulating glass. Assume that its emissivity for visible light is 0.900 and its emissivity for infrared light is 0.700. Assume that the noon Sun shines in perpendicular to the glass, with intensity 1 000 W/m^2, and that no water is then entering or leaving the box. Find the steady-state temperature of the interior of the box. (b) The couple have built an identical box with no water tubes. It lies flat on the ground in front of the house. They use it as a cold frame, where they plant seeds in early spring. If the same noon Sun is at an elevation angle of 50.0°, find the steady-state temperature of the interior of this box, assuming that the ventilation slots are tightly closed.

Figure P34.60 *(©Bill Banaszewski/Visuals Unlimited)*

61. An astronaut, stranded in space 10.0 m from his spacecraft and at rest relative to it, has a mass (including equipment) of 110 kg. Since he has a 100-W light source that forms a directed beam, he decides to use the beam as a photon rocket to propel himself continuously toward the spacecraft. (a) Calculate how long it takes him to reach the spacecraft by this method. (b) Suppose, instead, that he decides to throw the light source away in a direction opposite the spacecraft. If the light source has a mass of 3.00 kg and, after being thrown, moves at 12.0 m/s *relative to the recoiling astronaut,* how long does it take for the astronaut to reach the spacecraft?

62. The Earth reflects approximately 38.0% of the incident sunlight from its clouds and surface. (a) Given that the intensity of solar radiation is 1 340 W/m^2, what is the radiation pressure on the Earth, in pascals, when the Sun is straight overhead? (b) Compare this to normal atmospheric pressure at the Earth's surface, which is 101 kPa.

63. Lasers have been used to suspend spherical glass beads in the Earth's gravitational field. (a) If a bead has a mass of 1.00 μg and a density of 0.200 g/cm^3, determine the radiation intensity needed to support the bead. (b) If the beam has a radius of 0.200 cm, what power is required for this laser?

64. Lasers have been used to suspend spherical glass beads in the Earth's gravitational field. (a) If a bead has a mass m and a density ρ, determine the radiation intensity needed to support the bead. (b) If the beam has a radius r, what power is required for this laser?

65. Review Problem. A 1.00-m-diameter mirror focuses the Sun's rays onto an absorbing plate 2.00 cm in radius, which holds a can containing 1.00 L of water at 20.0°C. (a) If the solar intensity is 1.00 kW/m^2, what is the intensity on the absorbing plate? (b) What are the maximum magnitudes of the fields **E** and **B**? (c) If 40.0% of the energy is absorbed, how long would it take to bring the water to its boiling point?

66. A microwave source produces pulses of 20.0-GHz radiation, with each pulse lasting 1.00 ns. A parabolic reflector ($R = 6.00$ cm) is used to focus these pulses into a parallel beam of radiation, as shown in Figure P34.66. The average power during each pulse is 25.0 kW. (a) What is the wavelength of these microwaves? (b) What is the total energy contained in each pulse? (c) Compute the average energy density inside each pulse. (d) Determine the amplitude of the electric and magnetic fields in these microwaves. (e) Compute the force exerted on the surface during the 1.00-ns duration of each pulse if the pulsed beam strikes an absorbing surface.

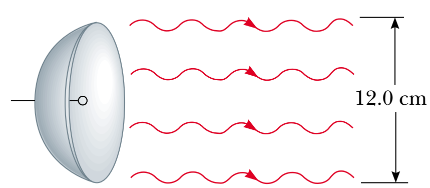


Figure P34.66

67. The electromagnetic power radiated by a nonrelativistic moving point charge q having an acceleration a is

$$\mathcal{P} = \frac{q^2a^2}{6\pi\epsilon_0c^3}$$

where ϵ_0 is the permittivity of vacuum (free space) and c is the speed of light in vacuum. (a) Show that the right side of this equation is in watts. (b) If an electron is placed in a constant electric field of 100 N/C, determine the acceleration of the electron and the electromagnetic power radiated by this electron. (c) If a proton is placed in a cyclotron with a radius of 0.500 m and a magnetic field of magnitude 0.350 T, what electromagnetic power is radiated by this proton?

68. A thin tungsten filament with a length of 1.00 m radiates 60.0 W of power in the form of electromagnetic waves. A perfectly absorbing surface, in the form of a hollow cylinder with a radius of 5.00 cm and a length of 1.00 m, is placed concentrically with the filament. Calculate the radiation pressure acting on the cylinder. (Assume that the radiation is emitted in the radial direction, and neglect end effects.)

69. The torsion balance shown in Figure 34.8 is used in an experiment to measure radiation pressure. The suspension fiber exerts an elastic restoring torque. Its torque constant is 1.00×10^{-11} N · m/degree, and the length of the horizontal rod is 6.00 cm. The beam from a 3.00-mW helium–neon laser is incident on the black disk, and the mirror disk is completely shielded. Calculate the angle between the equilibrium positions of the horizontal bar when the beam is switched from "off" to "on."

70. **Review Problem.** The study of Creation suggests a Creator with a remarkable liking for beetles and for small red stars. A red star, typical of the most common kind, radiates electromagnetic waves with a power of 6.00×10^{23} W, which is only 0.159% of the luminosity of the Sun. Consider a spherical planet in a circular orbit around this star. Assume that the emissivity of the planet, as defined in Section 20.7, is equal for infrared and visible light. Assume that the planet has a uniform surface temperature. Identify the projected area over which the planet absorbs starlight, and the radiating area of the planet. If beetles thrive at a temperature of 310 K, what should the radius of the planet's orbit be?

71. A "laser cannon" of a spacecraft has a beam of cross-sectional area A. The maximum electric field in the beam is E. At what rate a will an asteroid accelerate away from the spacecraft if the laser beam strikes the asteroid perpendicularly to its surface, and the surface is nonreflecting? The mass of the asteroid is m. Neglect the acceleration of the spacecraft.

72. A plane electromagnetic wave varies sinusoidally at 90.0 MHz as it travels along the $+x$ direction. The peak value of the electric field is 2.00 mV/m, and it is directed along the $\pm y$ direction. (a) Find the wavelength, the period, and the maximum value of the magnetic field. (b) Write expressions in SI units for the space and time variations of the electric field and of the magnetic field. Include numerical values, and include subscripts to indicate coordinate directions. (c) Find the average power per unit area that this wave propagates through space. (d) Find the average energy density in the radiation (in joules per cubic meter). (e) What radiation pressure would this wave exert upon a perfectly reflecting surface at normal incidence?

ANSWERS TO QUICK QUIZZES

34.1 Zero. Figure 34.3b shows that the **B** and **E** vectors reach their maximum and minimum values at the same time.

34.2 (b) Along the y axis because that is the orientation of the electric field. The electric field moves electrons in the antenna, thus inducing a current that is detected and amplified.

34.3 The AM wave, because its amplitude is changing, would appear to vary in brightness. The FM wave would have changing colors because the color we perceive is related to the frequency of the light.